Lump solutions of a nonlinear PDE containing a third-order derivative of time

Liyuan Ding\textsuperscript{a,d,*}, Wen-Xiu Ma\textsuperscript{b,c,d,e,**}, Qingxian Chen\textsuperscript{d,f}, Yehui Huang\textsuperscript{d,g}

\textsuperscript{a} School of Science, Nanjing University of Science & Technology, Nanjing 210094, Jiangsu, PR China
\textsuperscript{b} Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, PR China
\textsuperscript{c} Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
\textsuperscript{d} Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620, USA
\textsuperscript{e} School of Mathematics, South China University of Technology, Guangzhou 510640, PR China
\textsuperscript{f} Information School, Zhejiang University of Finance & Economics Dongfang College, Zhejiang 314408, PR China
\textsuperscript{g} School of Mathematics and Physics, North China Electric Power University, Beijing, 102206, PR China

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\textbf{A B S T R A C T}

A nonlinear partial differential equation combining with a third-order derivative of the time variable $D_x D_t^3$ is studied. By adding a new fourth-order derivative term, its lump solutions are explicitly constructed by the Hirota bilinear method and symbolic computation. Furthermore, the effect of the new fourth-order derivative term on the solution is discussed. The dynamical behaviors of two particular lump solutions are analyzed with different choices of the parameters.

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1. Introduction

In the last several years with the rapid development of nonlinear science, scientists and engineers have been interested in the analytical asymptotic techniques for nonlinear problems such as solid state physics, plasma physics, fluid mechanics and applied sciences. In many different fields of science and engineering, it is important to obtain exact or numerical solution of the nonlinear partial differential equations (PDEs) \cite{1,2}. Soliton solutions, which are normally localized in the time and space, are exact ones determined by exponentially localized functions \cite{3,4}. In contrast to soliton solutions, lump solutions are another kind of important exact solutions of PDEs, which are localized in directions only in space, and they are, originated from solving integrable equations in (2+1)-dimensions \cite{5–7}, and obtained from soliton theory by taking...
long wave limits [2,8,9]. In soliton theory the crucial step in finding lump solutions is to construct positive quadratic function solutions to Hirota bilinear equations [5]. Then based on positive quadratic function solutions, the logarithmic transformations yield lump solutions to nonlinear PDEs.

Through a depended variable transformation, a partial differential equation can be mapped into a Hirota bilinear form. Suppose that \( P \) is a polynomial in \( x, y, \) and \( t \). Then a Hirota bilinear differential equation in \((2+1)\)-dimensions can be defined by

\[
P(D_x, D_y, D_t) f \cdot f = 0,
\]

where \( D_x, D_y, D_t \) are the Hirota bilinear derivatives [5],

\[
D_x^n D_y^m f(x, y, t) \cdot g(x, y, t) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n \left( \frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right)^m f(x, y, t) \cdot g(x', y', t')|_{x'=x, y'=y, t'=t}.
\]

When \( f \) solves (2), \( N \)-soliton solutions in \((2+1)\)-dimensions to the corresponding PDE are defined through the transformation \( u = 2(\ln f)_x \) or \( u = 2(\ln f)_{xx} \) with the form as [10]

\[
f = \sum_{\mu=0, 1} \exp \left( \sum_{i=1}^N \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij} \right),
\]

where \( \sum_{\mu=0, 1} \) denotes the sum over all possibilities for \( \mu_1, \ldots, \mu_N \) in 0, 1, and

\[
\begin{align*}
\xi_i &= k_i x + l_i y - \omega_i t + \xi_{i,0}, \\
e^{a_{ij}} &= \frac{P(k_i+k_j, l_i+l_j, \omega_i+\omega_j)}{P(k_i-k_j, l_i-l_j, \omega_i-\omega_j)}, \quad 1 \leq i < j \leq N,
\end{align*}
\]

with the wave numbers \( k_i, l_i \) and the frequencies \( \omega_i, 1 \leq i \leq N \) satisfying the corresponding dispersion relations

\[
P(k_i, l_i, -\omega_i) = 0, \quad 1 \leq i \leq N,
\]

but the phase shifts \( \xi_{i,0} \) being arbitrary.

As is well known, the KPI equation possesses lump solutions [11]: \( u = 2(\ln f)_{xx} \), where

\[
f = (a_1 x + a_2 y + \frac{a_1 a_2^2 - a_1 a_3^2 + 2a_2 a_5 a_6}{a_1^2 + a_5^2} t + a_4)^2 \\
+ (a_5 x + a_6 y + \frac{2a_1 a_2 a_6 - a_1 a_2 a_5 + a_5 a_6^2}{a_1^2 + a_5^2} t + a_8)^2 + \frac{3(a_1^2 + a_5^2)^3}{(a_1 a_6 - a_2 a_5)^2}.
\]

The condition \( a_1 a_6 - a_2 a_5 \neq 0 \) guarantees the rational localization in all directions in the \((x, y)\)-plane.

In the past few decades, many researchers have studied soliton solutions, lump solutions, and other classes of solutions to integrable equations. The KPI equation has abundant lump solutions [11], and its special lump solutions are constructed from its soliton solutions [12]. Other integrable equations which possess lump solutions contain the three-dimensional three-wave resonant interaction [13], the Davey–Stewartson II equation [8], the Ishimori-I equation [14], the BKP equation [15,16], and the KP equation with a self-consistent source [17]. Furthermore, nonintegrable equations can possess lump solutions, among which are a few generalized KP, BKP, KP-Boussinesq, Sawada–Kotera, Calogero–Bogoyavlenskii–Schiff and Bogoyavlensky–Konopelchenko equations in \((2+1)\)-dimensions [18–24]. It has seen that it is crucial in finding lump solutions to construct quadratic function solutions to Hirota bilinear equations and determine the sign of the resulting solutions [5].

In this paper, we would like to concern a nonlinear PDE combining with the third derivative of the time variable \( D_x D_y^3 \) in \((2+1)\)-dimensions and determine its diverse lump solutions. This new term \( D_x D_y^3 \) makes the calculation more complicated. It is also reflected that the structure of the solution is more complex,
and we pay more attention to deal with and analyze the dispersion relation of the solution. Adding three new fourth-order derivative terms and all second-order derivative terms, we formulate a combined fourth-order nonlinear partial differential equation, which possesses a Hirota bilinear form. Based on a bilinear transformation, lump solutions are obtained through symbolic computation with Maple. By choosing special values for the coefficients in equation, specific lump solutions to the equation in the corresponding cases are given. Finally, We also exhibit three-dimensional plots and contour plots of the lump solutions and discuss their dynamic behaviors. Some conclusions are given in the final section.

2. Bilinear form and lump solutions of a fourth-order PDE with a new term

In this section, we would like to consider the following nonlinear partial differential equation, which contains three fourth-order terms and six second-order terms:

\[
P(u) = \alpha [3(u_x u_t)_x + u_{xxxx}] + \beta [3(u_x u_y)_x + u_{xxyy}] + \theta (u_{xttt} + 3u_{ttt}) + 3(u_x v_t) + \gamma_1 u_{yt} + \gamma_2 u_{xx} + \gamma_3 u_{xt} + \gamma_4 u_{xy} + \gamma_5 u_{yy} + \gamma_6 u_{ttt} = 0,
\]

where \(v_x = u\), the coefficients \(\alpha, \beta\) and \(\theta\) satisfy \(\alpha^2 + \beta^2 + \theta^2 \neq 0\), but \(\gamma_i, 1 \leq i \leq 6\), are arbitrary constants.

This \((2+1)\)-dimensional nonlinear equation (5) possesses a Hirota bilinear form:

\[
B(f) = (\alpha D_x^3 D_t + \beta D_x^3 D_y + \theta D_x D_t^3 + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2) f \cdot f = 0,
\]

under the logarithmic transformation

\[
u = 2(ln f)_x = \frac{2f_x}{f}, \quad v = 2(ln f),
\]

and we have the relation \(P(u) = (B(f))_x\).

If we get a solution to the Hirota bilinear form Eq. (6), we can generate the corresponding solution to the nonlinear equation (5).

Eq. (6) with \(\theta = 0\) has been studied in [21]. In comparison with the equation in [21], \(D_x D_t^3\) is a new term and so in this paper we assume that \(\theta \neq 0\). Because Eq. (5) has the bilinear form (6), we start to construct the solution of the combined bilinear equation (6) in a positive quadratic form as follows:

\[
f = (a_1 x + a_2 y + a_3 t + a_4)^2 + (a_5 x + a_6 y + a_7 t + a_8)^2 + a_9,
\]

where \(a_i, 1 \leq i \leq 9\), are the real constant parameters to be determined, in order to generate lump solutions to the combined fourth-order nonlinear equation (5).

We insert the formula (8) into Eq. (6) to obtain a system of algebraic equations on the parameters \(a_i, 1 \leq i \leq 9\), and try to solve them by Maple symbolic computation system except for some necessary sorting and simplification of the results. In order to facilitate the calculation and expression of the results, we need to make some setting for \(\gamma_i, 1 \leq i \leq 6\), in Eq. (6), or Eq. (5).

First, let \(\gamma_6 = 0\). Eq. (6) becomes:

\[
B(f) = (\alpha D_x^3 D_t + \beta D_x^3 D_y + \theta D_x D_t^3 + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2) f \cdot f = 0.
\]

After some calculations, we obtain a set of solutions for the parameters:

\[
\begin{align*}
    a_3 &= -\frac{b_1}{(a_2 \gamma_1 + a_1 \gamma_3)^2 + (a_6 \gamma_1 + a_5 \gamma_3)^2}, \\
    a_7 &= -\frac{b_2}{(a_2 \gamma_1 + a_1 \gamma_3)^2 + (a_6 \gamma_1 + a_5 \gamma_3)^2}, \\
    a_9 &= \frac{3(a_1^2 + a_2^2)(b_3 a_1 + b_4 \beta) + b_5 \theta}{(a_1 a_6 - a_2 a_5)^2 [(a_1 \gamma_3 + a_2 \gamma_1)^2 + (a_5 \gamma_3 + a_6 \gamma_1)^2] (\gamma_1^2 + \gamma_2^2 - \gamma_1 \gamma_3 \gamma_4 + \gamma_2^2 \gamma_5)},
\end{align*}
\]

...
and all the other \( a_i \) are arbitrary. The above involved constants \( b_i, 1 \leq i \leq 5 \), are defined as follows:

\[
\begin{align*}
\begin{aligned}
b_1 &= [(a_1^2a_2 + 2a_1a_5a_6 - a_2a_5^2)γ_2 + a_1(a_2^3 + a_5^2)γ_4 + a_2(a_2^5 + a_5^2)γ_5]γ_1 \\
&+ (a_1(a_1^2 + a_5^2)γ_2 + a_2(a_1^2 + a_5^2)γ_4 + (a_1a_2^3 + 2a_2a_5a_6 - a_1a_5^2)γ_5)γ_3,
\end{aligned} \\
b_2 &= [(-a_1^2a_6 + 2a_1a_2a_5 + a_5^2a_6)γ_2 + a_5(a_2^5 + a_5^2)γ_4 + a_6(a_2^3 + a_5^2)γ_5]γ_1 \\
&+ (a_5(a_1^2 + a_5^2)γ_2 + a_6(a_1^2 + a_5^2)γ_4 + (-a_2^5a_5 + 2a_1a_2a_6 + a_5a_6^2)γ_5)γ_3,
\end{align*}
\begin{align*}
b_3 &= (a_1^2 + a_5^2)(a_1a_2 + a_5a_6)(γ_1γ_2 + γ_3γ_4) + (a_2^3 + a_5^2)(a_2^5 + a_5^2)γ_1γ_4 \\
&+ (a_1^2 + a_5^2)^2γ_2γ_3 + (a_1a_2 + a_5a_6)γ_1γ_5 \\
&+ [(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2]γ_3γ_5,
\end{align*}
\begin{align*}
b_4 &= -(a_1a_2 + a_5a_6)[(a_2γ_1 + a_1γ_3)^2 + (a_6γ_1 + a_5γ_3)^2],
\end{align*}
\begin{align*}
b_5 &= 3c_1[c_3c_4(γ_1^2γ_7^2 + 2γ_1γ_3γ_7γ_5) + c_3c_5(γ_3^2γ_2^2 + 2γ_1γ_2γ_3γ_5) + c_4c_5(γ_1^2γ_7^2 + 2γ_1γ_2γ_3γ_5)] \\
&+ 3c_2[c_4^2(γ_1^2γ_7^2 + 2γ_1γ_3γ_7γ_5) + c_2^2(γ_1γ_3γ_7^2 + 2γ_1γ_2γ_3γ_5) + c_2^2(c_1^2γ_2^2 + c_3^2γ_1^2γ_3^2) + c_4c_5(γ_1γ_3γ_7^2 + 2γ_1γ_2γ_3γ_5)] \\
&+ 3c_3[c_3^2(γ_1^2γ_7^2 + 2γ_1γ_3γ_7γ_5) + c_3^2(γ_1γ_3γ_7^2 + 2γ_1γ_2γ_3γ_5) + c_3^2(c_1^2γ_2^2 + c_3^2γ_1^2γ_3^2) + c_4c_5(γ_1γ_3γ_7^2 + 2γ_1γ_2γ_3γ_5)] \\
&+ 3c_5[c_5^4(γ_1^2γ_7^2 + 2γ_1γ_3γ_7γ_5) + c_5^4(γ_1γ_3γ_7^2 + 2γ_1γ_2γ_3γ_5) + c_5^4(c_1^2γ_2^2 + c_3^2γ_1^2γ_3^2) + c_4c_5(γ_1γ_3γ_7^2 + 2γ_1γ_2γ_3γ_5)] \\
&+ 3c_1^4(c_1^2a_2^4 + a_5^2a_6^2)(a_1^2a_2^4 + a_5^2a_6^2) + a_1a_2a_5a_6(a_1^2a_2^4 - a_1^2a_5^2) \\
&+ 3c_1^4(c_1^2a_2^4 + a_5^2a_6^2)γ_2γ_7γ_5/γ_2γ_7γ_5.
\end{align*}
\]

The parameters \( c_i, 1 \leq i \leq 6 \), involved in \( b_5 \) are defined as follows:

\[
\begin{align*}
c_1 &= [3(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2],
\end{align*}
\begin{align*}
c_2 &= [3(a_1a_2 + a_5a_6)^2 + (a_1a_6 - a_2a_5)^2],
\end{align*}
\begin{align*}
c_3 &= (a_1^2 + a_5^2),
\end{align*}
\begin{align*}
c_4 &= (a_1a_2 + a_5a_6),
\end{align*}
\begin{align*}
c_5 &= (a_2^2 + a_5^2),
\end{align*}
\begin{align*}
c_6 &= [(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2].
\end{align*}
\]

Directly comparing our results with the context in [21], where \( γ_6 = 0 \), we have some new terms contained with \( θ \) in the denominator of \( a_9 \).

Secondly, let \( γ_5 = 0 \). Eq. (6) becomes:

\[
B(f) = (αD_x^3D_y + δD_x^3D_y + θD_xD_y^2 + γ_1D_yD_t + γ_2D_x^2 + γ_3D_xD_t + γ_4D_xD_y + γ_6D_y^2)f \cdot f = 0.
\]

After some calculations, we obtain:

\[
\begin{align*}
a_2 &= -\frac{c_1}{(a_3γ_1 + a_1γ_4)^2 + (a_7γ_1 + a_5γ_4)^2},
\end{align*}
\begin{align*}
a_6 &= -\frac{c_2}{(a_3γ_1 + a_1γ_4)^2 + (a_7γ_1 + a_5γ_4)^2},
\end{align*}
\begin{align*}
a_9 &= \frac{3(a_1^2 + a_5^2)(d_3α + d_4β + d_6θ)}{(a_1a_7 - a_3a_5^2)(γ_1^2γ_7^2 - γ_1γ_3γ_7γ_5 + γ_2^2γ_6^2)}.
\end{align*}
\]
where all the other \(a_i\) are arbitrary constants. The involved constants \(d_i\), \(1 \leq i \leq 5\), are defined as follows:

\[
\begin{align*}
    d_1 &= [(a_1^2 + 2a_1a_5a_7 - a_3a_2^2)\gamma_2 + a_1(a_3^2 + a_2^2)\gamma_3 + a_3(a_3^2 + a_2^2)\gamma_6]_1 + [a_1(a_1^2 + a_2^2)\gamma_2 + a_2(a_3^2 + a_2^2)\gamma_3 + (a_1a_2^2 + 2a_3a_5a_7 - a_3a_2^2)\gamma_6]_4, \\
    d_2 &= [( - a_1^2 + 2a_1a_3a_5 + a_3^2a_7)\gamma_2 + a_5(a_3^2 + a_2^2)\gamma_3 + a_7(a_3^2 + a_2^2)\gamma_6]_1 + [a_5(a_1^2 + a_2^2)\gamma_2 + a_7(a_1^2 + a_2^2)\gamma_3 + (- a_3^2a_5 + 2a_1a_3a_7 + a_5a_2^2)\gamma_6]_4, \\
    d_3 &= -(a_1a_3 + a_5a_7)(a_3\gamma_1 + a_1\gamma_1)^2 + (a_7\gamma_1 + a_5\gamma_4)^2, \\
    d_4 &= (a_1^2 + a_3^2)(a_1a_3 + a_5a_7)(\gamma_1\gamma_2 + \gamma_3\gamma_4) + (a_1^2 + a_3^2)(a_3^2 + a_2^2)\gamma_1\gamma_3 \\
    &+ ([a_1a_3 + a_5a_7]^2 - (a_1a_7 - a_3a_5)^2]\gamma_4\gamma_6, \\
    d_5 &= [3(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2][(a_1a_2 + a_5a_6)\gamma_1\gamma_2 + (a_2^2 + a_6^2)\gamma_3\gamma_5] \\
    &+ [3(a_1a_2 + a_5a_6)^2 + (a_1a_6 - a_2a_5)^2][(a_2^2 + a_6^2)\gamma_1\gamma_4 + (a_1^2 + a_5^2)\gamma_2\gamma_3] \\
    &+ 3(a_2^2 + a_6^2)(a_1a_2 + a_5a_6)(a_2^2 + a_6^2)\gamma_1\gamma_5 + (a_2^2 + a_6^2)\gamma_3\gamma_4].
\end{align*}
\]

We find that \(a_2\) and \(a_6\) are consistent with earlier findings in paper [21]. Although there is a new term \(D_xD_y^3\) in Eq. (5), the coefficient \(\theta\), as well as \(\alpha\) and \(\beta\), the other coefficients of the fourth-order terms in the nonlinear PDE (5), just appear in \(a_6\). So in each case, when \(\gamma_6\) or \(\gamma_5\) is zero, the new term only affects \(a_9\), the constant term of the positive quadratic form of solution (8).

To generate lump solutions for the case \(\gamma_5 = 0\), besides \(a_9\) should be positive to guarantee the analyticity of rational solutions, we require only one basic condition [11]:

\[
\frac{a_1a_6 - a_2a_5}{(a_1a_7 - a_3a_5)(a_1^2 + a_2^2)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - (a_3^2 + a_2^2)\gamma_1\gamma_6 - (a_1a_3 + a_5a_7)\gamma_4\gamma_6}. 
\]

So it follows that \(a_1a_6 - a_2a_5 \neq 0\) if and only if

\[
\begin{align*}
    a_1a_7 - a_3a_5 \neq 0, \\
    \gamma_1^2 + \gamma_4^2 \neq 0, \\
    (a_1^2 + a_3^2)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - (a_3^2 + a_2^2)\gamma_1\gamma_6 - (a_1a_3 + a_5a_7)\gamma_4\gamma_6. 
\end{align*}
\]

3. Two specific lump solutions and their profiles

In this section, we will take special sets of values for the coefficients in the combined fourth-order nonlinear equation (5) to obtain the corresponding lump solutions and discuss the dynamic behaviors of those solutions.

First, we take:

\[
\alpha = 1, \beta = 0, \theta = 1, \gamma_3 = \gamma_5 = 1, \gamma_1 = \gamma_2 = \gamma_4 = \gamma_6 = 0. 
\]

Eq. (5) is reduced to:

\[
3(u_xu_t)_x + u_{xxx} + u_{xttt} + 3u_tu_{tt} + 3u_xu_{tt} + u_{xt} + u_{yy} = 0, 
\]

which has a Hirota bilinear form:

\[
(D_x^3D_t + D_xD_y^3 + D_xD_t + D_y^2)f \cdot f = 0. 
\]

Substitute the above values (17) into the resulting parameters (10). Then, take the free parameters as follows:

\[
a_1 = -1, a_2 = 3, a_4 = 4, a_5 = 1, a_6 = -1, a_8 = -3, 
\]

\[
(\text{19})
\]

5
Fig. 1. Profiles of $u_1$ when $t = 0, 25, 50$: 3d plots (top) and contour plots (bottom).

we get: $a_3 = 7$, $a_7 = 1$, $a_9 = 468$. Putting all the $a_i$, $1 \leq i \leq 9$, into the formula (8), we obtain the corresponding positive quadratic form solution of Eq. (19)

$$f_1 = (-x + 3y + 7t + 4)^2 + (x - y + t - 3)^2 + 468.$$  \hspace{1cm} (21)

By the logarithmic transformation: $u = 2(\ln f)_x$, the lump solution of the special fourth-order nonlinear equation (18) is generated as follows:

$$u_1 = \frac{4(2x - 4y - 6t - 7)}{(-x + 3y + 7t + 4)^2 + (x - y + t - 3)^2 + 468}.$$ \hspace{1cm} (22)

The three-dimensional plots and contour plots of this lump solutions (22) are shown in Fig. 1, which are made via Maple plot tools.

Secondly, we take:

$$\alpha = 1, \beta = 1, \theta = 1, \gamma_4 = \gamma_6 = 1, \gamma_1 = \gamma_2 = \gamma_3 = \gamma_5 = 0.$$ \hspace{1cm} (23)

to get another fourth-order nonlinear equation:

$$3(u_x u_t)_x + u_{xxxt} + 3(u_x u_y)_x + u_{xxyy} + u_{xttt} + 3u_t u_{tt} + 3u_{xt} u_{tt} + u_{xy} + u_{tt} = 0.$$ \hspace{1cm} (24)

which has a Hirota bilinear form:

$$(D_x^3 D_t + D_x D_y + D_x D^3_t + D_x D_y + D^2_t) f \cdot f = 0.$$ \hspace{1cm} (25)

Substitute the above values (23) into the resulting parameters (14) and associated with the special value of the free parameters:

$$a_1 = 3, a_3 = -1, a_4 = -5, a_5 = -1, a_7 = 1, a_8 = 5,$$ \hspace{1cm} (26)

we get: $a_2 = -\frac{1}{5}$, $a_6 = \frac{3}{5}$, $a_9 = 450$.

The corresponding $f$ defined by (8) reads as follows:

$$f_2 = (3x - \frac{1}{5}y - t - 5)^2 + (-x + \frac{3}{5}y + t + 5)^2 + 450,$$ \hspace{1cm} (27)

which provides a lump solution of the special fourth-order nonlinear equation (25):

$$u_2 = \frac{4(10x - 4t + 15)}{(3x - \frac{1}{5}y - t - 5)^2 + (-x + \frac{3}{5}y + t + 5)^2 + 450}.$$ \hspace{1cm} (28)

Fig. 2 displays the three-dimensional plots and contour plots of the lump solution made by Maple.
4. Conclusion

In this paper, we have concerned a new combined fourth-order nonlinear PDE. Under two special cases $\gamma_5 = 0$ and $\gamma_6 = 0$, we obtained a class of lump solutions of the nonlinear equation (5) by Hirota bilinear method. All the above results offer us abundant new exact solutions. It is important to remark that the three nonlinear terms can be merged together into the considered nonlinear model. Through the symbolic computations with the Maple, we have worked out abundant lump solutions and found out that the coefficient of the new term just affects $a_9$, the constant term of the positive quadratic form of solution (8). Under the setting of $\gamma_5$ and $\gamma_6$, we determined the other coefficients of Eq. (5) to obtain the related specific lump solution and presented their profiles via the Maple plot tools. It is necessary to point out that we made the smoother and more regular contour plots than the ones in [21] and [25].

This research has enriched the category of nonlinear PDEs that possess lump solutions, and tried to figure out the relation between the lump solutions and the nonlinear terms contained in the new equation. It is well known that interaction solutions between lump solutions and soliton solutions can describe more nonlinear phenomena [21,26] and various studies have shown that the existence of interaction solutions between lumps and other kinds of exact solutions to nonlinear integrable equations [23,27–30], even in (3+1) dimension [31–33] and linear wave equations [34]. Since the interaction properties involve much more complicated mathematical computations, the further research for interaction solutions to other generalized bilinear differential equations is becoming more interesting and meaningful.

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