Applied Mathematics Letters 112 (2021) 106809

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Lump solutions of a nonlinear PDE containing a third-order derivative of time

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A R T I C L E I N F O

Article history:
Received 15 September 2020
Received in revised form 30 September 2020
Accepted 30 September 2020
Available online 8 October 2020

Keywords:
Lump solution
Hirota bilinear method
Soliton
Symbolic computation

A B S T R A C T

A nonlinear partial differential equation combining with a third-order derivative of the time variable $D_xD_t^3$ is studied. By adding a new fourth-order derivative term, its lump solutions are explicitly constructed by the Hirota bilinear method and symbolic computation. Furthermore, the effect of the new fourth-order derivative term on the solution is discussed. The dynamical behaviors of two particular lump solutions are analyzed with different choices of the parameters.

1. Introduction

In the last several years with the rapid development of nonlinear science, scientists and engineers have been interested in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics, fluid mechanics and applied sciences. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear partial differential equations (PDEs) [1,2]. Soliton solutions, which are normally localized in the time and space, are exact ones determined by exponentially localized functions [3,4]. In contrast to soliton solutions, lump solutions are another kind of important exact solutions of PDEs, which are localized in directions only in space, and they are, originated from solving integrable equations in (2+1)-dimensions [5–7], and obtained from soliton theory by taking

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https://doi.org/10.1016/j.aml.2020.106809

long wave limits [2,8,9]. In soliton theory the crucial step in finding lump solutions is to construct positive quadratic function solutions to Hirota bilinear equations [5]. Then based on positive quadratic function solutions, the logarithmic transformations yield lump solutions to nonlinear PDEs.

Through a depended variable transformation, a partial differential equation can be mapped into a Hirota bilinear form. Suppose that P is a polynomial in x, y, and t. Then a Hirota bilinear differential equation in (2+1)-dimensions can be defined by

$$P(D_x, D_y, D_t)f \cdot f = 0, \tag{1}$$

where D_x, D_y, D_t are the Hirota bilinear derivatives [5],

$$\begin{aligned} &D_x^l D_y^n D_t^m f(x,y,t) \cdot g(x,y,t) \\ &= (\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^l (\frac{\partial}{\partial y} - \frac{\partial}{\partial y'})^n (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^m f(x,y,t) \cdot g(x', y', t')|_{x'=x, y'=y, t'=t}. \end{aligned} \tag{2}$$

When f solves (2), N-soliton solutions in (2+1)-dimensions to the corresponding PDE are defined through the transformation $u = 2(lnf)_x$ or $u = 2(lnf)_{xx}$ with the form as [10]

$$f = \sum_{\mu=0,1} exp(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}), \tag{3}$$

where $\sum_{\mu=0,1}$ denotes the sum over all possibilities for $\mu_1, \ldots, \mu_N$ in $0, 1$, and

$$\begin{cases} \xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, & 1 \leq i \leq N, \\ e^{a_{ij}} = -\frac{P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)}, & 1 \leq i < j \leq N, \end{cases} \tag{4}$$

with the wave numbers k_i, l_i and the frequencies ω_i, $1 \leq i \leq N$ satisfying the corresponding dispersion relations

$$P(k_i, l_i, -\omega_i) = 0, \qquad 1 \leq i \leq N,$$

but the phase shifts $\xi_{i,0}$ being arbitrary.

As is well known, the KPI equation possesses lump solutions [11]: $u = 2(lnf)_{xx}$, where

$$\begin{aligned} f = &(a_1 x + a_2 y + \frac{a_1 a_2^2 - a_1 a_6^2 + 2a_2 a_5 a_6}{a_1^2 + a_5^2} t + a_4)^2 \\ &+ (a_5 x + a_6 y + \frac{2a_1 a_2 a_6 - a_2^2 a_5 + a_5 a_6^2}{a_1^2 + a_5^2} t + a_8)^2 + \frac{3(a_1^2 + a_5^2)^3}{(a_1 a_6 - a_2 a_5)^2}. \end{aligned}$$

The condition $a_1 a_6 - a_2 a_5 \neq 0$ guarantees the rational localization in all directions in the (x, y)-plane.

In the past few decades, many researchers have studied soliton solutions, lump solutions, and other classes of solutions to integrable equations. The KPI equation has abundant lump solutions [11], and its special lump solutions are constructed from its soliton solutions [12]. Other integrable equations which possess lump solutions contain the three-dimensional three-wave resonant interaction [13], the Davey–Stewartson II equation [8], the Ishimori-I equation [14], the BKP equation [15,16], and the KP equation with a self-consistent source [17]. Furthermore, nonintegrable equations can possess lump solutions, among which are a few generalized KP, BKP, KP-Boussinesq, Sawada–Kotera, Calogero–Bogoyavlenskii–Schiff and Bogoyavlensky–Konopelchenko equations in (2+1)-dimensions [18–24]. It has seen that it is crucial in finding lump solutions to construct quadratic function solutions to Hirota bilinear equations and determine the sign of the resulting solutions [5].

In this paper, we would like to concern a nonlinear PDE combining with the third derivative of the time variable $D_x D_t^3$ in (2+1)-dimensions and determine its diverse lump solutions. This new term $D_x D_t^3$ makes the calculation more complicated. It is also reflected that the structure of the solution is more complex, and we pay more attention to deal with and analyze the dispersion relation of the solution. Adding three new fourth-order derivative terms and all second-order derivative terms, we formulate a combined fourth-order nonlinear partial differential equation, which possesses a Hirota bilinear form. Based on a bilinear transformation, lump solutions are obtained through symbolic computation with Maple. By choosing special values for the coefficients in equation, specific lump solutions to the equation in the corresponding cases are given. Finally, We also exhibit three-dimensional plots and contour plots of the lump solutions and discuss their dynamic behaviors. Some conclusions are given in the final section.

2. Bilinear form and lump solutions of a fourth-order PDE with a new term

In this section, we would like to consider the following nonlinear partial differential equation, which contains three fourth-order terms and six second-order terms:

$$\begin{aligned} P(u) = {} & \alpha[3(u_x u_t)_x + u_{xxxt}] + \beta[3(u_x u_y)_x + u_{xxxy}] + \theta(u_{xttt} + 3u_t u_{tt} \\ & + 3u_{xt} v_{tt}) + \gamma_1 u_{yt} + \gamma_2 u_{xx} + \gamma_3 u_{xt} + \gamma_4 u_{xy} + \gamma_5 u_{yy} + \gamma_6 u_{tt} = 0, \end{aligned} \tag{5}$$

where $v_x = u$, the coefficients α, β and θ satisfy $\alpha^2 + \beta^2 + \theta^2 \neq 0$, but $\gamma_i, 1 \leq i \leq 6$, are arbitrary constants.

This (2+1)-dimensional nonlinear equation (5) possesses a Hirota bilinear form:

$$\begin{aligned} B(f) = {} & (\alpha D_x^3 D_t + \beta D_x^3 D_y + \theta D_x D_t^3 + \gamma_1 D_y D_t \\ & + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2) f \cdot f = 0, \end{aligned} \tag{6}$$

under the logarithmic transformation

$$u = 2(lnf)_x = \frac{2f_x}{f}, \qquad v = 2(lnf), \tag{7}$$

and we have the relation $P(u) = (\frac{B(f)}{f^2})_x$.

If we get a solution to the Hirota bilinear form Eq. (6), we can generate the corresponding solution to the nonlinear equation (5).

Eq. (6) with $\theta = 0$ has been studied in [21]. In comparison with the equation in [21], $D_x D_t^3$ is a new term and so in this paper we assume that $\theta \neq 0$. Because Eq. (5) has the bilinear form (6), we start to construct the solution of the combined bilinear equation (6) in a positive quadratic form as follows:

$$f = (a_1 x + a_2 y + a_3 t + a_4)^2 + (a_5 x + a_6 y + a_7 t + a_8)^2 + a_9, \tag{8}$$

where $a_i, 1 \leq i \leq 9$, are the real constant parameters to be determined, in order to generate lump solutions to the combined fourth-order nonlinear equation (5).

We insert the formula (8) into Eq. (6) to obtain a system of algebraic equations on the parameters $a_i, 1 \leq i \leq 9$, and try to solve them by Maple symbolic computation system except for some necessary sorting and simplification of the results. In order to facilitate the calculation and expression of the results, we need to make some setting for $\gamma_i, 1 \leq i \leq 6$, in Eq. (6), or Eq. (5).

First, let $\gamma_6 = 0$. Eq. (6) becomes:

$$\begin{aligned} B(f) = {} & (\alpha D_x^3 D_t + \beta D_x^3 D_y + \theta D_x D_t^3 + \gamma_1 D_y D_t \\ & + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2) f \cdot f = 0. \end{aligned} \tag{9}$$

After some calculations, we obtain a set of solutions for the parameters:

$$\begin{cases} a_3 = -\frac{b_1}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}, \\ a_7 = -\frac{b_2}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}, \\ a_9 = \frac{3(a_1^2 + a_5^2)(b_3\alpha + b_4\beta) + b_5\theta}{(a_1 a_6 - a_2 a_5)^2[(a_1\gamma_3 + a_2\gamma_1)^2 + (a_5\gamma_3 + a_6\gamma_1)^2](\gamma_1^2\gamma_2 - \gamma_1\gamma_3\gamma_4 + \gamma_3^2\gamma_5)}, \end{cases} \tag{10}$$

and all the other a_i are arbitrary. The above involved constants $b_i, 1 \leq i \leq 5$, are defined as follows:

$$\begin{cases} b_1 = [(a_1^2 a_2 + 2a_1 a_5 a_6 - a_2 a_5^2)\gamma_2 + a_1(a_2^2 + a_6^2)\gamma_4 + a_2(a_2^2 + a_6^2)\gamma_5]\gamma_1 \\ \quad +[a_1(a_1^2 + a_5^2)\gamma_2 + a_2(a_1^2 + a_5^2)\gamma_4 + (a_1 a_2^2 + 2a_2 a_5 a_6 - a_1 a_6^2)\gamma_5]\gamma_3, \\ b_2 = [(-a_1^2 a_6 + 2a_1 a_2 a_5 + a_5^2 a_6)\gamma_2 + a_5(a_2^2 + a_6^2)\gamma_4 + a_6(a_2^2 + a_6^2)\gamma_5]\gamma_1 \\ \quad +[a_5(a_1^2 + a_5^2)\gamma_2 + a_6(a_1^2 + a_5^2)\gamma_4 + (-a_2^2 a_5 + 2a_1 a_2 a_6 + a_5 a_6^2)\gamma_5]\gamma_3, \\ b_3 = (a_1^2 + a_5^2)(a_1 a_2 + a_5 a_6)(\gamma_1\gamma_2 + \gamma_3\gamma_4) + (a_1^2 + a_5^2)(a_2^2 + a_6^2)\gamma_1\gamma_4 \\ \quad +(a_1^2 + a_5^2)^2\gamma_2\gamma_3 + (a_2^2 + a_6^2)(a_1 a_2 + a_5 a_6)\gamma_1\gamma_5 \\ \quad +[(a_1 a_2 + a_5 a_6)^2 - (a_1 a_6 - a_2 a_5)^2]\gamma_3\gamma_5, \\ b_4 = -(a_1 a_2 + a_5 a_6)[(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2], \\ b_5 = 3c_1[c_3 c_4(\gamma_1\gamma_2^2\gamma_5 + 2\gamma_2\gamma_3\gamma_4\gamma_5) + c_3 c_5(\gamma_3\gamma_4^2\gamma_5 + 2\gamma_1\gamma_2\gamma_4\gamma_5) + c_4 c_5(\gamma_1\gamma_2\gamma_5^2 + \gamma_3\gamma_4\gamma_5^2)] \\ \quad +3c_2[c_3^2(\gamma_1\gamma_2^2\gamma_4 + \gamma_2\gamma_3\gamma_4^2) + c_5^2\gamma_1\gamma_4\gamma_5^2] + 3c_3^2[(c_3^2\gamma_2^3\gamma_3 + 3c_6\gamma_2^2\gamma_3\gamma_5) + c_4(3c_5\gamma_1\gamma_2\gamma_4^2 \\ \quad +3\gamma_2^2\gamma_3\gamma_4 + d_3\gamma_1\gamma_2^3)] + 3c_5^2[(c_3^2\gamma_1\gamma_4^3 + c_6\gamma_3\gamma_5^3) + c_4(3c_3\gamma_1\gamma_4^2\gamma_5 + c_3^2\gamma_3\gamma_4^3 + c_5\gamma_1\gamma_5^3)] \\ \quad +[3(a_1^4 + a_5^4)(a_2^4 + a_6^4) - 2(a_1^2 a_2^2 + a_5^2 a_6^2)(a_1^2 a_6^2 + a_2^2 a_5^2) + 4a_1 a_2 a_5 a_6(4a_1^2 a_2^2 - 4a_1^2 a_6^2 \\ \quad +11a_1 a_2 a_5 a_6 - 4a_2^2 a_5^2 + 4a_5^2 a_6^2)]\gamma_2\gamma_3\gamma_5^2. \end{cases} \tag{11}$$

The parameters c_i, $1 \leq i \leq 6$, involved in b_5 are defined as follows:

$$\begin{cases} c_1 = [3(a_1 a_2 + a_5 a_6)^2 - (a_1 a_6 - a_2 a_5)^2], \\ c_2 = [3(a_1 a_2 + a_5 a_6)^2 + (a_1 a_6 - a_2 a_5)^2], \\ c_3 = (a_1^2 + a_5^2), \\ c_4 = (a_1 a_2 + a_5 a_6), \\ c_5 = (a_2^2 + a_6^2), \\ c_6 = [(a_1 a_2 + a_5 a_6)^2 - (a_1 a_6 - a_2 a_5)^2]. \end{cases} \tag{12}$$

Directly comparing our results with the context in [21], where $\gamma_6 = 0$, we have some new terms contained with θ in the denominator of a_9.

Secondly, let $\gamma_5 = 0$. Eq. (6) becomes:

$$\begin{aligned} B(f) = (&\alpha D_x^3 D_t + \beta D_x^3 D_y + \theta D_x D_t^3 + \gamma_1 D_y D_t \\ &+ \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_6 D_t^2) f \cdot f = 0. \end{aligned} \tag{13}$$

After some calculations, we obtain:

$$\begin{cases} a_2 = -\frac{c_1}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}, \\ a_6 = -\frac{c_2}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}, \\ a_9 = \frac{3(a_1^2 + a_5^2)(d_3\alpha + d_4\beta) + d_5\theta}{(a_1 a_7 - a_3 a_5)^2(\gamma_1^2\gamma_2 - \gamma_1\gamma_3\gamma_4 + \gamma_4^2\gamma_6)}, \end{cases} \tag{14}$$

where all the other a_i are arbitrary constants. The involved constants $d_i, 1 \le i \le 5$, are defined as follows:

$$
\begin{cases}
d_1 = [(a_1^2 a_3 + 2a_1 a_5 a_7 - a_3 a_5^2)\gamma_2 + a_1(a_3^2 + a_7^2)\gamma_3 + a_3(a_3^2 + a_7^2)\gamma_6]\gamma_1 \\
\quad +[a_1(a_1^2 + a_5^2)\gamma_2 + a_3(a_1^2 + a_5^2)\gamma_3 + (a_1 a_3^2 + 2a_3 a_5 a_7 - a_1 a_7^2)\gamma_6]\gamma_4, \\
d_2 = [(-a_1^2 a_7 + 2a_1 a_3 a_5 + a_5^2 a_7)\gamma_2 + a_5(a_3^2 + a_7^2)\gamma_3 + a_7(a_3^2 + a_7^2)\gamma_6]\gamma_1 \\
\quad +[a_5(a_1^2 + a_5^2)\gamma_2 + a_7(a_1^2 + a_5^2)\gamma_3 + (-a_3^2 a_5 + 2a_1 a_3 a_7 + a_5 a_7^2)\gamma_6]\gamma_4, \\
d_3 = -(a_1 a_3 + a_5 a_7)[(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2], \\
d_4 = (a_1^2 + a_5^2)(a_1 a_3 + a_5 a_7)(\gamma_1\gamma_2 + \gamma_3\gamma_4) + (a_1^2 + a_5^2)(a_3^2 + a_7^2)\gamma_1\gamma_3 \\
\quad +(a_1^2 + a_5^2)^2\gamma_2\gamma_4 + (a_3^2 + a_7^2)(a_1 a_3 + a_5 a_7)\gamma_1\gamma_6 \\
\quad +[(a_1 a_3 + a_5 a_7)^2 - (a_1 a_7 - a_3 a_5)^2]\gamma_4\gamma_6, \\
d_5 = [3(a_1 a_2 + a_5 a_6)^2 - (a_1 a_6 - a_2 a_5)^2][(a_1 a_2 + a_5 a_6)\gamma_1\gamma_2 + (a_2^2 + a_6^2)\gamma_3\gamma_5] \\
\quad +[3(a_1 a_2 + a_5 a_6)^2 + (a_1 a_6 - a_2 a_5)^2][(a_2^2 + a_6^2)\gamma_1\gamma_4 + (a_1^2 + a_5^2)\gamma_2\gamma_3] \\
\quad +3(a_2^2 + a_6^2)(a_1 a_2 + a_5 a_6)[(a_2^2 + a_6^2)\gamma_1\gamma_5 + (a_1^2 + a_5^2)\gamma_3\gamma_4].
\end{cases}
\tag{15}
$$

We find that a_2 and a_6 are consistent with earlier findings in paper [21]. Although there is a new term $D_x D_t^3$ in Eq. (5), the coefficient θ, as well as α and β, the other coefficients of the fourth-order terms in the nonlinear PDE (5), just appear in a_9. So in each case, when γ_6 or γ_5 is zero, the new term only affects a_9, the constant term of the positive quadratic form of solution (8).

To generate lump solutions for the case of $\gamma_5 = 0$, besides a_9 should be positive to guarantee the analyticity of rational solutions, we require only one basic condition [11]:

$$
\begin{aligned}
& a_1 a_6 - a_2 a_5 \\
& = \frac{(a_1 a_7 - a_3 a_5)[(a_1^2 + a_5^2)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - (a_3^2 + a_7^2)\gamma_1\gamma_6 - (a_1 a_3 + a_5 a_7)\gamma_4\gamma_6]}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}.
\end{aligned}
$$

So it follows that $a_1 a_6 - a_2 a_5 \ne 0$ if and only if

$$
\begin{cases}
a_1 a_7 - a_3 a_5 \ne 0, \gamma_1^2 + \gamma_4^2 \ne 0 \\
(a_1^2 + a_5^2)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - (a_3^2 + a_7^2)\gamma_1\gamma_6 - (a_1 a_3 + a_5 a_7)\gamma_4\gamma_6 \ne 0.
\end{cases}
\tag{16}
$$

3. Two specific lump solutions and their profiles

In this section, we will take special sets of values for the coefficients in the combined fourth-order nonlinear equation (5) to obtain the corresponding lump solutions and discuss the dynamic behaviors of those solutions.

First, we take:

$$\alpha = 1, \beta = 0, \theta = 1, \gamma_3 = \gamma_5 = 1, \gamma_1 = \gamma_2 = \gamma_4 = \gamma_6 = 0. \tag{17}$$

Eq. (5) is reduced to:

$$3(u_x u_t)_x + u_{xxxt} + u_{xttt} + 3u_t u_{tt} + 3u_{xt} v_{tt} + u_{xt} + u_{yy} = 0, \tag{18}$$

which has a Hirota bilinear form:

$$(D_x^3 D_t + D_x D_t^3 + D_x D_t + D_y^2) f \cdot f = 0. \tag{19}$$

Substitute the above values (17) into the resulting parameters (10). Then, take the free parameters as follows :

$$a_1 = -1, a_2 = 3, a_4 = 4, a_5 = 1, a_6 = -1, a_8 = -3, \tag{20}$$

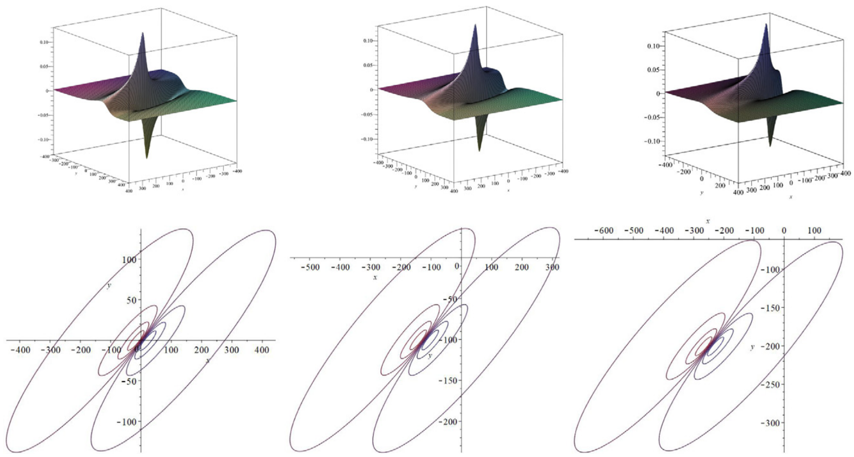

Fig. 1. Profiles of u_1 when $t = 0, 25, 50$: 3d plots (top) and contour plots (bottom).

we get: $a_3 = 7$, $a_7 = 1$, $a_9 = 468$. Putting all the a_i, $1 \le i \le 9$, into the formula (8), we obtain the corresponding positive quadratic form solution of Eq. (19)

$$f_1 = (-x + 3y + 7t + 4)^2 + (x - y + t - 3)^2 + 468. \tag{21}$$

By the logarithmic transformation: $u = 2(lnf)_x$, the lump solution of the special fourth-order nonlinear equation (18) is generated as follows:

$$u_1 = \frac{4(2x - 4y - 6t - 7)}{(-x + 3y + 7t + 4)^2 + (x - y + t - 3)^2 + 468}. \tag{22}$$

The three-dimensional plots and contour plots of this lump solutions (22) are shown in Fig. 1, which are made via Maple plot tools.

Secondly, we take:

$$\alpha = 1, \beta = 1, \theta = 1, \gamma_4 = \gamma_6 = 1, \gamma_1 = \gamma_2 = \gamma_3 = \gamma_5 = 0. \tag{23}$$

to get another fourth-order nonlinear equation:

$$3(u_x u_t)_x + u_{xxxt} + 3(u_x u_y)_x + u_{xxxy} + u_{xttt} + 3u_t u_{tt} + 3u_{xt} v_{tt} + u_{xy} + u_{tt} = 0. \tag{24}$$

which has a Hirota bilinear form:

$$(D_x^3 D_t + D_x^3 D_y + D_x D_t^3 + D_x D_y + D_t^2) f \cdot f = 0. \tag{25}$$

Substitute the above values (23) into the resulting parameters (14) and associated with the special value of the free parameters:

$$a_1 = 3, a_3 = -1, a_4 = -5, a_5 = -1, a_7 = 1, a_8 = 5, \tag{26}$$

we get: $a_2 = -\frac{1}{5}$, $a_6 = \frac{3}{5}$, $a_9 = 450$.

The corresponding f defined by (8) reads as follows:

$$f_2 = (3x - \frac{1}{5}y - t - 5)^2 + (-x + \frac{3}{5}y + t + 5)^2 + 450, \tag{27}$$

which provides a lump solution of the special fourth-order nonlinear equation (25):

$$u_2 = \frac{4(10x - 4t + 15)}{(3x - \frac{1}{5}y - t - 5)^2 + (-x + \frac{3}{5}y + t + 5)^2 + 450}. \tag{28}$$

Fig. 2 displays the three-dimensional plots and contour plots of the lump solution made by Maple.

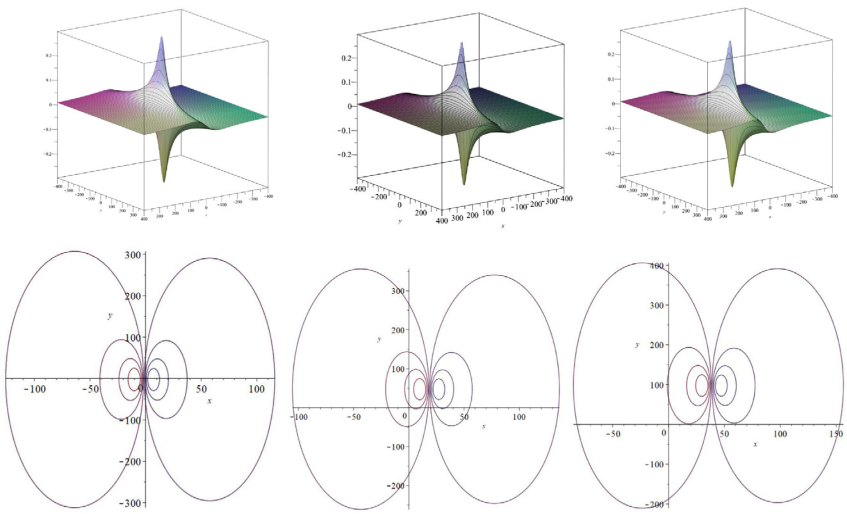

Fig. 2. Profiles of u_2 when $t = 0, 50, 100$: 3d plots (top) and contour plots (bottom).

4. Conclusion

In this paper, we have concerned a new combined fourth-order nonlinear PDE. Under two special cases $\gamma_5 = 0$ and $\gamma_6 = 0$, we obtained a class of lump solutions of the nonlinear equation (5) by Hirota bilinear method. All the above results offer us abundant new exact solutions. It is important to remark that the three nonlinear terms can be merged together into the considered nonlinear model. Through the symbolic computations with the Maple, we have worked out abundant lump solutions and found out that the coefficient of the new term just affects a_9, the constant term of the positive quadratic form of solution (8). Under the setting of γ_5 and γ_6, we determined the other coefficients of Eq. (5) to obtain the related specific lump solution and presented their profiles via the Maple plot tools. It is necessary to point out that we made the smoother and more regular contour plots than the ones in [21] and [25].

This research has enriched the category of nonlinear PDEs that possess lump solutions, and tried to figure out the relation between the lump solutions and the nonlinear terms contained in the new equation. It is well known that interaction solutions between lump solutions and soliton solutions can describe more nonlinear phenomena [21,26] and various studies have shown that the existence of interaction solutions between lumps and other kinds of exact solutions to nonlinear integrable equations [23,27–30], even in (3+1) dimension [31–33] and linear wave equations [34]. Since the interaction properties involve much more complicated mathematical computations, the further research for interaction solutions to other generalized bilinear differential equations is becoming more interesting and meaningful.

Acknowledgments

This work was finished when L.Y. Ding visited the Department of Mathematics and Statistics at University of South Florida in 2020. This research is supported by the China Scholarship Council (NO. 201906840068) and in part by National Natural Science Foundation of China (NO. 11961080).

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