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Rational solutions to two Sawada–Kotera-like equations

Ya-Hong Du and Yin-Shan Yun*

College of Sciences, Inner Mongolia University of Technology, Hohhot 010051, China
Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, USA
*yunyinshan@126.com

Wen-Xiu Ma

Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China
College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, Shandong, China
College of Mathematics and Physics, Shanghai University of Electric Power, Shanghai 200090, China
International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, Mmabatho 2735, South Africa

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Two Sawada–Kotera-like equations are introduced by the generalized bilinear operators D_p associated with two prime numbers $p = 3$ and $p = 5$, respectively. Rational solutions of the two presented Sawada–Kotera-like equations are generated by searching polynomial solutions of the corresponding two generalized bilinear equations.

Keywords: Generalized bilinear differential operator; rational solution; Sawada–Kotera-like equation.

1. Introduction

Japan's famous mathematician and physicist Ryogo Hirota proposed the bilinear derivative method,[1] and defined a new kind of differential operators — Hirota bilinear operators, as follows:

$$D_x^m f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m f(x)g(x')|_{x'=x} = \frac{\partial^m}{\partial {x'}^m} f(x+x')g(x-x')|_{x'=0}$$

*Corresponding author.

and

$$D_x^m D_t^n f\cdot g = \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n f(x,t)g(x',t')|_{x'=x,t'=t}$$
$$= \frac{\partial^m}{\partial x'^m}\frac{\partial^n}{\partial t'^n} f(x+x',t+t')g(x-x',t-t')|_{x'=0,t'=0}\,. \tag{1}$$

If an equation is written as a bilinear equation, many kinds of solutions of this equation can be generated, such as the (2+1)-dimensional Sawada–Kotera (SK) equation,[2] the (2+1)-dimensional generalized fifth-order KdV equation,[3] the (2+1)-dimensional bSK equation,[4] the Kadomtsev–Petviashvili equation,[5] the generalized Kadomtsev–Petviashvili-Boussinesq equation,[6] the nonlinear partial differential equations,[7] the (3+1)-dimensional linear PDEs,[8] the generalized Calogero–Bogoyavlenskii–Schiff equation,[9] the (2+1)-dimensional Ito equation[10] the BKP equation,[11,12] the dimensionally reduced p-gKP and p-gBKP equations,[13] the (4+1)-dimensional Fokas equation,[14] the (3+1)-dimensional nonlinear evolution equations[15,16] and the dimensionally reduced Hirota bilinear equation.[17,18] Based on Hirota bilinear operators, Ma[19–21] proposed the generalized bilinear differential operators as follows:

$$\begin{aligned} &D_{p,x}^m D_{p,t}^n f\cdot f\\ &= \left(\frac{\partial}{\partial x}+\alpha_p\frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t}+\alpha_p\frac{\partial}{\partial t'}\right)^n f(x,t)f(x',t')|_{x'=x,t'=t}\\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\alpha_p^i\alpha_p^j \frac{\partial^{m-i}}{\partial x^{m-i}}\frac{\partial^i}{\partial x'^i}\frac{\partial^{n-j}}{\partial t^{n-j}}\frac{\partial^j}{\partial t'^j} f(x,t)f(x',t')|_{x'=x,t'=t}\\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\alpha_p^i\alpha_p^j \frac{\partial^{m+n-i-j}f(x,t)}{\partial x^{m-i}t^{n-j}}\frac{\partial^{i+j}f(x,t)}{\partial x^i t^j},\quad m,n\geq 0\,, \end{aligned} \tag{2}$$

where the α_p^s satisfies

$$\alpha_p^s = (-1)^{r_p(s)},\quad s = r_p(s) \bmod p \tag{3}$$

and

$$\alpha_p^i\alpha_p^j \neq \alpha_p^{i+j},\quad i,j\geq 0\,,$$

when the prime number $p \geq 2$.

By involving different prime numbers p, Hirota bilinear equations have been generalized to generate diverse nonlinear differential equations possessing potential applications. On the situation of $p = 3$, the rational solutions of equations has been done, such as the (1+1)-dimensional KdV-like equation,[22] the (2+1)-dimensional Korteweg–de-Vries-like model,[23] the Hirota–Satsuma-like equation,[24] the (2+1)-dimensional KP-like equation,[25] the (3+1)-dimensional eKP-like equation,[26] the

(1+1)-dimensional Boussinesq-like equation[27] and the (3+1)-dimensional Jimbo–Miwa-like equation.[28] Rogue wave solutions also belong to a class of rational solution, the studies of rational solution are as important and practical as the other forms of soliton solutions.[29–31]

In this paper, we will consider the cases of generalized bilinear operators with prime numbers $p = 3$ and $p = 5$, and the fifth-order evolution equation SK equation[32,33]

$$u_{xxxxx} - 15u_{xxx}u - 15u_{xx}u_x + 45u^2u_x + u_t = 0\,. \tag{4}$$

For this equation, the following (2+1)-dimensional Sawada-Kotera equation[34]

$$u_t - (u_{xxxx} + 5uu_{xx} + \frac{5}{3}u^3 + 5u_{xy})_x - 5uu_y + 5\int u_{yy}dx - 5u_x\int u_y dx = 0\,, \tag{5}$$

developed from two KdV equations was originally proposed by B. G. Konopelchenko and V. G. Dubrovsky. When $u(x,y,t) \equiv u(x,t)$, the Eq. (5) evolves into the Eq. (4).

In this paper, from Hirota bilinear operators (1), we would like to introduce two SK-like equations by the generalized bilinear operators based on prime numbers $p = 3$ and $p = 5$. Then we will consider rational solutions of the two SK-like equations based on the polynomial solutions of the associated generalized bilinear equations respectively.

2. Two SK-Like Equations

Based on Hirota operators (1), under the variable transformation

$$u(x,t) = -2(\ln f)_{xx}\,,$$

the bilinear form of SK equation (4) is as follows:

$$\begin{aligned} D_{2,x}(D_{2,x}^5 + D_{2,t})f\cdot f = 2(&f_{xxxxxx}f - 6f_{xxxxx}f_x + 15f_{xxxx}f_{xx} \\ &- 10f_{xxx}^2 + f_{xt}f - f_xf_t) = 0\,. \end{aligned} \tag{6}$$

Based on the generalized bilinear operators, an extended bilinear form of SK equation (4) with prime number $p = 3$ is as follows:

$$D_{3,x}(D_{3,x}^5 + D_{3,t})f\cdot f = 2(f_{xxxxxx}f + 10f_{xxx}^2 + f_{xt}f - f_xf_t) = 0\,, \tag{7}$$

where

$$\alpha_3^1 = -1,\quad \alpha_3^2 = 1,\quad \alpha_3^3 = 1,\quad \alpha_3^4 = -1,\quad \alpha_3^5 = 1,$$
$$\alpha_3^6 = 1,\quad \alpha_3^7 = -1,\quad \alpha_3^8 = 1,\quad \alpha_3^9 = 1$$

and similarly, the extended bilinear form when $p = 5$ is

$$D_{5,x}(D_{5,x}^5 + D_{5,t})f\cdot f = 2(15f_{xxxx}f_{xx} - 10f_{xxx}^2 + f_{xt}f - f_xf_t) = 0\,. \tag{8}$$

Based on Bell polynomial theories,[20,21,35] by a dependent variable transformation

$$u(x,t) = -(\ln f)_x\,, \tag{9}$$

we directly find that the generalized bilinear equation (7) is linked to a SK-like equation as follows:

$$\begin{aligned} &2(66u^5u_x - u_{xt} - 75u^4u_{xx} - 165u_x^2u_{xx} + 55u_{xx}u_{xxx} + 40u^3u_{xxx} \\ &\quad - 300u^3u_x^2 + 390u^2u_xu_{xx} - 15u^2u_{xxxx} + 21u_xu_{xxxx} + 270uu_x^3 \\ &\quad - 120uu_{xx}^2 - 150uu_xu_{xxx} + 6uu_{xxxxx} - u_{xxxxxx}) = 0\,. \end{aligned} \tag{10}$$

Under the dependent variable transformation (9), the generalized bilinear equation (8) is linked to another SK-like equation as follows:

$$\begin{aligned} &2(30u^5u_x - u_{xt} - 45u^4u_{xx} - 180u^3u_x^2 + 40u^3u_{xxx} + 90uu_x^3 \\ &\quad - 30uu_xu_{xxx} + 210u^2u_xu_{xx} - 15u^2u_{xxxx} - 135u_x^2u_{xx} \\ &\quad - 5u_{xxx}u_{xx} + 15u_{xxxx}u_x) = 0\,. \end{aligned} \tag{11}$$

3. Rational Solutions of Eq. (10)

Case (1). Degree $(f, x, t) = (6, 1)$

In order to cast about for polynomial solutions of (7), we work with the computer algebra system. There is a polynomial solution with symbolic computation

$$f(x,t) = \sum_{i=0}^{6}\sum_{j=0}^{1} c_{i,j}x^it^j\,, \tag{12}$$

to (7). Two classes of polynomial solutions to (7) are generated

$$\begin{aligned} f_1(x,t) &= c_{5,0}x^5 + c_{4,0}x^4 + \frac{2c_{4,0}^2x^3}{5c_{5,0}} + \frac{2c_{4,0}^3x^2}{25c_{5,0}^2} + \frac{c_{4,0}^4x}{125c_{5,0}^3} \\ &\quad + 7200c_{5,0}t + c_{0,0} \end{aligned} \tag{13}$$

and

$$f_2(x,t) = c_{2,1}tx^2 + c_{1,1}tx + c_{2,0}x^2 + \frac{c_{1,1}c_{2,0}x}{c_{2,1}} + c_{0,1}t + \frac{c_{0,1}c_{2,0}}{c_{2,1}} \tag{14}$$

and two classes of rational solutions to (10) are given

$$u_1(x,t) = -\frac{p}{q}\,, \tag{15}$$

where

$$\begin{aligned} p &= 625c_{5,0}^4x^4 + 500c_{5,0}^3c_{4,0}x^3 + 150c_{5,0}^2c_{4,0}^2x^2 + 20c_{5,0}c_{4,0}^3x + c_{4,0}^4\,, \\ q &= 900000c_{5,0}^4t + 125c_{5,0}^4x^5 + 125c_{5,0}^3c_{4,0}x^4 + 50c_{5,0}^2c_{4,0}^2x^3 + 10c_{5,0}c_{4,0}^3x^2 \\ &\quad + c_{4,0}^4x + 125c_{5,0}^3c_{0,0} \end{aligned}$$

and

$$u_2(x,t) = -\frac{2c_{2,1}x + c_{1,1}}{x(c_{1,1} + xc_{2,1}) + c_{0,1}}\,. \tag{16}$$

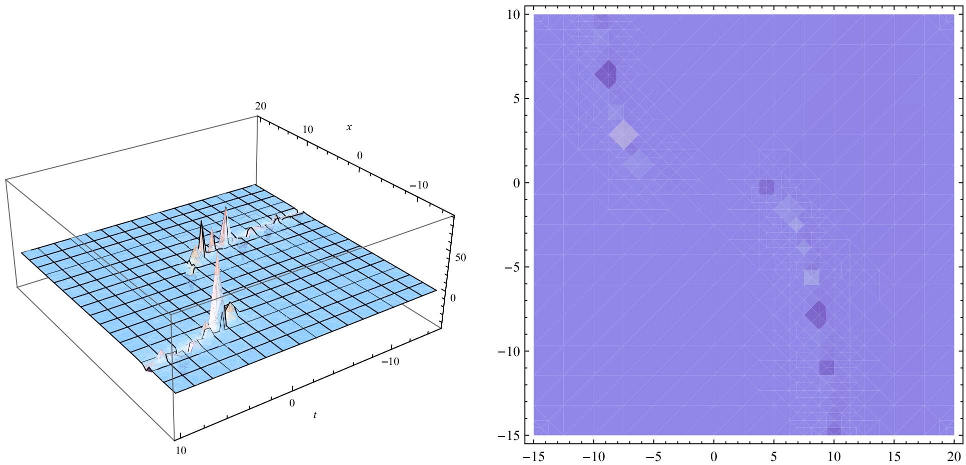

Fig. 1. Pictures of (18): 3D plot (left) and density plot (right).

A special solution of (15) with

$$c_{i,j} = 1 + ij, \quad 0 \le i \le 6, \ \ 0 \le j \le 1 \tag{17}$$

is given by

$$u(x,t) = -\frac{(5x+1)^4}{125 + 900000t + 125x^5 + 125x^4 + 50x^3 + 10x^2 + x}\,. \tag{18}$$

The solutions in (18) are depicted in Fig. 1.

Case (2). Degree $(f, x, t) = (6, 2)$

When $0 \le i \le 6, 0 \le j \le 2$, the polynomial solution becomes

$$f(x,t) = \sum_{i=0}^{6}\sum_{j=0}^{2} c_{i,j}x^i t^j\,, \tag{19}$$

to (7). Furthermore, two classes of polynomial solutions to (7) are generated

$$f_1(x,t) = \frac{\xi}{\eta}\,, \tag{20}$$

where

$$\begin{aligned}\xi &= 115200c_{3,1}^2c_{4,0}^4t^2 + 16c_{3,1}^2c_{4,0}^4tx^5 + 40c_{3,0}c_{3,1}^2c_{4,0}^3tx^4 + 40c_{3,0}^2c_{3,1}^2c_{4,0}^2tx^3\\ &\quad + 20c_{3,0}^3c_{3,1}^2c_{4,0}tx^2 + 5c_{3,0}^4c_{3,1}^2tx + 40c_{0,1}c_{3,0}^2c_{3,1}c_{4,0}^2t + 16c_{3,0}c_{3,1}c_{4,0}^4x^5\\ &\quad + 40c_{3,0}^2c_{3,1}c_{4,0}^3x^4 + 40c_{3,0}^3c_{3,1}c_{4,0}^2x^3 + 20c_{3,0}^4c_{3,1}c_{4,0}x^2 + 5c_{3,0}^5c_{3,1}x\\ &\quad - 115200c_{3,0}^2c_{4,0}^4 + 40c_{0,1}c_{3,0}^3c_{4,0}^2\,,\\ \eta &= 40c_{3,0}^2c_{3,1}c_{4,0}^2\end{aligned}$$

and

$$f_2(x,t) = \frac{(t\,(tc_{1,2} + c_{1,1}) + c_{1,0})\,(x\,(xc_{2,2} + c_{1,2}) + c_{0,2})}{c_{1,2}}\,. \tag{21}$$

Correspondingly, two rational solutions to (10) are

$$u_1(x,t) = -\frac{p}{q}, \tag{22}$$

where

$$\begin{aligned} p &= 80c_{3,1}c_{4,0}^4x^4 + 160c_{3,1}c_{4,0}^3c_{3,0}x^3 + 120c_{3,1}c_{4,0}^2c_{3,0}^2x^2 \\ &\quad + 40c_{3,1}c_{4,0}c_{3,0}^3x + 5c_{3,1}c_{3,0}^4\,, \\ q &= 115200c_{3,1}c_{4,0}^4t + 16c_{3,1}c_{4,0}^4x^5 + 40c_{3,0}c_{3,1}c_{4,0}^3x^4 + 40c_{3,0}^2c_{3,1}c_{4,0}^2x^3 \\ &\quad + 20c_{3,0}^3c_{3,1}c_{4,0}x^2 + 5c_{3,0}^4c_{3,1}x - 115200c_{3,0}c_{4,0}^4 + 40c_{0,1}c_{3,0}^2c_{4,0}^2 \end{aligned}$$

and

$$u_2(x,t) = -\frac{2xc_{2,2} + c_{1,2}}{x^2c_{2,2} + xc_{1,2} + c_{0,2}}\,. \tag{23}$$

A special solution of (22) with

$$c_{i,j} = i + j, \quad 0 \le i \le 6, \;\; 0 \le j \le 2 \tag{24}$$

is given by

$$u(x,t) = -\frac{p}{q}\,, \tag{25}$$

where

$$\begin{aligned} p &= 20480x^4 + 30720x^3 + 17280x^2 + 4320x + 405\,, \\ q &= 29491200t + 4096x^5 + 7680x^4 + 5760x^3 + 2160x^2 + 405x - 22116960\,. \end{aligned}$$

The solutions in (25) are depicted in Fig. 2.

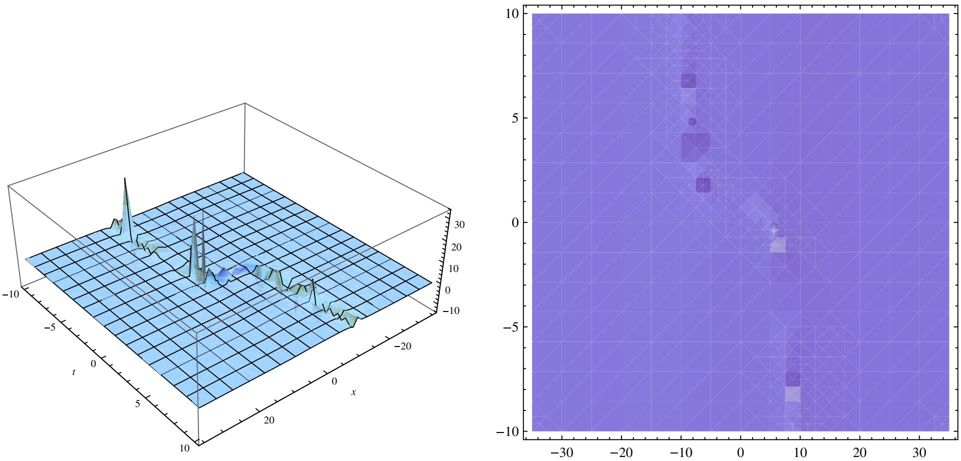

Fig. 2. Pictures of (25): 3D plot (left) and density plot (right).

4. Rational Solutions of Eq. (11)

About the polynomial solutions to (8), we work with the computer algebra system and make

$$f(x,t)=\sum_{i=0}^{6}\sum_{j=0}^{5}c_{i,j}x^i t^j\,. \tag{26}$$

One class of rational solution to (11) is

$$u(x,t)=-\frac{p}{q}\,, \tag{27}$$

where

$$\begin{aligned}p&=2430000c_{6,4}^3t+4050c_{6,4}^3x^5-1350\sqrt{15}\sqrt{c_{4,4}}c_{6,4}^{5/2}x^4+2700c_{4,4}c_{6,4}^2x^3\\&\quad-180\sqrt{15}c_{4,4}^{3/2}c_{6,4}^{3/2}x^2+90c_{4,4}^2c_{6,4}x+675c_{1,4}c_{6,4}^2-2430000c_{6,3}c_{6,4}^2\,,\\q&=2430000c_{6,4}^3tx-162000\sqrt{15}\sqrt{c_{4,4}}c_{6,4}^{5/2}t+675c_{6,4}^3x^6-270\sqrt{15}\sqrt{c_{4,4}}c_{6,4}^{5/2}x^5\\&\quad+675c_{4,4}c_{6,4}^2x^4-60\sqrt{15}c_{4,4}^{3/2}c_{6,4}^{3/2}x^3+45c_{4,4}^2c_{6,4}x^2+675c_{1,4}c_{6,4}^2x\\&\quad-2430000c_{6,3}c_{6,4}^2x-c_{4,4}^3+162000\sqrt{15}\sqrt{c_{4,4}}c_{6,3}c_{6,4}^{3/2}\\&\quad-45\sqrt{15}c_{1,4}\sqrt{c_{4,4}}c_{6,4}^{3/2}\,.\end{aligned}$$

A special solution of (27) with

$$c_{i,j}=e^{i+j},\quad 0\le i\le 6,\;\; 0\le j\le 5 \tag{28}$$

is given by

$$u(x,t)=-\frac{p}{q}\,, \tag{29}$$

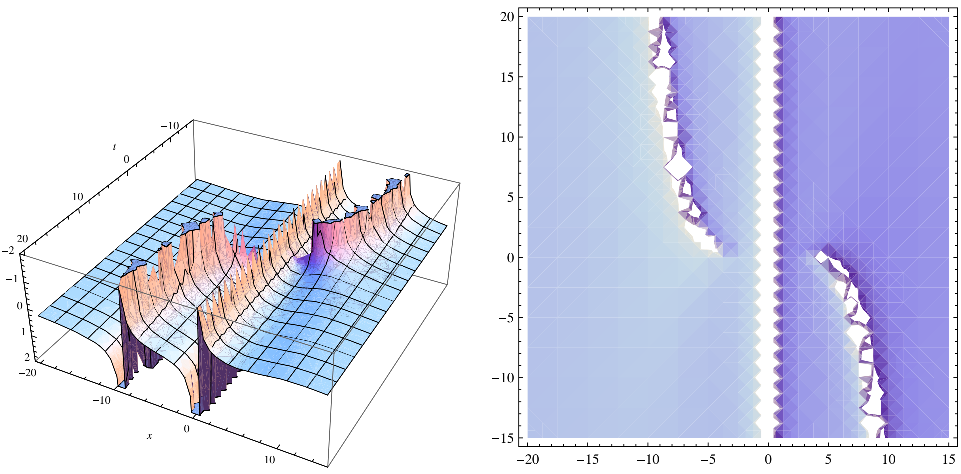

Fig. 3. Pictures of (29): 3D plot (left) and density plot (right).

where

$$p = 2430000e^6t + 4050e^6x^5 - 1350\sqrt{15}e^5x^4 + 2700e^4x^3 - 180\sqrt{15}e^3x^2 + 90e^2x - 2430000e^5 + 675e\,,$$

$$q = 2430000e^6tx - 162000\sqrt{15}e^5t + 675e^6x^6 - 270\sqrt{15}e^5x^5 + 675e^4x^4 - 60\sqrt{15}e^3x^3 + 45e^2x^2 - 2430000e^5x + 675ex + 162000\sqrt{15}e^4 - 45\sqrt{15} - 1\,.$$

The solutions in (29) are depicted in Fig. 3.

5. Discussion and Conclusions

Based on the generalized bilinear operators (2), two extended SK-like equations with $p = 3$ and $p = 5$ are generated, respectively. By using symbolic computation, two classes of rational solutions to (10) and one class of rational solution to (11) are generated.

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