

## Research paper

## On the Riemann-Hilbert Problem for the Nonlocal Reverse-Space Multi-Component Higher-Order Chen-Lee-Liu System

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## ABSTRACT

In this paper, the main work is to explore the nonlocal reverse-space multi-component higher-order Chen-Lee-Liu system through nonlocal symmetry reductions of matrix spectral problems and construct its soliton solutions via inverse scattering transforms. A formulation for solving specific Riemann-Hilbert problems is presented, where the jump matrix is taken to be the identity matrix, enabling explicit computations of  $N$ -soliton solutions for the nonlocal reverse-space multi-component higher-order Chen-Lee-Liu system under the non-zero boundary condition. Finally, the one- and two-soliton solutions are explicitly presented, and the dynamics of these solitons is further analyzed through visual verification.

## 1. Introduction

Nonlocal integrable equations have garnered growing attention in recent years, categorized into reverse-space, reverse-time and reverse-spacetime types. Notably, a couple of scalar nonlocal nonlinear Schrödinger (NLS) equations and modified Korteweg-de Vries (mKdV) equations have been identified as significant models for characterizing nonlocal nonlinear physical phenomena [1,2]. On one hand, the inverse scattering technique has been successfully applied to solve those nonlocal nonlinear equations, under either zero or nonzero boundary conditions [3–5]. On the other hand, Darboux transformations [6–9] and the Hirota bilinear method [10,11] are shown to be powerful in constructing their  $N$ -soliton solutions. Some multi-component [5,12] and higher dimensional [13–15] generalizations of nonlocal integrable equations have also been proposed and studied. Such nonlocal nonlinear integrable equations share the PT symmetry [16,17], i.e., invariance under the parity-time transformation ( $x \rightarrow -x$ ,  $t \rightarrow -t$ ,  $i \rightarrow -i$ ).

Nonlinear effects in optical fibers have been the subject of extensive research, arising from the complex interplay of dispersion, self-phase modulation, self-steepening, self-frequency shift, and other related phenomena. The nonlinear Schrödinger (NLS) equation [18] serves as a fundamental lowest-order approximation to describe the nonlinear effects in optical fibers without the inclusion of loss and gain. Naturally, to gain a deeper understanding of higher-order nonlinear effects, it is essential to incorporate several additional higher-order terms, such as third-order dispersion and quintic nonlinear terms. Among the well-established equations is the derivative nonlinear Schrödinger (DNLS) equation [19]. This equation effectively describes the propagation of short optical pulses and is widely applied in nonlinear optics and other scientific fields, holding significant importance in the field of physics. The

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generalized derivative nonlinear Schrödinger equation can be expressed in the following form [20]:

$$iq_t + q_{xx} - 2i(2\beta - 1)|q|^2 q_x - i(4\beta - 1)|q|^2 q_x^* + \beta(4\beta - 1)|q|^4 q = 0, \quad (1.1)$$

where the superscript  $*$  denotes the complex conjugate.

When  $\beta = 0$ , Eq. (1.1) can be reduced to the first DNLS (DNLS I) equation, i.e., Kaup-Newell(KN) equation [21]:

$$iq_t + q_{xx} + 2i|q|^2 q_x + i|q|^2 q_x^* = 0,$$

which is a canonical dispersive equation derived from the Magneto-hydrodynamic equations in the presence of the Hall effect.

When  $\beta = \frac{1}{4}$ , equation(1.1) is reduced to the second DNLS (DNLS II) equation, i.e., Chen-Lee-Liu(CLL) equation [22]:

$$iq_t + q_{xx} + i|q|^2 q_x = 0,$$

which appears in optical models of ultrashort pulses, where its final term represents the self-steepening effect in physics, arising from pulse propagation in a medium with intensity-dependent refractive index. In 2007, Moses et al. [23] experimentally demonstrated that pulse propagation involves self-steepening without self-phase modulation, providing the first experimental evidence for the CLL equation.

When  $\beta = \frac{1}{2}$ , Eq. (1.1) is reduced to the third DNLS (DNLS III) equation, i.e., Gerdjikov-Ivanov(GI) equation [24]:

$$iq_t + q_{xx} - i|q|^2 q_x^* + \frac{1}{2}|q|^4 q = 0,$$

which describes Alfvén wave propagation parallel to an ambient magnetic field in plasma physics, with behavior modulated by higher-order nonlinear effects. These three DNLS types are interconvertible via gauge transformations

$$q(x, t) = \tilde{q}(x, t) \exp\left(\frac{i}{2} \int_{-\infty}^x |\tilde{q}(y, t)|^2 dy\right),$$

where  $\tilde{q}$  denotes the potential function in the given equation, thus  $\text{CLL} \rightarrow \text{KN} \rightarrow \text{GI}$ . From a physical perspective, the first two types of DNLS equations exhibit similarities to the NLS equation with second-order dispersion and cubic nonlinearity, while the third type (GI) incorporates second-order dispersion and quintic nonlinearity. Mathematically, all these three equations possess distinct Lax pairs. Research on these equations has primarily focused on solution construction and long-time asymptotic analysis. Nevertheless, the increasing significance of higher-order nonlinear effects in nonlinear optics motivates the study of integrable models with third-order dispersion and quintic nonlinearity.

In this paper, we consider the nonlocal reverse-space multi-component higher-order Chen-Lee-Liu equation:

$$\begin{aligned} q_t(x, t) = & -q_{xxx}(x, t) - \frac{3}{2}q(x, t)\Sigma^{-1}q^\dagger(-x, t)q_{xx}(x, t) - \frac{3}{2}q_x(x, t)\Sigma^{-1}q^\dagger(-x, t)q_x(x, t) \\ & - \frac{3}{4}q(x, t)\Sigma^{-1}q^\dagger(-x, t)q(x, t)\Sigma^{-1}q^\dagger(-x, t)q_x(x, t). \end{aligned}$$

Firstly, we consider the higher-order generalized CLL equation with third dispersion and quintic nonlinear term

$$q_{j,t_2} = -q_{j,xxx} - \frac{3}{2}i|q|^2 q_{j,xx} - \frac{3}{2}iq_x q^* q_{j,x} + \frac{3}{4}|q|^4 q_{j,x}, \quad 1 \leq j \leq n. \quad (1.2)$$

where  $q = (q_1, q_2, \dots, q_n)$ ,  $q^* = (q_1^*, q_2^*, \dots, q_n^*)^T$ ,  $q_j^* = q_j^*(x, t)$ ,  $1 \leq j \leq n$ ,  $\dagger$  is the Hermitian transpose and  $\Sigma$  denotes an invertible constant Hermitian matrix. Soliton solutions are constructed from the specific Riemann-Hilbert problem.

## 2. Multi-component higher-order Chen-Lee-Liu system

Here, we construct a multi-component higher-order CLL system and prove its integrability by deriving its bi-Hamiltonian structure.

Firstly, we generalize the scalar potential to a vector potential via a matrix spectral problem:

$$\varphi_x = U\varphi = U(u, \lambda)\varphi, \quad U = (U_{jl})_{(n+1) \times (n+1)} = \begin{bmatrix} -i\lambda^2 + \frac{i}{4}|q|^2 & \lambda q \\ -\lambda q^* & (i\lambda^2 - \frac{i}{4}q^*q)I_n \end{bmatrix}, \quad (2.1)$$

where  $i$  is the unit imaginary number,  $\varphi$  is a  $m \times m$  matrix eigenfunction,  $\lambda$  is an eigenvalue,  $u$  is a  $2n$ -dimensional matrices potential:  $u = (q, q^*)^T$ . By reducing the matrix order, we observe that when  $q_j = q_j^* = 0$ ,  $(2 \leq j \leq n)$ , (2.1) reduces to the standard CLL matrix spectral problem [25]. Thus, (2.1) is termed the multi-component higher-order CLL spectral problem, and the associated system is referred to as the multi-component higher-order CLL integrable system.

To derive the integrable system, we solve the zero-curvature equation  $W_x = [U, W]$ . Let us consider a solution  $W$  of the following form

$$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad (2.2)$$

where  $a$  is a scalar,  $\mathbf{b}^T$  and  $\mathbf{c}$  are  $n$ -dimensional columns, and  $\mathbf{d}$  is a square matrix of size  $n \times n$ . Then we can directly calculate according to the stationary zero curvature equation to get

$$\begin{aligned} a_x &= \lambda(q\mathbf{c} + \mathbf{b}q^*), \\ b_x &= \lambda(q\mathbf{d} - a\mathbf{q}) - 2i\lambda^2\mathbf{b} + \frac{i}{4}(|q|^2\mathbf{b} + \mathbf{b}q^*q), \\ c_x &= \lambda(\mathbf{d}q^* - q^*a) + 2i\lambda^2\mathbf{c} - \frac{i}{4}(\mathbf{c}|q|^2 + q^*q\mathbf{c}), \\ d_x &= -\lambda(q^*\mathbf{b} + \mathbf{c}q) + \frac{i}{4}(\mathbf{d}q^*q - q^*q\mathbf{d}). \end{aligned} \quad (2.3)$$

Then, we make an expansion for  $W$  as follows

$$W = \begin{bmatrix} a & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} = \sum_{m=0}^{\infty} W_m \lambda^{-m}, \quad W_m = W_m(\mathbf{u}) = \begin{bmatrix} a^{[m]} & \mathbf{b}^{[m]} \\ \mathbf{c}^{[m]} & \mathbf{d}^{[m]} \end{bmatrix}, \quad m \geq 0, \quad (2.4)$$

where  $\mathbf{b}^{[m]}, \mathbf{c}^{[m]}, \mathbf{d}^{[m]}$  being defined by

$$\mathbf{b}^{[m]} = (b_1^{[m]}, b_2^{[m]}, \dots, b_n^{[m]}), \quad \mathbf{c}^{[m]} = (c_1^{[m]}, c_2^{[m]}, \dots, c_n^{[m]})^T, \quad \mathbf{d}^{[m]} = (d_{ij}^{[m]})_{n \times n}, \quad m \geq 0. \quad (2.5)$$

It then follows that the system (2.3) precisely yields the following recursion relations

$$\begin{aligned} a^{[2m+1]} &= 0, \quad \mathbf{b}^{[2m]} = 0, \quad \mathbf{c}^{[2m]} = 0, \quad \mathbf{d}^{[2m+1]} = 0, \quad a_x^{[0]} = 0, \quad \mathbf{d}_x^{[0]} = 0, \\ a_x^{[2m]} &= q\mathbf{c}^{[2m+1]} + \mathbf{b}^{[2m+1]}q^*, \\ \mathbf{b}^{[2m+1]} &= \frac{i}{2}(a^{[2m]}q - q\mathbf{d}^{[2m]} + \mathbf{b}_x^{[2m-1]} - \frac{i}{4}|q|^2\mathbf{b}^{[2m-1]} - \frac{i}{4}\mathbf{b}^{[2m-1]}q^*q), \\ \mathbf{c}^{[2m+1]} &= \frac{i}{2}(\mathbf{d}^{[2m]}q^* - q^*\mathbf{a}^{[2m]} - \mathbf{c}_x^{[2m-1]} - \frac{i}{4}\mathbf{c}^{[2m-1]}|q|^2 - \frac{i}{4}q^*q\mathbf{c}^{[2m-1]}), \\ \mathbf{d}_x^{[2m]} &= -\mathbf{c}^{[2m+1]}q - q^*\mathbf{b}^{[2m+1]} + \frac{i}{4}(\mathbf{d}^{[2m]}q^*q - q^*q\mathbf{d}^{[2m]}), \quad m \geq 1. \end{aligned} \quad (2.6)$$

From above equations, it is easy to get  $a^{[2m]}$  and  $\mathbf{d}^{[2m]}$

$$\begin{aligned} a^{[2m]} &= \frac{i}{2}\partial^{-1}(-q\mathbf{c}_x^{[2m-1]} + \mathbf{b}_x^{[2m-1]}q^* - \frac{i}{2}q^*q\mathbf{c}^{[2m-1]} - \frac{i}{2}\mathbf{b}^{[2m-1]}q^*q^*), \\ \mathbf{d}^{[2m]} &= \frac{i}{2}\partial^{-1}(\mathbf{c}_x^{[2m-1]}q - q^*\mathbf{b}_x^{[2m-1]} + \frac{i}{2}q^*q\mathbf{c}^{[2m-1]} + \frac{i}{2}q^*q\mathbf{b}^{[2m-1]}), \end{aligned} \quad (2.7)$$

where  $\partial^{-1}$  is an antiderivative in  $x$  which can be taken as either  $\partial^{-1} = \int_{-\infty}^x dy$  or  $\partial^{-1} = -\int_x^{\infty} dy$ . Besides, the integration constant is set to zero.

Next, without loss of generality, we take the initial values

$$a^{[0]} = \alpha_0, \quad \mathbf{d}^{[0]} = \alpha_1 I_n, \quad \mathbf{b}^{[1]} = \alpha_2 q, \quad \mathbf{c}^{[1]} = -\alpha_2 q^*, \quad (2.8)$$

where  $\alpha_1, \alpha_2$  are arbitrary real constants. According to this set of initial values, all matrices  $W_m (m \geq 0)$  are uniquely determined through the recursion relations (2.6). The coefficients present that

$$\begin{aligned} a^{[2]} &= \frac{\alpha_2}{2}i|q|^2, \quad d_{jl}^{[2]} = -\frac{\alpha_2}{2}iq_j^*q_l, \\ b_l^{[3]} &= \frac{\alpha_2}{2}iq_{l,x} - \frac{\alpha_2}{4}|q|^2q_l, \quad c_j^{[3]} = \frac{\alpha_2}{2}iq_{j,x}^* + \frac{\alpha_2}{4}q_j^*|q|^2, \\ a^{[4]} &= \frac{\alpha_2}{8}i|q|^4 + \frac{\alpha_2}{4}(q\mathbf{q}_x^* - \mathbf{q}_x q^*), \quad d_{jl}^{[4]} = -\frac{\alpha_2}{8}i(q_j^*q_l)^2 - \frac{\alpha_2}{4}(q_{j,x}^*q_l - q_j^*q_{l,x}), \\ b_l^{[5]} &= -\frac{\alpha_2}{4}q_{l,xx} + \frac{\alpha_2}{16}|q|^4q_l + \frac{\alpha_2}{8}iq\mathbf{q}_x^*q_l - \frac{3\alpha_2}{16}i(q_x\mathbf{q}^*q_l + q\mathbf{q}^*q_{l,x}), \\ c_j^{[5]} &= \frac{\alpha_2}{4}q_{j,xx}^* - \frac{\alpha_2}{16}q_j^*|q|^4 + \frac{\alpha_2}{8}iq_j^*\mathbf{q}_x q^* - \frac{3\alpha_2}{16}i(q_{j,x}^*q\mathbf{q}^* + q_j^*q\mathbf{q}_x^*), \\ &\vdots \end{aligned} \quad (2.9)$$

where  $1 \leq j, l \leq n$ . At the same time, we derive the recursion relation for  $\mathbf{b}^{[m]}$  and  $\mathbf{c}^{[m]}$

$$\begin{bmatrix} \mathbf{b}^{[2m+1]T} \\ \mathbf{c}^{[2m+1]} \end{bmatrix} = \Psi \begin{bmatrix} \mathbf{b}^{[2m-1]T} \\ \mathbf{c}^{[2m-1]} \end{bmatrix}, \quad m \geq 1, \quad (2.10)$$

where  $\Psi = \Psi_1 \Psi_2$  is a  $2n \times 2n$  matrix integro-differential operator and

$$\Psi_1 = \frac{1}{4} \begin{bmatrix} q^*\partial^{-1}q + \sum_{j=1}^n q_j^*\partial^{-1}q_j - 2iI_n & q^T\partial^{-1}q + (q^T\partial^{-1}q)^T \\ -q^*\partial^{-1}q^{*T} - (q^*\partial^{-1}q^{*T})^T & -q^T\partial^{-1}q^{*T} - \sum_{j=1}^n q\partial^{-1}q_j^* - 2iI_n \end{bmatrix},$$

$$\Psi_2 = \begin{bmatrix} (-\partial + \frac{i}{4}|q|^2)I_n + \frac{1}{4}q^T q^{*T} & 0 \\ 0 & (\partial + \frac{i}{4}|q|^2)I_n + \frac{1}{4}q^* q \end{bmatrix}.$$

Then, in order to obtain the multi-component higher-order CLL integrable hierarchies, we consider the temporal matrix spectral problems:

$$\varphi_t = V^{[r]}(u, \lambda)\varphi, \quad r \geq 0, \quad (2.11)$$

where

$$V^{[r]} = (V_{jl}^{[r]})_{(n+1) \times (n+1)} = (\lambda^r W)_+ + \Delta_r \quad (2.12)$$

$$= \sum_{m=0}^r \begin{bmatrix} a^{[2m]} \lambda^{2(r-m)+2} & b^{[2m+1]} \lambda^{2(r-m)+1} \\ c^{[2m+1]} \lambda^{2(r-m)+1} & d^{[2m]} \lambda^{2(r-m)+2} \end{bmatrix} + \begin{bmatrix} \Delta_r^1 & 0 \\ 0 & \Delta_r^2 \end{bmatrix}, \quad r \geq 0, \quad (2.13)$$

and  $\Delta_r$  is the modification term. According to the compatibility conditions of (2.1) and (2.13)

$$U_{t_r} - V_x^{[r]} + [U, V^{[r]}] = 0, \quad r \geq 0, \quad (2.14)$$

we have

$$\begin{aligned} \Delta_{r,x}^1 &= \frac{i}{4}(q_t q^* + q^* q_t), \\ \Delta_{r,x}^2 &= -\frac{i}{4}(q_t^* q + q^* q_t), \\ q_{t_r} &= b_x^{[2r+1]} - \frac{i}{4}q q^* b^{[2r+1]} - \frac{i}{4}b^{[2r+1]} q^* q - q \Delta_r^2 + \Delta_r^1 q, \\ q_{t_r}^* &= -c_x^{[2r+1]} - \frac{i}{4}q^* q c^{[2r+1]} - \frac{i}{4}c^{[2r+1]} q q^* + \Delta_r^2 q^* - q^* \Delta_r^1. \end{aligned} \quad (2.15)$$

Based on (2.15), we obtain

$$\Delta_r^1 = \frac{1}{2}a^{[2(r+1)]}, \quad \Delta_r^2 = \frac{1}{2}d^{[2(r+1)]}, \quad r \geq 0, \quad (2.16)$$

and

$$\begin{bmatrix} q^T \\ q^* \end{bmatrix}_{t_r} = \Psi_3 \Psi_2 \begin{bmatrix} b^{[2r+1]T} \\ c^{[2r+1]} \end{bmatrix}, \quad r \geq 1, \quad (2.17)$$

where

$$\Psi_3 = \begin{bmatrix} -\frac{i}{4}q^T \partial^{-1} q^{*T} - \frac{i}{4} \sum_{j=1}^n q_j \partial^{-1} q_j^* - I_n & -\frac{i}{4}q^T \partial^{-1} q - \frac{i}{4}(q^T \partial^{-1} q)^T \\ \frac{i}{4}q^* \partial^{-1} q^{*T} + \frac{i}{4}(q^* \partial^{-1} q^{*T})^T & \frac{i}{4}q^* \partial^{-1} q + \frac{i}{4} \sum_{j=1}^n q_j^* \partial^{-1} q_j - I_n \end{bmatrix}.$$

In this way, the multi-component higher-order CLL integrable hierarchy can be presented in following form:

$$u_r = \begin{bmatrix} q^T \\ q^* \end{bmatrix}_{t_r} = K_r = \Psi_3 \Psi_2 \begin{bmatrix} b^{[2r+1]T} \\ c^{[2r+1]} \end{bmatrix}, \quad r \geq 1. \quad (2.18)$$

When  $r = 2$ , we obtain the multi-component higher-order CLL equations:

$$q_{j,t_2} = -q_{j,xxx} - \frac{3}{2}i|q|^2 q_{j,xx} - \frac{3}{2}i q_x q^* q_{j,x} + \frac{3}{4}|q|^4 q_{j,x}, \quad 1 \leq j \leq n. \quad (2.19)$$

Finally, the multi-component higher-order CLL integrable hierarchy possess bi-Hamiltonian structures, which can be presented by the trace identity or the variational identity [26–28]. We can gain from the matrix  $U$

$$\begin{aligned} \text{tr}(W \frac{\partial U}{\partial \lambda}) &= (-2ia + 2i \sum_{j=1}^n d_{jj})\lambda + q c - b q^*, \\ \text{tr}(W \frac{\partial U}{\partial q}) &= c \lambda - \frac{i}{4}d q^* + \frac{i}{4}a q^*, \\ \text{tr}(W \frac{\partial U}{\partial q^*}) &= -b \lambda + \frac{i}{4}a q - \frac{i}{4}d q, \end{aligned}$$

i.e.,

$$\text{tr}(W \frac{\partial U}{\partial u}) = \begin{bmatrix} b^T \\ c \end{bmatrix} = \begin{bmatrix} -b^T \lambda + \frac{i}{4}a q^T - \frac{i}{4}d q^T \\ c \lambda + \frac{i}{4}a q^* - \frac{i}{4}d q^* \end{bmatrix}.$$

Based on the trace identity, we have

$$\frac{\delta}{\delta \mathbf{u}} \int \left( (-2ia + 2i \sum_{j=1}^n d_{jj})\lambda + \mathbf{q}\mathbf{c} - \mathbf{b}\mathbf{q}^* \right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \left( \lambda^\gamma \begin{bmatrix} -\mathbf{b}^T \lambda + \frac{i}{4} a \mathbf{q}^T - \frac{i}{4} \mathbf{d} \mathbf{q}^T \\ \mathbf{c} \lambda + \frac{i}{4} a \mathbf{q}^* - \frac{i}{4} \mathbf{d} \mathbf{q}^* \end{bmatrix} \right). \quad (2.20)$$

Plugging these  $a, b, c, d$

$$a = \sum_{r \geq 0} a^{[2r]} \lambda^{-2r}, \quad b = \sum_{r \geq 0} b^{[2r+1]} \lambda^{-2r-1}, \quad c = \sum_{r \geq 0} c^{[2r+1]} \lambda^{-2r-1}, \quad d = \sum_{r \geq 0} d^{[2r]} \lambda^{-2r}, \quad (2.21)$$

into the trace identity (2.20), we obtain:

$$\frac{\delta}{\delta \mathbf{u}} \int \left( -2ia^{[2r+2]} + 2i \sum_{j=1}^n d_{jj}^{[2r+2]} + \mathbf{q}\mathbf{c}^{[2r+1]} - \mathbf{b}^{[2r+1]} \mathbf{q}^* \right) dx = (-2r - \gamma) G_{2r+1},$$

with

$$G_{2r+1} = \begin{bmatrix} -\mathbf{b}^{[2r+1]T} + \frac{i}{4} a^{[2r]} \mathbf{q}^T - \frac{i}{4} \mathbf{d}^{[2r]} \mathbf{q}^T \\ \mathbf{c}^{[2r+1]} + \frac{i}{4} a^{[2r]} \mathbf{q}^* - \frac{i}{4} \mathbf{d}^{[2r]} \mathbf{q}^* \end{bmatrix}.$$

Setting  $r = 0$ , it is easy to get  $\gamma = 0$ . Thus,

$$\frac{\delta \tilde{H}_r}{\delta \mathbf{u}} = G_{2r+1},$$

and

$$\tilde{H}_r = -\frac{1}{2r} \int \left( -2ia^{[2r+2]} + 2i \sum_{j=1}^n d_{jj}^{[2r+2]} + \mathbf{q}\mathbf{c}^{[2r+1]} - \mathbf{b}^{[2r+1]} \mathbf{q}^* \right) dx, \quad r \geq 1.$$

The bi-Hamiltonian structure for the multi-component higher-order CLL integrable hierarchy is presented as following:

$$\mathbf{u}_{t_r} = \begin{bmatrix} \mathbf{q}^T \\ \mathbf{q}^* \end{bmatrix}_{t_r} = K_r = J_1 \frac{\delta \tilde{H}_r}{\delta \mathbf{u}} = J_2 \frac{\delta \tilde{H}_{r-1}}{\delta \mathbf{u}}, \quad r \geq 1, \quad (2.22)$$

where the Hamiltonian pairs  $(J_1, J_2)$ ,  $J_1 = \Psi_3 \Psi_2 \Psi_3^*$ ,  $J_2 = \Psi_3 \Psi_2 \Psi_1 \Psi_2 \Psi_3^*$ , and the symbol  $\Psi_3^*$  denotes the complex conjugate of  $\Psi_3$ .

Thus, each of the operators  $J_1^{-1} J_2$  presents a recursion operator for every hierarchy with a fixed integer  $n \geq 1$ . Adjoint symmetry constraints (or equivalently symmetry constraints) decompose each multi-component CLL system into two commuting finite-dimensional Liouville integrable Hamiltonian systems [29].

### 3. Nonlocal reverse-space multi-component higher-order Chen-Lee-Liu system

In this section, we construct the Riemann-Hilbert problem of the nonlocal reverse-space multi-component higher-order multi-component CLL system. According to the previous section, the  $r$ th flow of this integrable hierarchy can be presented in following form:

$$\varphi_x = U(\mathbf{u}, \lambda) \varphi, \quad (3.1a)$$

$$\varphi_{t_r} = V^{[\bar{r}]}(\mathbf{u}, \lambda) \varphi, \quad (3.1b)$$

where

$$U(\mathbf{u}, \lambda) = i\lambda^2 \Lambda + U_1(\mathbf{u}, \lambda), \quad V^{[\bar{r}]}(\mathbf{u}, \lambda) = 4i\lambda^{2\bar{r}} \Lambda + V_1^{[\bar{r}]}(\mathbf{u}, \lambda), \quad (3.2)$$

with  $\Lambda = \text{diag}(-1, I_n)$ ,  $\bar{r} = r + 1$ , and

$$U_1(\mathbf{u}, \lambda) = \lambda U_2 + \frac{i}{4} \Lambda U_2^2, \quad U_2 = \begin{bmatrix} 0 & \mathbf{q} \\ -\mathbf{q}^* & 0 \end{bmatrix}, \quad (3.3)$$

$$V_1^{[\bar{r}]}(\mathbf{u}, \lambda) = \begin{bmatrix} V_{11}^{[\bar{r}]} & V_{12}^{[\bar{r}]} \\ V_{21}^{[\bar{r}]} & V_{22}^{[\bar{r}]} \end{bmatrix}, \quad (3.4)$$

$$V_{11}^{[\bar{r}]} = \sum_{m=1}^{\bar{r}-1} a^{[2m]} \lambda^{2(\bar{r}-m)} + \frac{1}{2} a^{[2\bar{r}]}, \quad V_{12}^{[\bar{r}]} = \sum_{m=1}^{\bar{r}} b^{[2m-1]} \lambda^{2(\bar{r}-m)+1},$$

$$V_{21}^{[\bar{r}]} = \sum_{m=1}^{\bar{r}} c^{[2m-1]} \lambda^{2(\bar{r}-m)+1}, \quad V_{22}^{[\bar{r}]} = \sum_{m=1}^{\bar{r}-1} d^{[2m]} \lambda^{2(\bar{r}-m)} + \frac{1}{2} d^{[2\bar{r}]}.$$

In addition, the definition of  $q, q^*$  are the same as in the previous section, and  $a^{[m]}, b^{[m]}, c^{[m]}, d^{[m]}$  are defined by (2.5).

Motivated by the classical local reductions [30], we introduce a specific kind of nonlocal group reductions for the eigenvalue matrix  $U$ :

$$U^\dagger(-x, t, -i\lambda) = -CU(x, t, \lambda)C^{-1}, \quad (3.5)$$

where

$$C = \begin{bmatrix} 1 & 0 \\ 0 & \Sigma \end{bmatrix},$$

$\Sigma^\dagger = \Sigma$  is an invertible constant Hermitian matrix. Equivalently, it leads to

$$U_1^\dagger(-x, t, -i\lambda) = -CU_1(x, t, \lambda)C^{-1}, \quad (3.6)$$

then, we have

$$U_2^\dagger(-x, t) = -CU_2(x, t)C^{-1}. \quad (3.7)$$

Thus, it is easy to obtain

$$q^*(x, t) = -i\Sigma^{-1}q^\dagger(-x, t). \quad (3.8)$$

The vector function  $c$  in (2.3) under such a kind of reductions may be taken as

$$c(x, t, \lambda) = \Sigma^{-1}b^\dagger(-x, t, -i\lambda), \quad (3.9)$$

and those non-local reduction relations guarantee that

$$a^*(-x, t, -i\lambda) = a(x, t, \lambda), \quad d^\dagger(-x, t, -i\lambda) = \Sigma d(x, t, \lambda)\Sigma^{-1},$$

where  $a$  and  $d$  satisfy (2.3). For instance, under (3.8) and (3.9), we can compute that

$$\begin{aligned} (a^*(-x, t, -i\lambda))_x &= -a_x^*(-x, t, -i\lambda) \\ &= -i\lambda \left[ c^\dagger(-x, t, -i\lambda)q^\dagger(-x, t) + q^{*\dagger}(-x, t)b^\dagger(-x, t, -i\lambda) \right] \\ &= -i\lambda \left[ (b(x, t, \lambda)\Sigma^{-1})(i\Sigma q^*(x, t)) + (iq(x, t)\Sigma^{-1})(\Sigma c(x, t, \lambda)) \right] \\ &= \lambda \left[ b(x, t, \lambda)q^*(x, t) + q(x, t)c(x, t, \lambda) \right] \\ &= a_x(x, t, \lambda). \end{aligned}$$

Therefore, we have

$$\begin{cases} (a^{[m]})^*(-x, t) = (-1)^m a^{[m]}(x, t), \\ (b^{[m]})^\dagger(-x, t) = (-1)^{m+1} \Sigma c^{[m]}(x, t), \\ (d^{[m]})^\dagger(-x, t) = (-1)^m \Sigma d^{[m]}(x, t)\Sigma^{-1}, \end{cases}$$

where  $a^{[m]}, b^{[m]}, c^{[m]}$  and  $d^{[m]}, m \geq 1$  are defined the same as in the previous section. Then, we have

$$(V^{[\bar{r}]})^\dagger(-x, t, -i\lambda) = CV^{[\bar{r}]}(x, t, \lambda)C^{-1}, \quad (V_1^{[\bar{r}]})^\dagger(-x, t, -i\lambda) = CV_1^{[\bar{r}]}(x, t, \lambda)C^{-1}, \quad (3.10)$$

where  $V^{[\bar{r}]}$  and  $V_1^{[\bar{r}]}$  are defined in (3.2) and (3.4), respectively.

The above analysis guarantees that the non-local reduction (3.6) does not require any new condition for the compatibility of the spatial and temporal matrix spectral problems. Therefore, under the nonlocal group reductions in (3.6), the multi-component higher-order CLL equations in (2.19) are reduced to the following nonlocal reverse-space multi-component higher-order CLL equations:

$$\begin{aligned} q_t(x, t) &= -q_{xxx}(x, t) - \frac{3}{2}q(x, t)\Sigma^{-1}q^\dagger(-x, t)q_{xx}(x, t) - \frac{3}{2}q_x(x, t)\Sigma^{-1}q^\dagger(-x, t)q_x(x, t) \\ &\quad - \frac{3}{4}q(x, t)\Sigma^{-1}q^\dagger(-x, t)q(x, t)\Sigma^{-1}q^\dagger(-x, t)q_x(x, t). \end{aligned} \quad (3.11)$$

Obviously, the nonlocal reverse-space multi-component higher-order CLL systems can be expressed as:

$$q_t = K_{r,1}|_{q^*(x,t)=-i\Sigma^{-1}q^\dagger(-x,t)}, \quad (3.12)$$

where  $K_r = (K_{r,1}^\top, K_{r,2})$ . It is visible that if  $q(x, t)$  is a solution to Eq. (3.11),  $iq^*(-x, t)$  and  $iq(x, -t)$  are also solutions.

#### 4. The Riemann-Hilbert problem for multi-component higher-order Chen-Lee-Liu system

In this section, the RH problem of the nonlocal reverse-space multi-component higher-order CLL equation is constructed with the zero boundary condition.

#### 4.1. Property of eigenfunctions

Without loss of generality, we set  $\alpha_0 = -4i$ ,  $\alpha_1 = 4i$ ,  $\alpha_2 = 4$ . Assume that each potential rapidly vanishes as  $x \rightarrow \pm\infty$  or  $t \rightarrow \pm\infty$  and satisfy the integrable conditions:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x|^k |t|^l \sum_{j=1}^n |q_j| dx dt < \infty, \quad 0 \leq k, l \leq 1.$$

**Property 4.1.** Potential functions have the asymptotic behavior:  $\varphi \sim e^{i\lambda^2 \Lambda x + 4i\lambda^{2F} \Lambda t_F}$ , as  $x, t \rightarrow \pm\infty$ .

**Property 4.2.** If one takes the transformation

$$\varphi = \hat{\phi} E_g, \quad E_g = e^{i\lambda^2 \Lambda x + 4i\lambda^{2F} \Lambda t_F}, \quad (4.1)$$

one can obtain the canonical normalization condition  $\hat{\phi}: \hat{\phi} \rightarrow I_{n+1}$ , as  $x, t \rightarrow \pm\infty$ .

Substituting (4.1) into the matrix spectral problems in (3.1a)–(3.1b), it is easy to obtain the equivalent pair of matrix spectral problems

$$\hat{\phi}_x = i\lambda^2 [\Lambda, \hat{\phi}] + U_1 \hat{\phi}, \quad (4.2)$$

$$\hat{\phi}_{t_F} = 4i\lambda^{2F} [\Lambda, \hat{\phi}] + V_1^{[F]} \hat{\phi}. \quad (4.3)$$

In order to satisfy the regularization condition of the corresponding RH problem, that is, the spectral problem tends to the identity matrix as  $\lambda \rightarrow \infty$ . To achieve this condition, we need to introduce a transformation to guarantees asymptotics. From this, we can naturally define:

$$\hat{\phi}_0 = \begin{bmatrix} \exp(-\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx') & 0 \\ 0 & M \end{bmatrix}, \quad (4.4)$$

where  $M$  is an  $n \times n$  matrix, satisfying:

$$M_x = \frac{1}{4} i q^* q M.$$

In fact, we consider expanding the solution of the spectral problem (4.2)–(4.3) in the following form:

$$\hat{\phi}(x, \lambda) = \hat{\phi}_0 + \frac{\hat{\phi}_1}{\lambda} + \frac{\hat{\phi}_2}{\lambda^2} + \frac{\hat{\phi}_3}{\lambda^3} + O\left(\frac{1}{\lambda^4}\right), \quad (4.5)$$

where  $\hat{\phi}_k (k = 0, 1, 2, 3)$  are independent of the spectral parameter  $\lambda$ . Substituting Eq. (4.5) into Eqs. (4.2)–(4.3) and comparing the same order of  $\lambda$ , it is easy to obtain

$$\begin{aligned} O(1) : \hat{\phi}_{0x} &= i[\Lambda, \hat{\phi}_2] + U_2 \hat{\phi}_1 + \frac{1}{4} i \Lambda U_2^2 \hat{\phi}_0, \\ O(\lambda) : i[\Lambda, \hat{\phi}_1] + U_2 \hat{\phi}_0 &= 0, \\ O(\lambda^2) : i[\Lambda, \hat{\phi}_0] &= 0, \end{aligned} \quad (4.6)$$

it is easy to know that  $\hat{\phi}_0$  is diagonal and satisfies

$$\hat{\phi}_{0x} = -\frac{1}{4} i U_2^2 \Lambda \hat{\phi}_0. \quad (4.7)$$

Similarly, the expression of  $\hat{\phi}_{0t}$  can also be obtained in the same way. According to the conservation law of the  $r$ th flow in a multi-component high-order CLL system, it is easy to get  $\hat{\phi}_{0xt} = \hat{\phi}_{0tx}$ . Hence the  $\hat{\phi}_0 = \exp(-\frac{1}{4}i \int_{-\infty}^x U_2^2 \Lambda dx')$ . Noticed that  $\text{tr}(\hat{\phi}_0) = 0$ , as  $x \rightarrow -\infty$ , we gain that  $\hat{\phi}_0 \rightarrow I_{n+1}$ . According to the Abel formula, it can be derived that  $\det \hat{\phi}_0 = 1$ .

In the following we introduce the gauge transformation  $\phi = \hat{\phi}_0^{-1} \hat{\phi}$  and, by direct calculation, the matrix spectral problem (4.2)–(4.3) is equivalent to

$$\phi_x = i\lambda^2 [\Lambda, \phi] + \tilde{U}_1 \phi, \quad (4.8)$$

$$\phi_{t_F} = 4i\lambda^{2F} [\Lambda, \phi] + \tilde{V}_1^{[F]} \phi, \quad (4.9)$$

where  $\tilde{U}_1 = \hat{\phi}_0^{-1} U_1 \hat{\phi}_0 - \hat{\phi}_0^{-1} \hat{\phi}_{0x}$ ,  $\tilde{V}_1^{[F]} = \hat{\phi}_0^{-1} V_1^{[F]} \hat{\phi}_0 - \hat{\phi}_0^{-1} \hat{\phi}_{0t}$ , we have

$$\tilde{U}_1 = \frac{1}{2} i U_2^2 \Lambda + \lambda \tilde{U}_2, \quad (4.10)$$

$$\tilde{U}_1 = \begin{bmatrix} \frac{1}{2} i |q|^2 & \lambda \exp(\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx') q M \\ -\lambda M^{-1} q^* \exp(-\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx') & -\frac{1}{2} i q^* q \end{bmatrix}, \quad (4.11)$$

$$\tilde{U}_2 = \begin{bmatrix} 0 & \exp(\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx') q M \\ -M^{-1} q^* \exp(-\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx') & 0 \end{bmatrix}, \quad (4.12)$$

since  $\text{tr}(\tilde{U}_1) = \text{tr}(\tilde{V}_1^{[I]}) = 0$ . Based on a generalized Liouville's formula [31] one can obtain:

$$\det \phi = 1. \quad (4.13)$$

To develop associated Riemann-Hilbert problems, we consider the following adjoint equation of the (2.1) and (4.8)

$$\tilde{\phi}_x = -\tilde{\phi}U, \quad (4.14)$$

and

$$\tilde{\phi}_x = -i\lambda^2[\tilde{\phi}, \Lambda] - \tilde{\phi}\tilde{U}_1. \quad (4.15)$$

**Property 4.3.** If  $\phi(x, t, \lambda)$  is a solution to the spatial eigenvalue problem (4.8), obviously,  $C\phi^{-1}(x, t, \lambda)$  is a matrix adjoint eigenfunction and is associated with the given eigenvalue  $\lambda$ .

In fact, calculate directly

$$\begin{aligned} [\phi^\dagger(-x, t, -i\lambda)C]_x &= -(\phi_x)^\dagger(-x, t, -i\lambda) \\ &= -\left\{(-i)(-i\lambda)^2[\phi^\dagger(-x, t, -i\lambda), \Lambda] - \phi^\dagger(-x, t, -i\lambda)\tilde{U}_1^\dagger(-x, t)\right\}C \\ &= -\left\{i\lambda^2[\phi^\dagger(-x, t, -i\lambda)C, \Lambda] - \phi^\dagger(-x, t, -i\lambda)CC^{-1}\tilde{U}_1^\dagger(-x, t)C\right\} \\ &= -i\lambda^2[\phi^\dagger(-x, t, -i\lambda)C, \Lambda] - \phi^\dagger(-x, t, -i\lambda)C\tilde{U}_1(x, t), \end{aligned}$$

and thus,

$$\tilde{\phi}(x, t, \lambda) := \phi^\dagger(-x, t, -i\lambda)C, \quad (4.16)$$

represents another matrix adjoint eigenfunction associated with the original eigenvalue  $\lambda$ , which means that  $\phi^\dagger(-x, t, -i\lambda)C$  solves the adjoint matrix spectrum problem (4.15).

**Property 4.4.** From the asymptotic property and the uniqueness of the solution of  $\phi$ , it can be known that if  $x$  or  $t \rightarrow -\infty$  or  $+\infty$ ,  $\phi \rightarrow I_{n+1}$ , then

$$\phi^\dagger(-x, t, -i\lambda) = C\phi^{-1}(x, t, \lambda)C^{-1}. \quad (4.17)$$

It follows that if  $\lambda$  is an eigenvalue of (4.8) or (4.15), then  $-i\lambda$  is another eigenvalue of (4.8) or (4.15), and the corresponding eigenfunction  $\phi$  satisfies the Eq. (4.17).

## 4.2. Riemann-Hilbert problems

Let us now formulate a class of associated Riemann-Hilbert problems with the space variable  $x$ . In the direct scattering problem, we first consider the two matrix eigenfunctions  $\phi_\pm(x, \lambda)$  of (4.8) with the asymptotic conditions

$$\phi_\pm \rightarrow I_{n+1}, \quad x \rightarrow \pm\infty, \quad (4.18)$$

respectively. From (4.13), we can readily find that

$$\det \phi_\pm = 1, \quad (4.19)$$

for all  $\lambda \in \mathbb{R}$ .

In the following we denote

$$E(x, \lambda) = e^{i\lambda^2 \Lambda x}, \quad (4.20)$$

$$\psi_- = \phi_- E, \quad \psi_+ = \phi_+ E, \quad (4.21)$$

since  $\psi_-(x, \lambda)$  and  $\psi_+(x, \lambda)$  are both matrix eigenfunctions of the spectral problems (4.8), they must be linearly dependent, and accordingly, we have

$$\psi_-(x, \lambda) = \psi_+(x, \lambda)S(\lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}, \quad (4.22)$$

that is

$$\phi_- E = \phi_+ ES(\lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}, \quad (4.23)$$

where  $S(\lambda) = (s_{jl})_{(n+1) \times (n+1)}$  is traditionally called the scattering matrix and satisfies  $\det S(\lambda) = 1$ .

The scattering matrix  $S(\lambda)$  contains important scattering data from which we can reconstruct the potential function  $u$ . This reconstruction largely depends on the analytical properties of the matrix characteristic function  $\phi_\pm(x, \lambda)$  and the scattering matrix  $S(\lambda)$  in

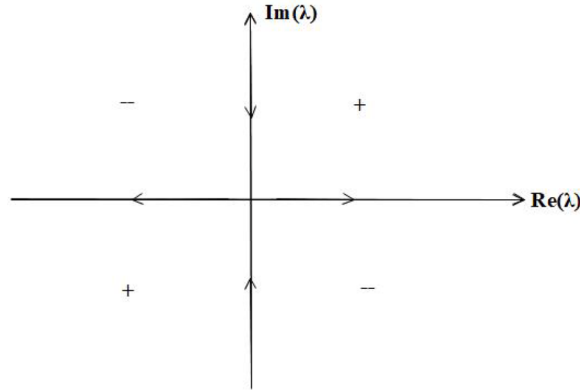


Fig. 1. The jump contour in the complex  $\lambda$ -plane.

the complex plane of  $\lambda$ . Therefore, we first study these analytical properties. Applying the method of variation of parameters as well as the boundary conditions (4.18), we can transform Eq. (4.8) into the following Volterra integral equations for  $\phi_{\pm}$ :

$$\phi_{-}(x, \lambda) = I_{n+1} + \int_{-\infty}^x e^{i\lambda^2 \Lambda(x-y)} \tilde{U}_1(y) \phi_{-}(y, \lambda) e^{-i\lambda^2 \Lambda(x-y)} dy, \quad (4.24)$$

$$\phi_{+}(x, \lambda) = I_{n+1} - \int_x^{\infty} e^{i\lambda^2 \Lambda(x-y)} \tilde{U}_1(y) \phi_{+}(y, \lambda) e^{-i\lambda^2 \Lambda(x-y)} dy. \quad (4.25)$$

Thus, the two eigenfunctions  $\phi_{\pm}$  exist, and allow analytical continuations off the real axis  $\lambda \in \mathbb{R} \cup i\mathbb{R}$  as long as the both integrals on their right hand sides converge.

**Lemma 4.1.** *The analytical property of  $\phi_{\pm}$*

(1) *the first column of  $\phi_{-}$  and the last  $n$  columns of  $\phi_{+}$  are analytical with respect to  $\lambda \in \Gamma_{+} = \{\lambda \in \mathbb{C} \mid \arg \lambda \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})\}$ ;*

(2) *the last  $n$  columns of  $\phi_{-}$  and the first column of  $\phi_{+}$  are analytical with respect to  $\lambda \in \Gamma_{-} = \{\lambda \in \mathbb{C} \mid \arg \lambda \in (\frac{\pi}{2}, \pi) \cup (\frac{3\pi}{2}, 2\pi)\}$ .*

**Proof.** According to the diagonal form of  $\Lambda$  and the structure of  $U$ , we can observe that the integral equation for the first column of  $\phi_{-}$  contains only the exponential factor  $e^{2i\lambda^2(x-y)}$ , which also decays exponentially because of  $y < x$  in the integral, when  $\lambda \in \Gamma_{+}$ , and the integral equation for the last  $n$  columns of  $\phi_{+}$  contains only the exponential factor  $e^{-2i\lambda^2(x-y)}$ , which decays exponentially because of  $y > x$  in the integral. It thus follows that those  $n+1$  (i.e., the first column of  $\phi_{-}$  and the last  $n$  columns of  $\phi_{+}$ ) columns are analytical in the upper half-plane  $\lambda \in \Gamma_{+}$  and allow for analytical continuations to  $\Gamma_0 = \{\mathbb{R} \cup i\mathbb{R}\}$ . Similarly, we can see that the last  $n$  columns of  $\phi_{-}$  and the first column of  $\phi_{+}$  are analytical in the lower half-plane  $\lambda \in \Gamma_{-}$  and analytically extended to  $\Gamma_0$  (Fig. 1).

□

First, in order to determine two generalized Jost solutions, denoted by  $J_{+}(x, \lambda)$  and  $J_{-}(x, \lambda)$ , which are analytic in  $\Gamma_{+}$  and  $\Gamma_{-}$  (the upper and lower half-planes) and continuous in  $\Gamma_0$ , respectively, we express

$$\phi_{\pm} = (\phi_{\pm}^1, \phi_{\pm}^2, \dots, \phi_{\pm}^{n+1}),$$

where  $\phi_{\pm}^j (1 \leq j \leq n+1)$  denotes the  $j$ th column of  $\phi_{\pm}$ . Then, we take the generalized matrix Jost solution  $J_{+}(x, \lambda)$  as

$$J_{+}(x, \lambda) = (\phi_{-}^1, \phi_{+}^2, \dots, \phi_{+}^{n+1}) = \phi_{-} H_1 + \phi_{+} H_2, \quad (4.26)$$

Here we denote  $H_1$  and  $H_2$  as

$$H_1 = \text{diag}(1, \underbrace{0, \dots, 0}_n), \quad H_2 = \text{diag}(0, \underbrace{1, \dots, 1}_n).$$

Then, according to Lemma 4.1

**Theorem 4.1.**  $J_{+}(x, \lambda)$  is analytic in  $\lambda \in \Gamma_{+}$  and continuous in  $\Gamma_0$ . The other generalized matrix Jost eigenfunction

$$(\phi_{+}^1, \phi_{-}^2, \dots, \phi_{-}^{n+1}) = \phi_{+} H_1 + \phi_{-} H_2,$$

is analytic in  $\lambda \in \Gamma_{-}$  and continuous in  $\Gamma_0$ .

In addition, from the Volterra integral Eqs. (4.24) and (4.25), we see that  $J_{+}$  generate the canonical normalization conditions:

$$J_{+}(x, \lambda) \rightarrow I_{n+1}, \quad \lambda \in \Gamma_{+} \rightarrow \infty, \quad (4.27)$$

and

$$(\phi_{+}^1, \phi_{-}^2, \dots, \phi_{-}^{n+1}) \rightarrow I_{n+1}, \quad \lambda \in \Gamma_{-} \rightarrow \infty.$$

Second, to determine the other generalized Jost solution  $J_-$ , i.e., the analytic counterpart of  $J_+$  in the lower half-plane  $\Gamma_-$ , we adopt the adjoint matrix spectral problems. Notice that if  $\varphi$  and  $\phi$  are solutions to the matrix spectrum problem, then the inverse matrices  $\tilde{\varphi}_\pm(x, \lambda) = (\varphi_\pm(x, \lambda))^{-1}$  and  $\tilde{\phi}_\pm(x, \lambda) = (\phi_\pm(x, \lambda))^{-1}$  solve those two adjoint equations, respectively. In fact, by (4.15) as well

$$(\phi\phi^{-1})_x = \phi_x\phi^{-1} + \phi(\phi^{-1})_x = 0,$$

we have

$$(\phi^{-1})_x = -i\lambda^2[\phi^{-1}, \Lambda] - \phi^{-1}\tilde{U}_1, \quad (4.28)$$

that is  $(\phi_\pm(x, \lambda))^{-1}$  satisfies the adjoint matrix spectrum problem (4.15). Similarly,  $(\phi_\pm)^{-1}$  can be denoted as

$$\tilde{\phi}_\pm = (\tilde{\phi}_\pm^1, \tilde{\phi}_\pm^2, \dots, \tilde{\phi}_\pm^{n+1}).$$

Then, the corresponding matrix Jost eigenfunction  $J_-$  can be expressed as

$$J_-(x, \lambda) = (\tilde{\phi}_-^1, \tilde{\phi}_-^2, \dots, \tilde{\phi}_-^{n+1})^T = H_1\tilde{\phi}_- + H_2\tilde{\phi}_+ = H_1(\phi_-)^{-1} + H_2(\phi_+)^{-1}. \quad (4.29)$$

Obviously, we can obtain

**Theorem 4.2.**  $J_-(x, \lambda)$  is analytic in  $\lambda \in \Gamma_-$  and continuous in  $\Gamma_0$ , and

$$J_-(x, \lambda) \rightarrow I_{n+1}, \quad \lambda \in \Gamma_- \rightarrow \infty,$$

$$(\tilde{\phi}_+^1, \tilde{\phi}_+^2, \dots, \tilde{\phi}_+^{n+1}) \rightarrow I_{n+1}, \quad \lambda \in \Gamma_+ \rightarrow \infty.$$

Substituting  $J_+$  and  $J_-$  into Eq. (4.23), it is equivalent to

$$J_+ = \phi_+ E S_+ E^{-1} = \phi_+ E \begin{bmatrix} s_{11} & 0 & \cdots & 0 \\ s_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+1,1} & 0 & \cdots & 1 \end{bmatrix} E^{-1}, \quad (4.30)$$

and

$$J_- = E S_-^{-1} E^{-1} (\phi_+)^{-1} = E \begin{bmatrix} \hat{s}_{11} & \hat{s}_{12} & \cdots & \hat{s}_{1,n+1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} E^{-1} (\phi_+)^{-1}, \quad (4.31)$$

where  $S^{-1}(\lambda) = (S(\lambda))^{-1} = (\hat{s}_{ij})_{(n+1) \times (n+1)}$ .

Since the Jost eigenfunction  $J_\pm(x, \lambda)$  of the two matrices are analytic with relation to  $\lambda$  in  $\Gamma_+$  and  $\Gamma_-$ , respectively, combined with (4.23), (4.26) and (4.29), it is easy to obtain:

$$J_+(x, \lambda) = J_-(x, \lambda) G_0(x, \lambda), \quad \lambda \in \Gamma_0, \quad (4.32)$$

in which the jump matrix

$$G_0(x, \lambda) = E(H_1 + H_2 S(\lambda))(H_1 + S^{-1}(\lambda)E^{-1}) = E \begin{bmatrix} 1 & \hat{s}_{12} & \hat{s}_{13} & \cdots & \hat{s}_{1,n+1} \\ s_{21} & 1 & 0 & \cdots & 0 \\ s_{31} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n+1,1} & 0 & 0 & \cdots & 1 \end{bmatrix} E^{-1}. \quad (4.33)$$

Thus, (4.32) and (4.33) establish an RH problem for nonlocal reverse-space multi-component higher-order CLL systems. When  $\lambda \in \Gamma_\pm \rightarrow \infty$ , the asymptotic property

$$J_\pm(x, \lambda) \rightarrow I_{n+1}, \quad (4.34)$$

generates canonical normalization conditions for the above RH problem. The jump matrix  $G_0$  carries the basic scattering data from the scattering matrix  $S(\lambda)$ , and then by the relationship of (4.17),  $G_0$  satisfies

$$G_0^\dagger(-x, t, -i\lambda) = C G_0(x, t, \lambda) C^{-1}. \quad (4.35)$$

## 5. Solutions for the Riemann-Hilbert problem

### 5.1. Evolution of the scattering data

We first develop the evolution laws for the scattering data to formulate the inverse scattering transforms. According to the relation  $\phi_- E = \phi_+ E S(\lambda)$ , combined with the involution properties given in **Property 4.3** and **Property 4.4**, we obtain the involution property of the scattering matrix

$$S^\dagger(-i\lambda) = S^{-1}(\lambda). \quad (5.1)$$

Furthermore, the similar analysis shows that the Jost solutions satisfy another symmetry relation

$$\phi(-\lambda) = \sigma \phi(\lambda) \sigma,$$

where  $\sigma = \text{diag}(1, \underbrace{-1, \dots, -1}_n)$ . Then, we can easily get

$$S(-\lambda) = \sigma S(\lambda) \sigma, \quad (5.2)$$

Next, we study the property of  $s_{11}$  and  $\hat{s}_{11}$ , which play an important role in later analysis. From Eqs. (5.1) and (5.2), we obtain the relations

$$s_{11}^*(-i\lambda) = \hat{s}_{11}(\lambda), \quad (5.3)$$

and

$$s_{11}(\lambda) = s_{11}(-\lambda). \quad (5.4)$$

Suppose that  $\lambda \in \Gamma_+$  is a zero of  $s_{11}$ , then  $-\lambda$  is a zero as well by means of Eq. (5.4). From the relation Eq. (5.3), we obtain that  $\hat{s}_{11}$  has two zeros, namely  $\mp i\lambda$ .

Next, we complete the direct scattering transform. Take the derivative of (4.23) with respect to the temporal variable  $t$  and utilize the temporal matrix spectral problems for  $\phi_{\pm}$ . This way, it follows that the scattering matrix  $S(\lambda)$  obeys an evolution law:

$$S_{t_r} = 4i\lambda^{2r}[\Lambda, S], \quad (5.5)$$

This precisely yields that the time-dependent scattering coefficients obey the following evolution laws:

$$s_{1,j} = s_{1,j}(0, \lambda) e^{8i\lambda^{2r}t_r}, \quad s_{j,1} = s_{j,1}(0, \lambda) e^{-8i\lambda^{2r}t_r}, \quad 2 \leq j \leq n+1,$$

and all other scattering coefficients do not depend on the time variable  $t_r$ .

## 5.2. Solutions for the Riemann-Hilbert problem

In this section, we use the RH method to obtain the solutions to the nonlocal reverse-space multi-component higher-order CLL system. There are two cases of the RH problems: One is the regular RH problem; the other is the non-regular RH problem.

Firstly, recall (4.30) and (4.31), we have

$$\det J_+(x, \lambda) = s_{11}(\lambda), \quad \det J_-(x, \lambda) = \hat{s}_{11}(\lambda), \quad (5.6)$$

which

$$\hat{s}_{11} = (S^{-1})_{11} = \begin{vmatrix} s_{22} & s_{23} & \cdots & s_{2,n+1} \\ s_{32} & s_{33} & \cdots & s_{3,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+1,2} & s_{n+1,3} & \cdots & s_{n+1,n+1} \end{vmatrix}.$$

**Case 1.** the RH problem is regular: i.e.,  $\det J_+ = s_{11} \neq 0$  and  $\det J_- = \hat{s}_{11} \neq 0$ .

**Lemma 5.1.** If  $J_{\pm}$  satisfy the canonical normalization conditions:  $J_{\pm}(x, \lambda) \rightarrow I_{n+1}$ ,  $\lambda \rightarrow \infty$ , thus, the solution to the regular RH problem is unique.

**Proof.** In fact, we assume that both  $J_{\pm}^1$  and  $J_{\pm}^2$  are solutions to (4.32), then

$$(J_-^1)^{-1} J_+^1 = (J_-^2)^{-1} J_+^2, \quad J_+^1 (J_+^2)^{-1} = J_-^1 (J_-^2)^{-1}, \quad \lambda \in \mathbb{R} \cup i\mathbb{R}.$$

Since the RH problem is regular, neither  $\det J_{\pm}^1$  nor  $\det J_{\pm}^2$  is zero. Thus,  $J_+^1 (J_+^2)^{-1}$  and  $J_-^1 (J_-^2)^{-1}$  are analytic in  $\Gamma_+$  and  $\Gamma_-$ , respectively. Obviously, they define a matrix function that is analytic on the entire plane. It follows from the boundary conditions that the matrix function tends to the identity matrix  $I_{n+1}$  as  $\lambda$  tends to infinity. In complex analysis, Liouville's theorem tells that if a function is analytic and bounded in the entire complex plane, then this function must be a constant. Then, based on this theorem, it can be obtained:

$$J_+^1 (J_+^2)^{-1} = J_-^1 (J_-^2)^{-1} = I_{n+1}, \quad \lambda \in \mathbb{C}.$$

so  $J_{\pm}^1 = J_{\pm}^2$ , which means that the solution to the RH problem is unique.  $\square$

In fact, the unique solution of this regular matrix RH problem does not have an explicit expression, but its formal solution can be expressed by the Plemelj formula. Without loss of generality, we rewrite the boundary conditions of this RH problem. When  $\lambda \in \mathbb{R} \cup i\mathbb{R}$

$$\begin{cases} J_+(\lambda) - J_-(\lambda) = J_-(\lambda) J_0(\lambda), \\ J_{\pm}(\lambda) \rightarrow I_{n+1}, \quad \lambda \rightarrow \infty, \end{cases} \quad (5.7)$$

where

$$J_0(\lambda) = G_0(\lambda) - I_{n+1} = E \begin{bmatrix} 0 & \hat{s}_{12} & \hat{s}_{13} & \cdots & \hat{s}_{1,n+1} \\ s_{21} & 0 & 0 & \cdots & 0 \\ s_{31} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n+1,1} & 0 & 0 & \cdots & 0 \end{bmatrix} E^{-1},$$

$G_0(\lambda)$  is the jump matrix. According to (4.32) and the Plemelj formula, the formal solution of the regular matrix RH problem can be obtained:

$$(J_+)^{-1}(\lambda) = I_{n+1} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{J_0(\xi)(J_+)^{-1}(\xi)}{\xi - \lambda} d\xi, \quad \lambda \in \Gamma_+. \quad (5.8)$$

However, in many cases, the RH problem (4.32) is non-regular.

**Case 2.** the RH problem is non-regular: i.e.,  $\det J_+ = s_{11} = 0$  and  $\det J_- = \hat{s}_{11} = 0$ .

**Lemma 5.2.** If  $\det J_+ = s_{11} = 0$ ,  $\det J_- = \hat{s}_{11} = 0$ , and  $\det J_{\pm} = 0$  in  $\Gamma_{\pm}$ , then the expression for  $J_{\pm}$  at  $\det J_{\pm} = 0$  is determined.

**Proof.** If  $\det J_+ = s_{11} = 0$  and  $\det J_- = \hat{s}_{11} = 0$ , then the uniqueness of solutions for each associated RH problem fails. In general, suppose that  $s_{11}(\lambda)$  has  $2N$  zeros  $\{\pm\lambda_k \in \Gamma_+, 1 \leq k \leq N\}$  (where  $N$  is another arbitrarily given natural number),  $\hat{s}_{11}(\lambda)$  has  $2N$  zeros  $\{\pm\hat{\lambda}_k \in \Gamma_-, 1 \leq k \leq N\}$ . In this case, each of  $\ker J_+(\lambda_k)$ ,  $1 \leq k \leq N$  only contain a single basis column vector, which is denoted by  $\mathbf{v}_k$ ; each of  $\ker J_-(\hat{\lambda}_k)$ ,  $1 \leq k \leq N$  only contain a single basis row vector, which is denoted by  $\hat{\mathbf{v}}_k$ . Then, it is easy to know that

$$J_+(\lambda_k)\mathbf{v}_k = 0, \quad \hat{\mathbf{v}}_k J_-(\hat{\lambda}_k) = 0, \quad 1 \leq k \leq N. \quad (5.9)$$

It is known that under the standard normalization condition (4.34) and the special zero construction condition (5.9), we can obtain an explicit solution to the RH problem (4.32) [19]. If we take the jump matrix  $G_0$  as the identity matrix  $I_{n+1}$ , we can obtain the  $N$ -soliton solution of the RH problem. In this case, we can clearly see that  $G_0 = I_{n+1}$  can be achieved as long as  $s_{j,1} = \hat{s}_{1,j} = 0$ ,  $2 \leq j \leq n+1$ , which also means that there is no reflection (the reflection coefficient is zero) in the scattering problem. From this, the solution to the special RH problem (4.32) is obtained

$$J_+(x, \lambda) = I_{n+1} - \sum_{k,l=1}^N \left( \frac{\mathbf{v}_k(M^{-1})_{kl}\hat{\mathbf{v}}_l}{\lambda - \hat{\lambda}_l} - \frac{\sigma\mathbf{v}_k(M^{-1})_{kl}\hat{\mathbf{v}}_l\sigma}{\lambda + \hat{\lambda}_l} \right), \quad (5.10)$$

$$J_-(x, \lambda) = I_{n+1} - \sum_{k,l=1}^N \left( \frac{\mathbf{v}_l^{\dagger}(M^{-1})_{kl}^{\dagger}\hat{\mathbf{v}}_k^{\dagger}}{\lambda - \hat{\lambda}_l} - \frac{\sigma\mathbf{v}_l^{\dagger}(M^{-1})_{kl}^{\dagger}\hat{\mathbf{v}}_k^{\dagger}\sigma}{\lambda + \hat{\lambda}_l} \right), \quad (5.11)$$

where  $M = (m_{kl})_{N \times N}$  is a square matrix whose entries are determined by

$$m_{kl} = \frac{\hat{\mathbf{v}}_k\mathbf{v}_l}{\lambda_l - \hat{\lambda}_k} - \frac{\hat{\mathbf{v}}_k\sigma\mathbf{v}_l}{\lambda_l + \hat{\lambda}_k}, \quad \lambda_l \neq \hat{\lambda}_k. \quad (5.12)$$

□

Since both  $\lambda_k$  and  $\hat{\lambda}_k$  are constants, they are independent of space  $x$  and time  $t$ . Furthermore, the evolution of  $\mathbf{v}_k(x, t_{\bar{F}})$  and  $\hat{\mathbf{v}}_k(x, t_{\bar{F}})$ ,  $1 \leq k \leq N$  with respect to space and time can be obtained. For example, by taking the derivative of the first equation of (5.9) with respect to  $x$  and combining it with the Eq. (4.8), we can obtain

$$J_+(x, \lambda_k) \left( \frac{d\mathbf{v}_k}{dx} - i\lambda_k^2 \Lambda \mathbf{v}_k \right) = 0, \quad 1 \leq k \leq N. \quad (5.13)$$

It can be seen that for each  $1 \leq k \leq N$ ,  $\frac{d\mathbf{v}_k}{dx} - i\lambda_k^2 \Lambda \mathbf{v}_k$  is an element of  $\ker J_+(x, \lambda_k)$ , combined with the relation of (5.9), it is not difficult to find that  $\mathbf{v}_k$  is linearly dependent. Without loss of generality, we denote it as a simple linear relationship

$$\frac{d\mathbf{v}_k}{dx} = i\lambda_k^2 \Lambda \mathbf{v}_k, \quad 1 \leq k \leq N. \quad (5.14)$$

Similarly, we can also obtain the correlation of  $\mathbf{v}_k$  with respect to time  $t$

$$\frac{d\mathbf{v}_k}{dt_{\bar{F}}} = 4i\lambda_k^{2\bar{F}} \Lambda \mathbf{v}_k, \quad 1 \leq k \leq N. \quad (5.15)$$

Then, combining (5.14) and (5.15) we can obtain the correlation of  $\mathbf{v}_k$  with respect to space  $x$  and time  $t_{\bar{F}}$

$$\mathbf{v}_k(x, t_{\bar{F}}) = e^{i\lambda_k^2 \Lambda x + 4i\lambda_k^{2\bar{F}} \Lambda t_{\bar{F}}} \mathbf{v}_{k_0}, \quad 1 \leq k \leq N. \quad (5.16)$$

Similarly, we can also obtain the correlation of  $\hat{\mathbf{v}}_k$  with respect to space and time

$$\hat{\mathbf{v}}_k(x, t_{\bar{F}}) = \hat{\mathbf{v}}_{k_0} e^{-i\hat{\lambda}_k^2 \Lambda x - 4i\hat{\lambda}_k^{2\bar{F}} \Lambda t_{\bar{F}}}, \quad 1 \leq k \leq N. \quad (5.17)$$

where  $\mathbf{v}_{k_0}$  and  $\hat{\mathbf{v}}_{k_0}$  are arbitrary constant column and row vectors, respectively.

Finally,  $J_+$  is expanded at  $\lambda$  as

$$J_+(x, \lambda) = I_{n+1} + \frac{J_+^1(x)}{\lambda} + \frac{J_+^2(x)}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right), \quad \lambda \rightarrow \infty, \quad (5.18)$$

Substituting the above formula into (4.8)-(4.9), it is easy to obtain

$$\tilde{U}_2 = -i[\Lambda, J_+^1],$$

where  $J_+^1 = ((J_+^1)_{jl})_{(n+1) \times (n+1)}$ . Equivalently,

$$q_j = 2ie^{-\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx'} (J_+^1)_{1, j+1}, \quad (5.19)$$

$$q_j^* = -2i(J_+^1)_{j+1, 1} e^{\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx'}. \quad (5.20)$$

Meanwhile, we can also obtain the involution property of  $J_+^1$ :

$$(J_+^1)^\dagger(-x, t) = iCJ_+^1(x, t)C^{-1}, \quad (5.21)$$

Thus,  $q_j$  and  $q_j^*$  ( $j = 1, 2, \dots, n$ ) can be used to determine the solution of the nonlocal reverse-space multi-component higher-order CLL Eq. (3.11). There is the following theorem

**Theorem 5.1.** *Given that the solution in (5.10)-(5.11) is associated with  $\lambda$ , then*

$$J_+^1 = - \sum_{k,l=1}^N (v_k(M^{-1})_{kl} \hat{v}_l - \sigma v_k(M^{-1})_{kl} \hat{v}_l \sigma), \quad (5.22)$$

substitute it into (5.19)-(5.20), solutions of the multi-component high-order CLL equation can be obtained

$$q_j = -2ie^{-\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx'} \sum_{k,l=1}^N (v_k(M^{-1})_{kl} \hat{v}_l - \sigma v_k(M^{-1})_{kl} \hat{v}_l \sigma)_{1, j+1}, \quad 1 \leq j \leq n, \quad (5.23)$$

where matrices  $M$  are defined as before (5.12), vectors  $v_k = (v_{k,1}, v_{k,2}, \dots, v_{k,n+1})^T$ ,  $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \dots, \hat{v}_{k,n+1})$ ,  $1 \leq k \leq N$ .

### 5.3. Nonlocal reverse-space multi-component higher-order Chen-Lee-Liu system N-soliton solutions

To obtain soliton solutions for the nonlocal reverse-space multi-component higher-order CLL Eq. (3.11), which we summarize as follows.

**Theorem 5.2.** *Assume that  $\det J_-(\lambda) = \hat{s}_{11}(\lambda)$  has  $2N$  zeros, note  $\hat{\lambda}_k = -i\lambda_k \in \Gamma_-$ ,  $1 \leq k \leq N$ . Then the elements of  $\ker J_-(\lambda)$  can be represented as*

$$\hat{v}_k(x, t_{\hat{r}}) = \hat{v}_k(x, t_{\hat{r}}, \hat{\lambda}_k) = v_k^\dagger(-x, t_{\hat{r}}, -i\hat{\lambda}_k)C = v_{k_0}^\dagger e^{-i\hat{\lambda}_k^2 \Lambda x - 4i\hat{\lambda}_k^{2p} \Lambda t_{\hat{r}}} C, \quad 1 \leq k \leq N, \quad (5.24)$$

where  $v_{k_0}^\dagger$  is an arbitrary constant sequence vector. And then, we have the  $N$ -soliton solution of the nonlocal reverse-space multi-component higher-order CLL equation

$$q_j = -2ie^{-\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx'} \sum_{k,l=1}^N (v_k(M^{-1})_{kl} \hat{v}_l - \sigma v_k(M^{-1})_{kl} \hat{v}_l \sigma)_{1, j+1}, \quad 1 \leq j \leq n. \quad (5.25)$$

where  $v_k = (v_{k,1}, v_{k,2}, \dots, v_{k,n+1})^T$  and  $\hat{v}_k = (\hat{v}_{k,1}, \hat{v}_{k,2}, \dots, \hat{v}_{k,n+1})$ ,  $1 \leq k \leq N$  are defined in (5.16) and (5.24), respectively.

**Proof.** Based on the elements (5.24) of  $\ker J_-(\lambda)$ , we can easily see that the solutions of the special RH problem determined by (5.10) and (5.11) satisfies (4.35), that is,

$$(J_+^\dagger)(-x, t, -i\lambda) = CJ_+(x, t, \lambda)C^{-1}, \quad (5.26)$$

This indicates that  $J_+^1$  satisfies the involution property (5.21). Combined with Theorem 5.1, the  $N$ -soliton solution (5.25) of the nonlocal reverse-space multi-component higher-order CLL equation can be obtained.  $\square$

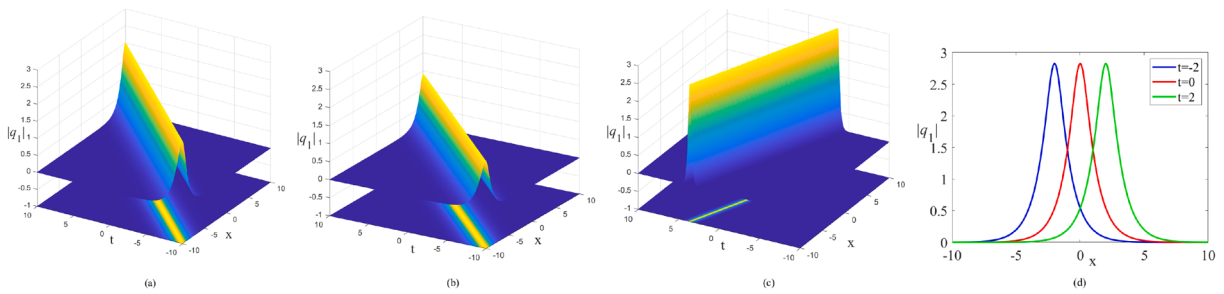
Finally, we present one and two-soliton solutions for the nonlocal reverse-space multi-component higher-order CLL equation, and analyze the properties and characteristics of their dynamics through plotting graphs.

**One-soliton solution** To obtain one-soliton solution, we set  $N = 1, r = 2$ . For simplicity, we let  $\theta_k = i\lambda_k^2 x + 4i\lambda_k^6 t$  and  $\hat{\theta}_k = -i\hat{\lambda}_k^2 x - 4i\hat{\lambda}_k^6 t$ , denote  $v_{1_0} = (c_{1,1}, c_{1,2}, \dots, c_{1,n+1})^T$  and  $v_{1_0}^\dagger = (c_{1,1}^*, c_{1,2}^*, \dots, c_{1,n+1}^*)$ . Through direct calculation, the explicit expression of one-soliton solution can be derived

$$q_j = \frac{-2i(\lambda_1^2 - \hat{\lambda}_1^2)c_{1,1}c_{1,j+1}^* e^{\hat{\theta}_1 - \theta_1}}{\hat{\lambda}_1 |c_{1,1}|^2 e^{-(\hat{\theta}_1 + \theta_1)} + \lambda_1 \sum_{m=1}^n |c_{1,m+1}|^2 e^{\hat{\theta}_1 + \theta_1}} \exp\left(-\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx'\right) \quad 1 \leq j \leq n. \quad (5.27)$$

Letting

$$\lambda_1 = \xi_1 + i\eta_1, \quad c_{1,1} = \sqrt{n}e^{-2\eta_1 x_0 + i\sigma_0}, \quad c_{1,m+1} = 1, \quad 1 \leq m \leq n,$$



**Fig. 2.** One-soliton solutions  $|q_1|$ : (a)  $n = 1, \lambda_1 = 0.5 - 0.5i, c_{1,1} = c_{1,2} = 1$ ; (b)  $n = 2, \lambda_1 = 0.5 - 0.5i, c_{1,1} = c_{1,2} = c_{1,3} = 1$ ; (c)  $n = 2, \lambda_1 = 1 - 0.5i, c_{1,1} = c_{1,2} = c_{1,3} = 1$ ; (d) when  $t = -2, 0, 2$  (under the condition of (c)).

we can rewrite (5.27) as

$$q_j = \frac{8\xi_1\eta_1 \exp(-2i(\xi_1^2 - \eta_1^2)x - 8i\tau_1 t + i\sigma_0)}{\sqrt{n}(\xi_1 \cosh \tau_2 + i\eta_1 \sinh \tau_2)} \exp\left(-\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx'\right), \quad (5.28)$$

where  $x_0$  and  $\sigma_0$  are real parameters,

$$\begin{aligned} \tau_1 &= \xi_1^6 + \eta_1^6 - 9\xi_1^2\eta_1^4 + 9\xi_1^4\eta_1^2 \\ \tau_2 &= -4\xi_1\eta_1 x - 16\xi_1\eta_1(3\xi_1^4 + 3\eta_1^4 - 10\xi_1^2\eta_1^2)t + 8\eta_1 x_0. \end{aligned}$$

This solution is a solitary wave in the multi-component CLL equation. Its amplitude function  $|q_j|$  has the shape of a hyperbolic secant with peak amplitude  $\frac{8\xi_1\eta_1}{\sqrt{n}}$ , and its velocity is  $-16\xi_1\eta_1(3\xi_1^4 + 3\eta_1^4 - 10\xi_1^2\eta_1^2)$ . The phase of this solution depends linearly on both space  $x$  and time  $t$ . The spatial gradient of the phase is proportional to the speed of the wave. Parameters  $x_0$  and  $\sigma_0$  are the initial location and phase of this solitary wave. We present the one-soliton solutions for  $n = 1$  and  $n = 2$  in Fig. 2. We can see that the amplitude of the solitary wave decreases as  $n$  increases.

**Two-soliton solution** When  $N = 2$ , the solution (5.25) is

$$q_j = 4i \frac{\det \hat{M}}{\det M} \exp\left(-\frac{1}{4}i \int_{-\infty}^x |q|^2(x', t) dx'\right), \quad 1 \leq j \leq n, \quad (5.29)$$

where

$$\hat{M} = \begin{bmatrix} 0 & c_{1,1}e^{-\theta_1} & c_{2,1}e^{-\theta_2} \\ c_{1,j+1}^*e^{\theta_1^*} & m_{11} & m_{12} \\ c_{2,j+1}^*e^{\theta_2^*} & m_{21} & m_{22} \end{bmatrix},$$

and

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix},$$

with

$$m_{kl} = \frac{2}{\lambda_l^2 - \hat{\lambda}_k^2} \left( \lambda_l \sum_{m=1}^n (c_{k,m+1}^* c_{l,m+1}) e^{\theta_k^* + \theta_l} + \hat{\lambda}_k c_{k,1}^* c_{l,1} e^{-(\theta_k^* + \theta_l)} \right), \quad k, l = 1, 2.$$

According to the factor of the exponential function in  $M$ , we can know that there are two different states of the two-soliton solution. One is  $3(\xi_1^2 - \eta_1^2)^2 - 4\xi_1^2\eta_1^2 \neq 3(\xi_2^2 - \eta_2^2)^2 - 4\xi_2^2\eta_2^2$ , the other one is  $3(\xi_1^2 - \eta_1^2)^2 - 4\xi_1^2\eta_1^2 = 3(\xi_2^2 - \eta_2^2)^2 - 4\xi_2^2\eta_2^2$ . Without loss of generality, we take

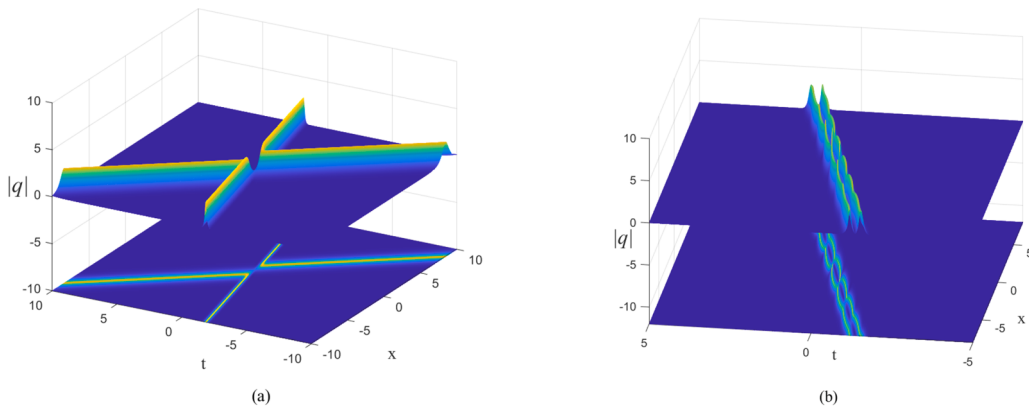
$$\lambda_1 = 1 + \frac{\sqrt{2}}{2}i, \quad \lambda_2 = 1 - \frac{\sqrt{3100}}{100}i, \quad c_{1,1} = c_{2,1} = c_{1,2} = c_{2,2} = 1,$$

which makes the two-soliton solution elastically collide as shown in Fig. 3 (a). Similarly, when we take

$$\lambda_1 = 1 + \sqrt{\frac{5}{3} - \frac{\sqrt{46}}{6}}i, \quad \lambda_2 = 1.1 + \sqrt{\frac{121}{60} - \frac{\sqrt{47314}}{150}}i, \quad c_{1,1} = c_{2,1} = c_{1,2} = c_{2,2} = 1,$$

the two-soliton solution appears in the bound state shown in Fig. 3 (b). Note that we only provide the figures for  $|q|$  with  $n = 1$ .

In this case, the two right-travelling solitons pass through each other and keep each amplitude and velocity. In the first case, Fig. 3 (a) describes an analytic regular two-soliton interaction, which are far apart and moving toward each other. However, these solitons reappear from the interaction without any change of shape and velocity as  $t \rightarrow \infty$ , and there is no energy radiation emitted to the far field. Thus the interaction of these solitons is elastic. This elastic interaction is a remarkable property which signals that the multi-component higher-order CLL system is integrable. There is still some trace of the interaction however. Indeed, after the interaction, each soliton acquires a position shift and a phase shift. The position of each soliton is always shifted forward, as if the soliton accelerates during interactions.



**Fig. 3.** Two-soliton solutions  $|q|$  : **(a)** collision, where  $\text{Re}(\lambda_1^4) + 2 \text{Re}^2(\lambda_1^2) \neq \text{Re}(\lambda_2^4) + 2 \text{Re}^2(\lambda_2^2)$ ; **(b)** bound state, where  $\text{Re}(\lambda_1^4) + 2 \text{Re}^2(\lambda_1^2) = \text{Re}(\lambda_2^4) + 2 \text{Re}^2(\lambda_2^2)$ .

## 6. Conclusions

In this paper, we mainly focused on the nonlocal multi-component reverse-space CLL equation with the zero boundary condition, and the  $N$ -soliton solutions to the equation have been derived. By formulating a specific Riemann-Hilbert problem from the corresponding spectral problem, we show that the  $N$ -soliton solutions can be expressed via the solution to this Riemann-Hilbert problem under reflectionless conditions. Compared with the local case, the nonlocal reverse-space CLL equation exhibits distinct symmetry relations, leading to independent scattering data  $\lambda_k$  and  $\hat{\lambda}_k$ . This independence gives rise to novel types of  $N$ -soliton solutions, as different nonlocal reductions generate varying symmetry structures. Compared with the zero boundary condition, the Riemann-Hilbert problem with non-zero boundary becomes more complex, especially for multi-component systems. The main reason is that the spectral distribution becomes more complex under the non-zero boundary condition, and the forms of solutions corresponding to spectral points become more diversified. The periodic wave solutions are constructed when the Riemann-Hilbert problem is non-regular. Conversely, when the Riemann-Hilbert problem is regular, we can analyze the asymptotic of the solutions. At present, some authors have studied the long-time asymptotic behavior for the nonlocal equations. Following this idea, the long-time asymptotic of the nonlocal multi-component higher-order CLL equation will be studied in future work.

## CRedit authorship contribution statement

**Liyuan Ding:** Writing – original draft, Software, Methodology, Formal analysis, Data curation; **Ming Duan:** Writing – review & editing, Conceptualization; **Wen-Xiu Ma:** Writing – review & editing, Visualization, Methodology.

## Data availability

No data was used for the research described in the article.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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