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基于李代数 $\mathrm{sl}(m+1,\mathbb{R})$ 的多分量扰动 AKNS 孤子梯队 *

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摘要：基于李代数 $\mathrm{sl}(m+1,\mathbb{R})$, 提出了一个新的多分量矩阵谱问题，进而利用零曲率公式构造了新的多分量扰动 AKNS 孤子梯队．利用迹恒等式构造了双哈密顿结构，同时给出了遗传递推算子.

关键词：矩阵谱问题；扰动 AKNS 孤子梯队；双哈密顿结构；递推算子； $\mathrm{sl}(m+1,\mathbb{R})$.

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1 引言

从矩阵谱问题或 Lax 对出发, 零曲率公式提供了一个构造孤子梯队的有效方法. 成功的例子包括 Ablowitz-Kaup-Newell-Segur (AKNS) 梯队，Kaup-Newell (KN) 梯队，Wadati-Konno-Ichikawa (WKI) 梯队， Korteweg-de Vries (KdV) 梯队， Dirac 梯队和 Boiti-Pempinelli-Tu (BPT) 梯队等 [1−38]. 基于矩阵谱问题，孤子梯队通常具有双哈密顿结构从而是 Liouville 可积的．特别是当矩阵圈代数是半单的时候，相应的双哈密顿结构能够利用迹恒等式得到，参见文献 $[9,11,18-20,22,27-29]$.

经典的 AKNS 关于空间变量的谱问题是

$$\phi_x=(\lambda e_1+qe_2+pe_3)\phi=\begin{pmatrix}-\lambda & q\\ p & \lambda\end{pmatrix}\phi, \tag{1.1}$$

这里 $e_1=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, $e_2=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $e_3=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ 是李代数 $\mathrm{sl}(2,\mathbb{R})$ 的基．文献 [6, 13] 的作者引入非线性项 αpq, 提出了如下的两分量扰动 AKNS 关于空间变量的谱问题

$$\phi_x=((\lambda+\alpha pq)e_1+qe_2+pe_3)\phi=\begin{pmatrix}-\lambda-\alpha pq & q\\ p & \lambda+\alpha pq\end{pmatrix}\phi, \tag{1.2}$$

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并构造了双哈密顿结构．我们将把上述谱问题 (1.2) 推广到多分量的情形，即构造了相应的多分量孤子梯队和双哈密顿结构．第二节中，我们考虑如下的基于李代数 sl$(m+1,\mathbb{R})$ 的多分量扰动 AKNS 关于空间变量的谱问题

$$\phi_x = U(u,\lambda)\phi = \begin{pmatrix} -m(\lambda+\alpha r) & q \\ p & (\lambda+\alpha r)I_m \end{pmatrix}\phi, \quad u = \begin{pmatrix} q^T \\ p \end{pmatrix}, \quad r = qp, \tag{1.3}$$

这里 $q=(q_1,q_2,\cdots,q_m)$, $p=(p_1,p_2,\cdots,p_m)^T$, λ 是谱参数， I_m 是 $m\times m$ 单位矩阵．当 $\alpha=0$ 时，相应的谱问题已经在文献 [12, 15, 20, 28, 34] 中得到解决．特别地， $m=2$ 的情形约化为四分量的 AKNS 谱问题 [16]．进而当 $m=1$ 时该谱问题约化为经典的 AKNS 情形．第三节将给出一些总结．

2 新的多分量扰动 AKNS 孤子梯队

首先我们求解稳态零曲率方程

$$W_x = [U,W]. \tag{2.1}$$

我们设 W 为

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.2}$$

这里 a 是标量， b^T 和 c 是 m- 维列向量， d 是 $m\times m$ 矩阵．从而零曲率方程 (2.1 等价于

$$\begin{cases} a_x = qc - bp, \\ d_x = pb - cq, \\ b_x = qd - aq - (m+1)(\lambda+\alpha r)b, \\ c_x = pa - dp + (m+1)(\lambda+\alpha r)c. \end{cases} \tag{2.3}$$

所以我们可以得到

$$\mathrm{tr}(d)_x = \mathrm{tr}(d_x) = \mathrm{tr}(pb - cq) = bp - qc = -a_x.$$

接下来我们构造 W 的形式解

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{k=0}^{\infty} W_k \lambda^{-k} = \sum_{k=0}^{\infty} \begin{pmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)} \end{pmatrix}\lambda^{-k}, \tag{2.4}$$

这里 $b^{(k)}, c^{(k)}$ 和 $d^{(k)}$ 设为

$$b^{(k)} = (b_1^{(k)}, b_2^{(k)}, \cdots, b_m^{(k)}), \quad c^{(k)} = (c_1^{(k)}, c_2^{(k)}, \cdots, c_m^{(k)})^T, \quad d^{(k)} = (d_{ij}^{(k)})_{m\times m}. \tag{2.5}$$

在上述假设下，我们有如下递推关系式

$$\begin{cases} b^{(0)} = 0, \quad c^{(0)} = 0, \quad a_x^{(0)} = 0, \quad d_x^{(0)} = 0, \\ a_x^{(k)} = qc^{(k)} - b^{(k)}p, \ d_x^{(k)} = pb^{(k)} - c^{(k)}q, \\ b^{(k+1)} = \dfrac{1}{m+1}[qd^{(k)} - a^{(k)}q - b_x^{(k)} - (m+1)\alpha r b^{(k)}], \\ c^{(k+1)} = \dfrac{1}{m+1}[c_x^{(k)} - pa^{(k)} + d^{(k)}p - (m+1)\alpha r c^{(k)}], \quad k \geq 0. \end{cases} \tag{2.6}$$

从而对于 $b^{(k)}$ 和 $c^{(k)}$ 有

$$\begin{pmatrix} c^{(k+1)} \\ {b^{(k+1)}}^T \end{pmatrix} = L \begin{pmatrix} c^{(k)} \\ {b^{(k)}}^T \end{pmatrix}, \qquad k \geq 0, \tag{2.7}$$

这里算子 L 可写为

$$L = \frac{1}{m+1}\begin{pmatrix} \left[\partial - \beta(m+1)r - \sum_{i=1}^{m} p_i\partial^{-1}q_i\right] I_m & p\partial^{-1}p^T + (p\partial^{-1}p^T)^T \\ -p\partial^{-1}q & \\ -q^T\partial^{-1}q - (q^T\partial^{-1}q)^T & \left[-\partial - \beta(m+1)r + \sum_{i=1}^{m} q_i\partial^{-1}p_i\right] I_m \\ & +q^T\partial^{-1}p^T \end{pmatrix}. \tag{2.8}$$

设初值为

$$a^{(0)} = -m, \qquad d^{(0)} = I_m, \tag{2.9}$$

并且令

$$W_k|_{u=0} = 0, \qquad k \geq 1, \tag{2.10}$$

则我们可以一致地确定序列 $\{a^{(k)}, b^{(k)}, c^{(k)}, d^{(k)} |\ k \geq 1\}$ 的表达式．例如前三组为

$$\begin{cases} a^{(1)} = 0, \quad b^{(1)} = q, \quad c^{(1)} = p, \quad d^{(1)} = (0)_{m\times m}; \\ a^{(2)} = \dfrac{r}{m+1},\ b^{(2)} = \dfrac{1}{m+1}[-q_x - (m+1)\alpha rq], \\ c^{(2)} = \dfrac{1}{m+1}[p_x - (m+1)\alpha rp],\ d^{(2)} = \dfrac{1}{m+1}(-pq); \\ a^{(3)} = \dfrac{1}{(m+1)^2}[qp_x - q_xp - 2(m+1)\alpha r^2], \\ d^{(3)} = \dfrac{1}{(m+1)^2}[pq_x - p_xq + 2(m+1)\alpha rpq], \\ b^{(3)} = \dfrac{1}{(m+1)^2}\left\{q_{xx} - 2rq + (m+1)\beta[r_xq + 2rq_x + (m+1)\alpha r^2q]\right\}, \\ c^{(3)} = \dfrac{1}{(m+1)^2}\left\{p_{xx} - 2pr - (m+1)\beta[r_xp + 2rp_x - (m+1)\alpha r^2p]\right\}. \end{cases}$$

显然，(2.6), (2.9) 和 (2.10) 式保证了 $W_k \in \mathrm{sl}(m+1, \mathbb{R}), \quad k \geq 0$.

更进一步，我们记关于时间变量的矩阵谱问题为

$$\begin{aligned} &\phi_{t_n} = V^{(n)}\phi, \\ &V^{(n)} = (\lambda^n W)_+ + \Delta_n = \sum_{j=0}^{n} W_j\lambda^{n-j} + \begin{pmatrix} e_n & 0 \\ 0 & h_n \end{pmatrix}, \qquad n \geq 0, \end{aligned} \tag{2.11}$$

这里 P_+ 定义为 λ 的多项式部分．(1.3) 和 (2.11) 式的相容性条件，即零曲率方程

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \qquad n \geq 0, \tag{2.12}$$

给出了

$$\begin{cases} e_{n_x} = -m\alpha r_{t_n}, \qquad h_{n_x} = \alpha r_{t_n} I_m, \\ q_{t_n} = -(m+1)b^{(n+1)} - qh_n + e_n q, \qquad n \geq 0. \\ p_{t_n} = (m+1)c^{(n+1)} - pe_n + h_n p, \end{cases} \tag{2.13}$$

取 $e_n = -mf_n$ 和 $h_n = f_n I_m$, 我们有

$$\begin{cases} f_{n_x} = \alpha r_{t_n} = \beta(m+1)(qc^{(n+1)} - b^{(n+1)}p), \\ q_{t_n} = -(m+1)(b^{(n+1)} + f_n q), \\ p_{t_n} = (m+1)(c^{(n+1)} + f_n p), \qquad n \geq 0. \end{cases} \tag{2.14}$$

从而解第一个方程可以得到 $f_n = \alpha(m+1)a^{(n+1)}$, 即有

$$\Delta_n = \alpha(m+1)a^{(n+1)} \begin{pmatrix} -m & 0 \\ 0 & I_m \end{pmatrix}, \qquad n \geq 0. \tag{2.15}$$

这样我们就得到了如下的多分量扰动 AKNS 孤子梯队

$$\begin{cases} q_{t_n} = -(m+1)[b^{(n+1)} + \alpha(m+1)\partial^{-1}(qc^{(n+1)} - b^{(n+1)}p)q], \\ p_{t_n} = (m+1)[c^{(n+1)} + \alpha(m+1)\partial^{-1}(qc^{(n+1)} - b^{(n+1)}p)p], \end{cases} \qquad n \geq 0. \tag{2.16}$$

(2.16) 式写成算子方程的形式有

$$u_{t_n} = \begin{pmatrix} q^T \\ p \end{pmatrix}_{t_n} = K_n = R \begin{pmatrix} c^{(n+1)} \\ b^{(n+1)^T} \end{pmatrix}, \qquad n \geq 0, \tag{2.17}$$

这里算子 R 定义为

$$R = (m+1) \begin{pmatrix} -\alpha(m+1)q^T\partial^{-1}q & -I_m + \alpha(m+1)q^T\partial^{-1}p^T \\ I_m + \alpha(m+1)p\partial^{-1}q & -\alpha(m+1)p\partial^{-1}p^T \end{pmatrix}. \tag{2.18}$$

在该孤子梯队 (2.17) 中，第一个非平凡的非线性系统为

$$\begin{cases} p_{t_2} = \dfrac{1}{m+1}(p_{xx} - 2rp) - \alpha[2(pq)_x p + (m+1)\alpha r^2 p], \\ q_{t_2} = -\dfrac{1}{m+1}(q_{xx} - 2rq) - \alpha[2q(pq)_x - (m+1)\alpha r^2 q]. \end{cases}$$

当 $m=2,\ \alpha=0$ 时，上述系统退化为著名的非线性 Burgers 系统. 类似地， mKdV 系统可以从 (2.16) 式的第二个非线性系统中得到.

最后，我们利用如下的迹恒等式 [9,11,27,29]

$$\frac{\delta}{\delta u}\int \mathrm{tr}\Big(W\frac{\partial U}{\partial \lambda}\Big)\mathrm{d}x = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\Big[\lambda^{\gamma}\mathrm{tr}\Big(W\frac{\partial U}{\partial u}\Big)\Big]$$

来构造孤子梯队 (2.17) 的双哈密顿结构.

直接计算有

$$\begin{cases} \operatorname{tr}\Big(W\dfrac{\partial U}{\partial \lambda}\Big)=-ma+\operatorname{tr}(d), \\ \operatorname{tr}\Big(W\dfrac{\partial U}{\partial u}\Big)=-\alpha ma\begin{pmatrix} p \\ q^T \end{pmatrix}+\begin{pmatrix} c \\ b^T \end{pmatrix}+\alpha\operatorname{tr}(d)\begin{pmatrix} p \\ q^T \end{pmatrix}. \end{cases} \tag{2.19}$$

把 (2.19) 式代入上述迹恒等式，平衡 λ 的次数，我们有

$$\begin{aligned} &\frac{\delta}{\delta u}\int\Big(-ma^{(k+1)}+\sum_{i=1}^{m}d_{ii}^{(k+1)}\Big)\mathrm{d}x \\ &=(\gamma-k)\Big[-\alpha m\begin{pmatrix} p \\ q^T \end{pmatrix}a^{(k)}+\begin{pmatrix} c^{(k)} \\ {b^{(k)}}^T \end{pmatrix}+\alpha\begin{pmatrix} p \\ q^T \end{pmatrix}\sum_{i=1}^{m}d_{ii}^{(k)}\Big]. \end{aligned}$$

为了确定常数 γ 的值，我们在上式中令 $k=1$ 即可得到 $\gamma=0$. 从而我们有

$$\frac{\delta}{\delta u}\mathcal{H}_k=\Big[-\alpha m\begin{pmatrix} p \\ q^T \end{pmatrix}a^{(k)}+\begin{pmatrix} c^{(k)} \\ {b^{(k)}}^T \end{pmatrix}+\alpha\begin{pmatrix} p \\ q^T \end{pmatrix}\sum_{i=1}^{m}d_{ii}^{(k)}\Big],\qquad k\geq 0. \tag{2.20}$$

这里的哈密顿泛函 $\mathcal{H}_k$ 定义为

$$\begin{aligned} &\mathcal{H}_0=\alpha m(m+1)\int r\mathrm{d}x, \\ &\mathcal{H}_k=\frac{\delta}{\delta u}\int\frac{\Big(ma^{(k+1)}-\sum\limits_{i=1}^{m}d_{ii}^{(k+1)}\Big)}{k}\mathrm{d}x,\qquad k\geq 1. \end{aligned} \tag{2.21}$$

为了给出哈密顿结构和哈密顿算子，我们记

$$G^{(k)}=\Big[-\alpha m\begin{pmatrix} p \\ q^T \end{pmatrix}a^{(k)}+\begin{pmatrix} c^{(k)} \\ {b^{(k)}}^T \end{pmatrix}+\alpha\begin{pmatrix} p \\ q^T \end{pmatrix}\sum_{i=1}^{m}d_{ii}^{(k)}\Big],\qquad k\geq 0, \tag{2.22}$$

则有

$$\begin{aligned} G^{(k)}&=-\alpha m\begin{pmatrix} p \\ q^T \end{pmatrix}a^{(k)}+\begin{pmatrix} c^{(k)} \\ {b^{(k)}}^T \end{pmatrix}+\alpha\begin{pmatrix} p \\ q^T \end{pmatrix}\partial^{-1}(b^{(k)}p-qc^{(k)}) \\ &=N^{-1}\begin{pmatrix} c^{(k)} \\ {b^{(k)}}^T \end{pmatrix},\qquad k\geq 0, \end{aligned} \tag{2.23}$$

这里算子 N^{-1} 是

$$N^{-1}=\begin{pmatrix} I_m-\alpha(m+1)p\partial^{-1}q & \alpha(m+1)p\partial^{-1}p^T \\ -\alpha(m+1)q^T\partial^{-1}q & I_m+\alpha(m+1)q^T\partial^{-1}p^T \end{pmatrix}. \tag{2.24}$$

从而我们有

$$
\begin{aligned}
&\begin{pmatrix} c^{(k)} \\ {b^{(k)}}^T \end{pmatrix} = NG^{(k)}, \\
&N = \begin{pmatrix} I_m + \alpha(m+1)p\partial^{-1}q & -\alpha(m+1)p\partial^{-1}p^T \\ \alpha(m+1)q^T\partial^{-1}q & I_m - \alpha(m+1)q^T\partial^{-1}p^T \end{pmatrix}, \qquad k \geq 0.
\end{aligned} \tag{2.25}
$$

这样我们已经可以得到孤子梯队 (2.17) 如下的哈密顿结构

$$
u_{t_n} = K_n = J\frac{\delta \mathcal{H}_{n+1}}{\delta u}. \tag{2.26}
$$

这里的哈密顿算子 J 定义为

$$
J = RN = (m+1)\begin{pmatrix} -2\alpha(m+1)q^T\partial^{-1}q & -I_m + 2\alpha(m+1)q^T\partial^{-1}p^T \\ I_m + 2\alpha(m+1)p\partial^{-1}q & -2\alpha(m+1)p\partial^{-1}p^T \end{pmatrix}. \tag{2.27}
$$

从关系式 $G^{(n+1)} = \Psi G^{(n)}$, $\Psi = N^{-1}LN$, $K_{n+1} = \Phi K_n$, $n \geq 0$ 和 $J\Psi = \Phi J$, 我们可以得到孤子梯队 (2.17) 的遗传递推算子

$$
\Phi = \Psi^{\dagger} = N^{\dagger}L^{\dagger}N^{-\dagger}, \tag{2.28}
$$

这里算子 $\Psi^{\dagger}$ 定义为 Ψ 的共轭算子. 从而 $\Phi = (\Phi_{ij})_{2\times 2}$ 的显式表达式可以算出

$$
\left\{
\begin{aligned}
\Phi_{11} = {} & \alpha^2(m+1)q^T\partial^{-1}q\partial p\partial^{-1}p^T \\
& +\alpha q^T\partial^{-1}p^T\{[\partial + \alpha(m+1)r]I_m + \alpha(m+1)\partial q^T\partial^{-1}p^T\} \\
& +\frac{1}{m+1}\Big\{q^T\partial^{-1}p^T - \Big[\partial + \alpha(m+1)r - \sum_{i=1}^{m} q_i\partial^{-1}p_i\Big]I_m \\
& -\alpha(m+1)[\partial + \alpha(m+1)r]q^T\partial^{-1}p^T\Big\}, \\
\Phi_{12} = {} & \frac{1}{m+1}\{q^T\partial^{-1}q + (q^T\partial^{-1}q)^T - \alpha(m+1)[\partial + \alpha(m+1)r]q^T\partial^{-1}q\} \\
& +\alpha^2(m+1)q^T\partial^{-1}p^T\partial q^T\partial^{-1}q \\
& -\alpha q^T\partial^{-1}q\{[\partial - \alpha(m+1)r]I_m - \alpha(m+1)\partial p\partial^{-1}q\}, \\
\Phi_{21} = {} & -\frac{1}{m+1}\{p\partial^{-1}p^T + (p\partial^{-1}p^T)^T + \alpha(m+1)[\partial - \alpha(m+1)r]p\partial^{-1}p^T\} \\
& -\alpha^2(m+1)p\partial^{-1}q\partial p\partial^{-1}p^T \\
& -\alpha p\partial^{-1}p^T\{[\partial + \alpha(m+1)r]I_m + \alpha(m+1)\partial q^T\partial^{-1}p^T\}, \\
\Phi_{22} = {} & -\alpha^2(m+1)p\partial^{-1}p^T\partial q^T\partial^{-1}q \\
& +\alpha p\partial^{-1}q\{[\partial - \alpha(m+1)r]I_m - \alpha(m+1)\partial p\partial^{-1}q\} \\
& +\frac{1}{m+1}\Big\{-p\partial^{-1}q + \Big[\partial - \alpha(m+1)r - \sum_{i=1}^{m} p_i\partial^{-1}q_i\Big]I_m \\
& -\alpha(m+1)[\partial - \alpha(m+1)r]p\partial^{-1}q\Big\}.
\end{aligned}
\right. \tag{2.29}
$$

所以我们得到了梯队 (2.17) 的双哈密顿结构

$$u_{t_n}=K_n=J\frac{\delta\mathcal{H}_{n+1}}{\delta u}=M\frac{\delta\mathcal{H}_n}{\delta u},\qquad n\geq 0. \tag{2.30}$$

这里第二个哈密顿算子 $M=(M_{ij})_{2\times 2}$ 由式子 $M=\Phi J$ 给出，其显式表达式为

$$\begin{cases}
M_{11}=\alpha(m+1)\left[(\partial+\alpha(m+1)r)I_m-\alpha(m+1)q^T\partial^{-1}p^T\partial\right]q^T\partial^{-1}q\\
\qquad\quad+q^T\partial^{-1}q+(q^T\partial^{-1}q)^T\\
\qquad\quad-\alpha(m+1)q^T\partial^{-1}q[(\partial-\alpha(m+1)r)I_m+\alpha(m+1)\partial p\partial^{-1}q],\\
M_{12}=-q^T\partial^{-1}p^T+\left[\partial+\alpha(m+1)r-\sum\limits_{i=1}^{m}q_i\partial^{-1}p_i\right]I_m\\
\qquad\quad-\alpha(m+1)[\partial+\alpha(m+1)r]q^T\partial^{-1}p^T+\alpha^2(m+1)^2q^T\partial^{-1}q\partial p\partial^{-1}p^T\\
\qquad\quad-\alpha(m+1)q^T\partial^{-1}p^T[(\partial+\alpha(m+1)r)I_m+\alpha(m+1)\partial q^T\partial^{-1}p^T],\\
M_{21}=-p\partial^{-1}q+\left[\partial-\alpha(m+1)r-\sum\limits_{i=1}^{m}p_i\partial^{-1}q_i\right]I_m\\
\qquad\quad+\alpha(m+1)[\partial-\alpha(m+1)r]p\partial^{-1}q+\alpha^2(m+1)^2p\partial^{-1}p^T\partial q^T\partial^{-1}q\\
\qquad\quad+\alpha(m+1)p\partial^{-1}q[(\partial-\alpha(m+1)r)I_m+\alpha(m+1)\partial p\partial^{-1}q],\\
M_{22}=-\alpha(m+1)\left[(\partial-\alpha(m+1)r)I_m+\alpha(m+1)p\partial^{-1}q\partial\right]p\partial^{-1}p^T\\
\qquad\quad+p\partial^{-1}p^T+(p\partial^{-1}p^T)^T\\
\qquad\quad+\alpha(m+1)p\partial^{-1}p^T[(\partial+\alpha(m+1)r)I_m-\alpha(m+1)\partial q^T\partial^{-1}p^T].
\end{cases} \tag{2.31}$$

双哈密顿结构意味着该多分量扰动 AKNS 孤子梯队 (2.17) 是 Liouville 可积的 [36]. 另外守恒泛函 $\{\mathcal{H}_k\}_{k=0}^{\infty}$ 和对称 $\{K_k\}_{k=0}^{\infty}$ 有关系式

$$\begin{cases}
\{\mathcal{H}_k,\mathcal{H}_l\}_J=\displaystyle\int\left(\frac{\delta\mathcal{H}_k}{\delta u}\right)^TJ\frac{\delta\mathcal{H}_l}{\delta u}\mathrm{d}x=0,\\
\{\mathcal{H}_k,\mathcal{H}_l\}_M=\displaystyle\int\left(\frac{\delta\mathcal{H}_k}{\delta u}\right)^TM\frac{\delta\mathcal{H}_l}{\delta u}\mathrm{d}x=0,
\end{cases}\qquad k,l\geq 0 \tag{2.32}$$

和

$$[K_k,K_l]=K_k'(u)[K_l]-K_l'(u)[K_k]=0,\qquad k,l\geq 0. \tag{2.33}$$

3 总结

通过引入一个非线性项 $r=qp=\sum\limits_{i=1}^{m}q_ip_i$, 我们基于李代数 $\mathrm{sl}(m+1,\mathbb{R})$ 提出了一个新的矩阵谱问题. 利用零曲率公式，我们构造了相应的多分量扰动 AKNS 孤子梯队. 最后，我们利用迹恒等式建立了双哈密顿结构. 在附录中我们列出了一些新的恒等式.

从如下的关于空间变量的矩阵谱问题

$$\phi_x=U(u,\lambda)\phi=\begin{pmatrix}0&-q&-m(\lambda+\alpha r)\\ q^T&0&-p\\ m(\lambda+\alpha r)&p^T&0\end{pmatrix}\phi,\quad u=\begin{pmatrix}q^T\\ p\end{pmatrix},\quad r=qq^T+p^Tp$$

和 $W=\begin{pmatrix} 0 & -c & -a \\ c^T & d & -b \\ a & b^T & 0 \end{pmatrix}$ 出发，我们也可以构造基于李代数 so$(m+2,\mathbb{R})$ 的多分量扰动 AKNS 孤子梯队．方便起见，我们省略了类似的计算过程.

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附录

设 $b^{(k)}$ 和 $q=(q_1,\cdots,q_m)$ 是 m - 维行向量， $c^{(k)}$ 和 $p=(p_1,\cdots,p_m)^T$ 是 m - 维列向量，容易证明下列恒等式成立.

$$\partial^{-1}(c^{(k)}q)p=\Big(\sum_{j=1}^{m}p_j\partial^{-1}q_j\Big)(c^{(k)}),\quad p\partial^{-1}(b^{(k)}p)=p\partial^{-1}(p^Tb^{(k)^T}),$$

$$\partial^{-1}(pb^{(k)})p=(p\partial^{-1}p^T)^T(b^{(k)^T}),\quad \partial^{-1}(b^{(k)^T}p^T)q^T=\Big(\sum_{j=1}^{m}q_j\partial^{-1}p_j\Big)(b^{(k)^T}),$$

$$q^T\partial^{-1}(c^{(k)^T}q^T)=q^T\partial^{-1}(qc^{(k)}),\quad \partial^{-1}(q^Tc^{(k)^T})q^T=(q^T\partial^{-1}q)^T(c^{(k)}),$$

$$\Big(\sum_{j=1}^{m}q_j\partial^{-1}p_j\Big)q^T\partial^{-1}p^T=(q^T\partial^{-1}q)^Tp\partial^{-1}p^T,$$

$$\Big(\sum_{j=1}^{m}q_j\partial^{-1}p_j\Big)q^T\partial^{-1}q=(q^T\partial^{-1}q)^Tp\partial^{-1}q,$$

$$\Big(\sum_{j=1}^{m} p_j\partial^{-1}q_j\Big)p\partial^{-1}p^T = (p\partial^{-1}p^T)^T q^T\partial^{-1}p^T,$$

$$\Big(\sum_{j=1}^{m} p_j\partial^{-1}q_j\Big)p\partial^{-1}q = (p\partial^{-1}p^T)^T q^T\partial^{-1}q,$$

$$q^T\partial^{-1}p^T\Big(\sum_{j=1}^{m} q_j\partial^{-1}p_j\Big) = q^T\partial^{-1}q(p\partial^{-1}p^T)^T,$$

$$q^T\partial^{-1}q\Big(\sum_{j=1}^{m} p_j\partial^{-1}q_j\Big) = q^T\partial^{-1}p^T(q^T\partial^{-1}q)^T,$$

$$p\partial^{-1}p^T\Big(\sum_{j=1}^{m} q_j\partial^{-1}p_j\Big) = p\partial^{-1}q(p\partial^{-1}p^T)^T,$$

$$p\partial^{-1}q\Big(\sum_{j=1}^{m} p_j\partial^{-1}q_j\Big) = p\partial^{-1}p^T(q^T\partial^{-1}q)^T.$$

Multicomponent Perturbed AKNS Soliton Hierarchy Associated with the Lie Algebra sl$(m+1,\mathbb{R})$

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Abstract: A new multicomponent matrix spectral problem associated with the Lie algebra sl$(m+1,\mathbb{R})$ is proposed, and new perturbed AKNS soliton hierarchy is generated by the corresponding zero curvature formulation. By virtue of the trace identity, a bi-Hamiltonian structure is established for the hierarchy, and a common hereditary recursion operator is explicitly worked out.

Key words: Matrix spectral problem; Perturbed AKNS soliton hierarchy; Bi-Hamiltonian structure; Recursion operator; sl$(m+1,\mathbb{R})$.
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