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ABSTRACT

The main topic of the paper is to investigate the generalized $(2+1)$-dimensional Date–Jimbo–Kashiwara–Miwa (DJKM) and Korteweg–de Vries (KdV) equations, which are widely used in many physical areas, especially in fluids. A new Wronskian formulation is presented for these two equations associated with the bilinear Bäcklund transformation. Based on Wronskian identities of the bilinear Kadomtsev–Petviashvili (KP) hierarchy, the Wronskian determinant solution is verified by a direct and concise calculation. The newly introduced Wronskian formulation provides a comprehensive way for building rational solutions. A few rational Wronskian solutions of lower order are computed for the generalized $(2+1)$-dimensional DJKM equation. Our work can show that the extended $(2+1)$-dimensional KdV equation possesses the similar rational Wronskian solutions through the corresponding logarithmic transformation.

I. INTRODUCTION

Nonlinear evolution equations (NLEEs), especially nonlinear integrable equations, have wide applications in fields such as biology, chemical engineering, fluid mechanics, and plasmas physics.[1–7] Studying on NLEEs helps us gain a deeper understanding of the phenomena they describe. In the past decades, nonlinear integrable equations have become increasingly important owing to their remarkable properties, such as Lax pairs, bilinear Bäcklund transformations, N-soliton solutions, and infinite symmetries.[8–11] In the soliton theory, the commuting Kadomtsev–Petviashvili (KP) hierarchy of equations[12] also known as the integrable positive KP hierarchy,[13]

$$u_{t_1} = u_x, \tag{1.1a}$$

$$u_{t_2} = -2u_y, \tag{1.1b}$$

$$u_{t_3} = -6uu_x - u_{xxx} + 3\partial_x^{-1}u_{yy}, \tag{1.1c}$$

$$u_{t_4} = 12(2u_x\partial_x^{-1}u_y - \partial_x^{-2}u_{yyy} + u_{xxy} + 4uu_y), \tag{1.1d}$$

has been introduced in earlier studies. When $t = t_3$, the third member of this KP hierarchy is just the normal Kadomtsev–Petviashvili I (KPI) equation,[14]

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0. \tag{1.2}$$

The fourth Eq. (1.1d) can be written as the $(2+1)$-dimensional Date–Jimbo–Kashiwara–Miwa (DJKM) equation,[15–18]

$$\varphi_{xxxxy} + 4\varphi_{xxy}\varphi_x + 2\varphi_{xxx}\varphi_y + 6\varphi_{xy}\varphi_{xx} - \varphi_{yyy} - 2\varphi_{xxt} = 0, \tag{1.3}$$

by taking the transformations $u = \varphi_x$ and $t_4 = \frac{1}{24}t$.

In nonlinear science, the KP and DJKM equations act as mathematical models for the propagation of 2D nonlinear solitary waves in fluid mechanics, plasma physics, ocean dynamics, nonlinear optics, and so on. These two equations are important integrable systems, which have many interesting characteristics in physical science, such as bilinear Bäcklund transformations, Lax representations, and infinitely many conservation laws.[15–19] As well known, through the logarithmic transformation $u = 2(\ln f)_{xx}$, the KPI Eq. (1.2) has the following Hirota bilinear form:

$$(D_x^4 + D_xD_t + 3\sigma^2 D_y^2)f\cdot f = 0, \quad \sigma^2 = -1 \tag{1.4}$$

with D_x, D_y, and D_t being Hirota's bilinear differential operators.[20] However, the $(2+1)$-dimensional DJKM Eq. (1.3) does not possess a direct bilinear form like the KP Eq. (1.2), and it has a trilinear representation,[15,16]

$$D_x\{(D_x^3D_y - 3D_xD_t)f\cdot f\}\cdot f^2 + \frac{1}{2}D_y\{(D_x^4 + 3\sigma^2D_y^2)f\cdot f\}\cdot f^2 = 0, \quad \sigma^2 = -1, \tag{1.5}$$

by the dependent variable transformation $\varphi = 2(\ln f)_x$. To our knowledge, the KPI Eq. (1.4) and the DJKM Eq. (1.5) possess abundant exact solutions, including N-soliton solutions, lump solutions with different kinds of rational dispersion relations,[20–22] and determinant-type solutions with a Wronskian structure.[18,20,23,24]

Recent studies show that there exist an extension of the $(2+1)$-dimensional DJKM Eq. (1.3),[23,25] expressed as follows:

$$b(\psi_{xxxxy} + 4\gamma\psi_{xxy}\psi_x + 2\gamma\psi_{xxx}\psi_y + 6\gamma\psi_{xx}\psi_{xy} + \delta\psi_{yyy}) + c\psi_{xxt} = 0, \tag{1.6}$$

where γ, δ, b, and c are all arbitrary non-zero real constants. Under the logarithmic transformation

$$\psi = \frac{2}{\gamma}(\ln f)_x, \tag{1.7}$$

the extended form (1.5) has the following trilinear form:

$$\begin{aligned}&b(f^2f_{xxxxy} - ff_{xxxx}f_y + 2ff_{xx}f_{xxy} - 4ff_xf_{xxxy} + 4f_xf_{xxx}f_y \\ &\quad - 2f_{xx}^2f_y - 4f_{xx}f_xf_{xy} + 4f_x^2f_{xxy} + \delta f^2f_{yyy} + 2\delta f_y^3 - 3\delta ff_yf_{yy}) \\ &\quad + cf^2f_{xxt} - cff_{xx}f_t - 2cff_xf_{xt} + 2cf_x^2f_t = 0,\end{aligned} \tag{1.8}$$

which is equivalent to

$$D_x\left[\left(bD_x^3D_y + \frac{3}{2}cD_xD_t\right)f\cdot f\right]\cdot f^2 + \frac{b}{2}D_y\left[\left(D_x^4 + 3\delta D_y^2\right)f\cdot f\right]\cdot f^2 = 0. \tag{1.9}$$

In addition, by combining the third member (1.1c) and fourth member (1.1d) in the positive KP hierarchy, a novel $(2+1)$-dimensional integrable Korteweg-de Vries (KdV) equation,

$$\begin{aligned}u_t = a(6uu_x + u_{xxx} - 3w_y) + b(2wu_x - z_y + u_{xxy} + 4uu_y),\\ u_y = w_x, \quad u_{yy} = z_{xx},\end{aligned} \tag{1.10}$$

has been systematically investigated by Lou.[26] The directly written Lax pairs and Painlevé property show that a linear combination in a soliton hierarchy is still integrable.[26,27]

This motivates us to explore an extended $(2+1)$-dimensional KdV equation, written as follows:

$$\begin{aligned}&a(6\gamma uu_x + u_{xxx} + 3\delta v_{xy}) + b(u_{xxy} + 2\gamma u_xv_x + 4\gamma uu_y + \delta v_{yy}) \\ &\quad + cu_t = 0, \quad v_{xx} = u_y,\end{aligned} \tag{1.11}$$

where the constants γ, δ, b, and c satisfy $\gamma\delta bc \neq 0$, but the constant a is arbitrary. Through the transformation

$$u = \frac{2}{\gamma}(\ln f)_{xx}, \quad v = \frac{2}{\gamma}(\ln f)_y, \tag{1.12}$$

this equation is transformed into

$$\begin{aligned}&\frac{3a}{2}D_x\left[\left(D_x^4 + 3\delta D_y^2\right)f\cdot f\right]\cdot f^2 \\ &\quad + D_x\left[\left(bD_x^3D_y + \frac{3}{2}cD_xD_t\right)f\cdot f\right]\cdot f^2 \\ &\quad + \frac{b}{2}D_y\left[\left(D_x^4 + 3\delta D_y^2\right)f\cdot f\right]\cdot f^2 = 0,\end{aligned} \tag{1.13}$$

which equivalently reads

$$\begin{aligned}&a(f^2f_{xxxxx} + 2ff_{xx}f_{xxx} - 5ff_xf_{xxxx} - 6f_{xx}^2f_x \\ &\quad + 8f_{xxx}f_x^2 + 3\delta f^2f_{xyy} - 3\delta ff_xf_{yy} \\ &\quad - 6\delta ff_yf_{xy} + 6\delta f_xf_y^2) + b(f^2f_{xxxxy} \\ &\quad - ff_{xxxx}f_y + 2ff_{xx}f_{xxy} - 4ff_xf_{xxxy} \\ &\quad + 4f_xf_{xxx}f_y - 2f_{xx}^2f_y - 4f_{xx}f_xf_{xy} + 4f_x^2f_{xxy} \\ &\quad + \delta f^2f_{yyy} + 2\delta f_y^3 - 3\delta ff_yf_{yy}) \\ &\quad + cf^2f_{xxt} - cff_{xx}f_t - 2cff_xf_{xt} + 2cf_x^2f_t = 0.\end{aligned} \tag{1.14}$$

Note that Eq. (1.13) becomes the trilinear form (1.9) of the generalized $(2+1)$-dimensional DJKM equation when $a = 0$. Recent work on integrability and exact solutions of Eq. (1.11) can be found in Refs. 28 and 29.

The investigation of exact solutions plays a significant role in revealing the physical mechanism of various natural phenomena characterized by integrable equations in the real world. As a type of exact solutions, rational solutions are attracting growing attention of numerous scholars due to their extensive applications in many fields.[21,22,30] For instance, a variety of wave behaviors found in the deep ocean, fluid dynamics, optical fibers, and Bose–Einstein condensates can be modeled by nonsingular rational solutions.[31,32] Consequently, an important fundamental work is how to explore rational solutions of integrable equations.

The Wronskian structure is a normal characteristic of integrable equations, which provides a useful tool for obtaining different kinds of exact solutions to integrable equations, such as rational solutions, solitons, negatons, complexitons, and interaction solutions.[33–37] The merit of this technique is that solutions can be verified via a direct substitution. In recent years, Nimmo and Zhao[38] put forward the partition notation for derivatives of the Wronskian determinant and presented the Wronskian-type determinant identities they satisfy. On this basis, we will provide a more concise and direct way to the Wronskian formulation by utilizing Wronskian identities of the bilinear KP hierarchy and properties of Hirota operators,[23,39] thereby greatly simplifying the verification process of the solution. In particular, the resulting Wronskian formulation paves a way for constructing abundant rational solutions.

Very recently, by introducing an auxiliary independent variable and applying the Wronskian identity of the bilinear KP equation, a set of sufficient conditions for Wronskian solutions was constructed for the extended $(2+1)$-dimensional KdV Eq. (1.11), and the resulting N-soliton solution is essentially of $(1+1)$-dimension due to the dimensional reduction.[23] However, rational solutions have not been given via the presented Wronskian formulation. In this paper, we aim to discuss a new Wronskian formulation for solutions of Eqs. (1.6) and (1.11), which particularly presents rational solutions to Eqs. (1.6) and (1.11). Our results will also demonstrate the trilinear Eqs. (1.9) and (1.13) have the same sufficient conditions for Wronskian solutions.

The following is the structure of the paper. In Sec. II, by means of the bilinear Bäcklund transformation, a new Wronskian formulation, different from the ones in Refs. 18, 23, and 39, is constructed to the trilinear Eqs. (1.9) and (1.13). Then, the Wronskian determinant solution is verified by a direct and concise substitution. In Sec. III, rational solutions are furnished by taking special cases in the resulting Wronskian formulation. Our conclusion and remarks will be given at the end.

II. A NEW WRONSKIAN FORMULATION

In this section, we first give the bilinear Bäcklund transformation of Eq. (1.13) obtained in Ref. 28. If f and f' are two different solutions of Eq. (1.13), then

$$(\tilde{\delta}D_y + D_x^2)f \cdot f' = 0, \tag{2.1a}$$

$$\left[aD_x^3 + bD_x^2D_y - 3a\tilde{\delta}D_xD_y - b\tilde{\delta}D_y^2 + cD_t\right]f \cdot f' = \lambda ff', \tag{2.1b}$$

where $\tilde{\delta}^2 = \delta$ and λ is an arbitrary constant, which is a bilinear Bäcklund transformation between f and f'. Next, setting $f' = 1$ as the seed solution in the system (2.1), corresponding to the zero solution $u = 0$ of (1.11), we have the following system:

$$\tilde{\delta}f_y + f_{xx} = 0, \tag{2.2a}$$

$$af_{xxx} + bf_{xxy} - 3a\tilde{\delta}f_{xy} - b\tilde{\delta}f_{yy} + cf_t = \lambda f, \tag{2.2b}$$

which is equivalent to

$$\tilde{\delta}f_y + f_{xx} = 0, \tag{2.3a}$$

$$4af_{xxx} - \frac{2b}{\tilde{\delta}}f_{xxxx} + cf_t = \lambda f. \tag{2.3b}$$

When the constant $\lambda \neq 0$, the above-mentioned system (2.3) can be rewritten as follows:

$$\tilde{\delta}f_y + f_{xx} = 0, \quad 4af_{xxx} = \lambda f, \quad -\frac{2b}{\tilde{\delta}}f_{xxxx} + cf_t = 0, \tag{2.4}$$

which helps us to give a new Wronskian formulation, different from the ones in Refs. 18, 28, and 39 for the trilinear Eqs. (1.9) and (1.13).

Let us now adopt the compact Freeman and Nimmo's notation[40,41]

$$W = W(\phi_1, \phi_2, \ldots, \phi_N) = \begin{vmatrix} \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2 & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_N & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix} = |0, 1, \ldots, N-1| = |\widehat{N-1}|, \tag{2.5}$$

where $\phi_i^{(j)} = \frac{\partial^j \phi_i}{\partial x^j}, i, j \geq 1$. We would like to furnish a set of sufficient conditions, which guarantee that the Wronskian determinant solves the trilinear Eqs. (1.9) and (1.13).

Theorem 2.1. Assume that a group of functions $\phi_i = \phi_i(x, y, t), 1 \leq i \leq N$, meets the following linear conditions:

$$\phi_{i,xxx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j, \tag{2.6a}$$

$$\phi_{i,y} = -\frac{1}{\tilde{\delta}}\phi_{i,xx}, \tag{2.6b}$$

$$\phi_{i,t} = \frac{2b}{\tilde{\delta}c}\phi_{i,xxxx}, \tag{2.6c}$$

where $\tilde{\delta}^2 = \delta$ and the coefficient matrix $A = (\lambda_{ij})_{1 \leq i,j \leq N}$ is an arbitrary real constant matrix. Then, the Wronskian determinant $f = f_N = |\widehat{N-1}|$ defined in Eq. (2.5) *solves the trilinear DJKM* Eq. (1.9) *and the trilinear KdV* Eq. (1.13).

As known to all, the first two numbers of the KP hierarchy[42] have the following Hirota bilinear forms:

$$(D_1^4 - 4D_1D_3 + 3D_2^2)f \cdot f = 0, \tag{2.7a}$$

$$\left[(D_1^3 + 2D_3)D_2 - 3D_1D_4\right]f \cdot f = 0, \tag{2.7b}$$

where f denotes a function related to variables x_j, $j = 1, 2, 3, \ldots$, and $D_j \equiv D_{x_j}$. Before providing a simple and clean proof of Theorem 2.1, we first describe two useful lemmas associated with Wronskian identities of the bilinear KP hierarchy.

Lemma 2.1. Assume that a group of functions $\phi_i = \phi_i(x_1, x_2, x_3, \ldots), (1 \leq i \leq N)$, satisfies the linear partial differential equations,

$$\partial_{x_j}\phi_i = \frac{\partial^j \phi_i}{\partial x^j}, \quad j = 1, 2, 3, \ldots \tag{2.8}$$

Then, the Wronskian determinant $f = f_N = |\widehat{N-1}|$ defined by (2.5) solves the bilinear forms (2.7a) and (2.7b).

The proof of Lemma 2.1 can be found in Refs. 20 and 43. This lemma also shows that the bilinear forms (2.7) yield the Plücker relations for determinants if the function f is written as the Wronskian determinant.[44] Namely, the Hirota bilinear Eqs. (2.7a) and (2.7b) become the following Wronskian identities:

$$\begin{aligned}&(D_1^4 - 4D_1D_3 + 3D_2^2)|\widehat{N-1}| \cdot |\widehat{N-1}| = 24(|\widehat{N-1}||\widehat{N-3}, N, N+1| \\ &- |\widehat{N-2}, N||\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+1| \\ &\times |\widehat{N-3}, N-1, N|) \equiv 0\end{aligned} \tag{2.9a}$$

and

$$\begin{aligned}&\left[(D_1^3 + 2D_3)D_2 - 3D_1D_4\right]|\widehat{N-1}| \cdot |\widehat{N-1}| \\ &= 12(|\widehat{N-1}||\widehat{N-3}, N, N+2| - |\widehat{N-2}, N||\widehat{N-3}, N-1, N+2| \\ &+ |\widehat{N-2}, N+2||\widehat{N-3}, N-1, N|) \\ &- 12(|\widehat{N-1}||\widehat{N-4}, N-2, N, N+1| \\ &- |\widehat{N-4}, N-2, N-1, N+1||\widehat{N-2}, N| \\ &+ |\widehat{N-4}, N-2, N-1, N||\widehat{N-2}, N+1|) \equiv 0,\end{aligned} \tag{2.9b}$$

respectively. The first identity (2.9a) is nothing but a Plücker relation, and the second identity (2.9b) is a linear combination of two Plücker relations.

Lemma 2.2. *Suppose the Wronskian entries* $\phi_i = \phi_i(x, y, t)$, $1 \le i \le N$, *in the Wronskian determinant* (2.5) *satisfy the expressions* (2.8) *and*

$$\sum_{j=1}^{N} \lambda_{ij}\phi_j = r\partial_{x_3}\phi_i = r\phi_{i,xxx}, \tag{2.10}$$

where r *is an arbitrary constant and the coefficient matrix* $A = (\lambda_{ij})_{1\le i,j\le N}$ *is an arbitrary real constant matrix. Then, we have*

$$rD_kD_3 f\cdot f = 0, \quad k = 1, 2, 3, \dots, \tag{2.11}$$

where the function f *is expressed by the Wronskian determinant* (2.5) *and* $D_k = D_{x_k}$. *Here, the Hirota's bilinear operators* D_{x_k} *are Hirota's bilinear differential operators.*

The proof of Lemma 2.2 can also be found in Ref. 23. It is generally known that the first member (2.7a) in the KP hierarchy, that is, the bilinear KP equation, is reduced to the following bilinear Boussinesq equation without velocity term,

$$(D_1^4 + 3D_2^2) f\cdot f = 0, \tag{2.12}$$

under the condition $D_3 = 0$, which is regarded as the "3-reduction" of the bilinear KP equation.[45] Obviously, if the Wronskian entries $\phi_i, 1 \le i \le N$, in the Wronskian determinant (2.5) meet the conditions (2.8) and (2.10), then a direct calculation yields

$$\begin{aligned}&(D_1^4 + 3D_2^2)|\widehat{N-1}|\cdot|\widehat{N-1}|\\ &\quad= \left[(D_1^4 - 4D_1D_3 + 3D_2^2) + 4D_1D_3\right]|\widehat{N-1}|\cdot|\widehat{N-1}| = 0,\end{aligned} \tag{2.13}$$

by utilizing the above-mentioned two lemmas. Thus, the Wronskian determinant $f = |\widehat{N-1}|$ defined by (2.5) is a solution of the bilinear Boussinesq Eq. (2.12). Similarly, the application of Lemmas 1.1 and 1.2 also leads to the Wronskian determinant $f = |\widehat{N-1}|$ defined by (2.5) that solves the bilinear equation,

$$(D_1^3D_2 - 3D_1D_4) f\cdot f = 0, \tag{2.14}$$

if both the conditions (2.8) and (2.10) hold.

Proof of Theorem 2.1. By using the conditions (2.6b) and (2.6c), the derivatives of the Wronskian determinant $f = f_N = |\widehat{N-1}|$ with respect to the variables x, y, and t can be expressed as follows:

$$\begin{aligned}&D_x^4 f\cdot f = D_1^4 f\cdot f, \quad D_y^2 = \frac{1}{\delta}D_2^2 f\cdot f,\\ &D_x^3D_y f\cdot f = -\frac{1}{\delta}D_1^3D_2 f\cdot f,\\ &D_xD_t f\cdot f = \frac{2b}{\delta c}D_1D_4 f\cdot f.\end{aligned} \tag{2.15}$$

Now, by making use of the condition (2.6a) and Lemma 2.2, we can easily get

$$(D_x^4 + 3\delta D_y^2) f\cdot f = (D_1^4 + 3D_2^2) f\cdot f = 0$$

and

$$\begin{aligned}\left(bD_x^3D_y + \frac{3}{2}cD_xD_t\right) f\cdot f &= \left(-\frac{b}{\delta}D_1^3D_2 + \frac{3b}{\delta}D_1D_4\right) f\cdot f\\ &= -\frac{b}{\delta}(D_1^3D_2 - 3D_1D_4) f\cdot f = 0.\end{aligned} \tag{2.16}$$

It follows then that

$$\begin{aligned}&D_x\left[\left(bD_x^3D_y + \frac{3}{2}cD_xD_t\right)|\widehat{N-1}|\cdot|\widehat{N-1}|\right]\cdot|\widehat{N-1}|^2\\ &\quad+\frac{b}{2}D_y\left[\left(D_x^4 + 3\delta D_y^2\right)|\widehat{N-1}|\cdot|\widehat{N-1}|\right]\cdot|\widehat{N-1}|^2 = 0\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}&\frac{3a}{2}D_x\left[\left(D_x^4 + 3\delta D_y^2\right)|\widehat{N-1}|\cdot|\widehat{N-1}|\right]\cdot|\widehat{N-1}|^2\\ &\quad+D_x\left[\left(bD_x^3D_y + \frac{3}{2}cD_xD_t\right)|\widehat{N-1}|\cdot|\widehat{N-1}|\right]\cdot|\widehat{N-1}|^2\\ &\quad+\frac{b}{2}D_y\left[\left(D_x^4 + 3\delta D_y^2\right)|\widehat{N-1}|\cdot|\widehat{N-1}|\right]\cdot|\widehat{N-1}|^2 = 0.\end{aligned} \tag{2.18}$$

This shows that $f = |\widehat{N-1}|$ solves the trilinear DJKM Eq. (1.9) and the trilinear KdV Eq. (1.13). The proof is finished.

III. WRONSKIAN RATIONAL SOLUTIONS

In this section, we would like to construct rational solutions to the generalized $(2+1)$-dimensional DJKM Eq. (1.6) via the presented Wronskian formulation. Since similar transformations of the coefficient matrix A may lead to the same Wronskian solutions to soliton equations, to construct rational solutions for Eq. (1.6), we only need to focus on the following case of the coefficient matrix A:

$$A = (\lambda_{ij})_{1\le i,j\le N} = \begin{pmatrix} 0 & & & & 0\\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0\end{pmatrix}_{N\times N}. \tag{3.1}$$

Let us consider the sequence of polynomials $\phi_i, i \ge 1$, determined by

$$\begin{aligned}&\phi_{1,xxx} = 0, \quad \phi_{i+1,xxx} = \phi_i, \quad \phi_{i,y} = -\frac{1}{\delta}\phi_{i,xx},\\ &\phi_{i,t} = \frac{2b}{\delta c}\phi_{i,xxxx}, \quad i \ge 1,\end{aligned} \tag{3.2}$$

which follows from the hypothesis for A in (3.1). For each $k \ge 1$, corresponding to the following Jordan block:

$$\begin{pmatrix} 0 & & & 0\\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0\end{pmatrix}_{k\times k},$$

a normal Wronskian solution

$$\psi = \frac{2}{\gamma}\partial_x \ln W(\phi_1, \phi_2, \dots, \phi_k)$$

is named as a rational Wronskian solution of order $k-1$.[34] Assume that the three linearly independent solutions of ϕ_1 are $\phi_{1,1}$, $\phi_{2,1}$, and $\phi_{3,1}$. Then, two other general rational Wronskian solutions,

$$\psi = \frac{2}{\gamma}\partial_x \ln W(\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,k_1}; \phi_{2,1}, \phi_{2,2}, \dots, \phi_{2,k_2}),$$

$$\begin{aligned}\psi = \frac{2}{\gamma}\partial_x \ln W(&\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,k_1}; \phi_{2,1}, \phi_{2,2}, \dots, \phi_{2,k_2};\\ &\phi_{3,1}, \phi_{3,2}, \dots, \phi_{3,k_3}),\end{aligned}$$

where $\phi_{i,1}, \phi_{i,2}, \ldots, \phi_{i,k_i}, 1 \le i \le 3$, are three groups of functions corresponding to three Jordan blocks of the above type, are called the rational Wronskian solutions of orders (k_1-1, k_2-1) and (k_1-1, k_2-1, k_3-1),[34] respectively. In what follows, we will present a few rational Wronskian solutions of lower order for the generalized (2 + 1)-dimensional DJKM Eq. (1.6).

Case 1 zero order: Let us consider $\phi_1 = c_1 + c_2 x + c_3\left(x^2 - \frac{2}{\tilde{\delta}}y\right)$. Thus, we have the following Wronskian determinant and the resulting rational Wronskian solution of zero order:

$$f = W(\phi_1) = \phi_1 = c_1 + c_2 x + c_3\left(x^2 - \frac{2}{\tilde{\delta}}y\right),$$
$$\psi = \frac{2}{\gamma}\partial_x \ln W(\phi_1) = \frac{2(c_2 + 2c_3 x)}{\gamma\left[c_1 + c_2 x + c_3\left(x^2 - \frac{2}{\tilde{\delta}}y\right)\right]}. \tag{3.3}$$

In particular, choosing $\phi_1 = x^2 - \frac{2}{\tilde{\delta}}y$, we obtain

$$\psi = \frac{4\tilde{\delta}x}{\gamma(\tilde{\delta}x^2 - 2y)}. \tag{3.4}$$

Case 2 first order: Setting $\phi_1 = 1$, a direct calculation leads to $\phi_2 = \frac{1}{6}x^3 - \frac{1}{\tilde{\delta}}xy$. Therefore, the corresponding Wronskian determinant and rational Wronskian solution of first order may be written as follows:

$$f = W(\phi_1, \phi_2) = \frac{1}{2}x^2 - \frac{1}{\tilde{\delta}}y,$$
$$\psi = \frac{2}{\gamma}\partial_x \ln W(\phi_1, \phi_2) = \frac{4\tilde{\delta}x}{\gamma(\tilde{\delta}x^2 - 2y)}, \tag{3.5}$$

which are the same as the zero-order solutions (3.4) above.

We choose $\phi_1 = x$, a direct expansion yields $\phi_2 = \frac{1}{24}x^4 - \frac{1}{2\tilde{\delta}}x^2 y + \frac{1}{2\delta}y^2 + \frac{2b}{\tilde{\delta}c}t$. Then, the corresponding Wronskian determinant and rational Wronskian solution of first order are

$$f = W(\phi_1, \phi_2) = \frac{1}{8}x^4 - \frac{1}{2\tilde{\delta}}x^2 y - \frac{1}{2\delta}y^2 - \frac{2b}{\tilde{\delta}c}t,$$
$$\psi = \frac{2}{\gamma}\partial_x \ln W(\phi_1, \phi_2) = \frac{8\delta cx(\tilde{\delta}x^2 - 2y)}{\gamma(\tilde{\delta}\delta cx^4 - 4\delta cx^2 y - 4\tilde{\delta}cy^2 - 16\delta bt)}. \tag{3.6}$$

In addition, setting $\phi_1 = \frac{1}{2}x^2 - \frac{1}{\tilde{\delta}}y$, we get $\phi_2 = \frac{1}{120}x^5 - \frac{1}{6\tilde{\delta}}x^3 y + \frac{1}{2\delta}xy^2 + \frac{2b}{\tilde{\delta}c}xt$. Further, the corresponding Wronskian determinant and rational Wronskian solution of first order can be expressed as follows:

$$f = W(\phi_1, \phi_2) = \frac{1}{80}x^6 - \frac{1}{8\tilde{\delta}}x^4 y + \frac{1}{4\delta}x^2 y^2 - \frac{b}{\tilde{\delta}c}x^2 t - \frac{1}{2\tilde{\delta}\delta}y^3 - \frac{2b}{\delta c}yt,$$
$$\psi = \frac{2}{\gamma}\partial_x \ln W(\phi_1, \phi_2) = \frac{12\tilde{\delta}\delta cx^5 - 80\delta cx^3 y + 80\tilde{\delta}cxy^2 - 320b\delta xt}{\gamma(\tilde{\delta}\delta cx^6 - 10\delta cx^4 y + 20\tilde{\delta}cx^2 y^2 - 80\delta bx^2 t - 40cy^3 - 160\tilde{\delta}byt)}. \tag{3.7}$$

Case 3 second order: If setting $\phi_1 = x$, we can compute that

$$\phi_2 = \frac{1}{24}x^4 - \frac{1}{2\tilde{\delta}}x^2 y + \frac{1}{2\delta}y^2 + \frac{2b}{\tilde{\delta}c}t$$

and

$$\phi_3 = \frac{1}{5040}x^7 - \frac{1}{120\tilde{\delta}}x^5 y + \frac{1}{12\delta}x^3 y^2 + \frac{b}{3\tilde{\delta}c}x^3 t - \frac{1}{6\delta\tilde{\delta}}xy^3 - \frac{2b}{\delta c}xyt.$$

A direct computation gives rise to the associated Wronskian determinant and rational Wronskian solution of second order as follows:

$$\begin{aligned} f &= W(\phi_1, \phi_2, \phi_3) \\ &= \frac{1}{2240}x^9 - \frac{1}{140\tilde{\delta}}x^7 y - \frac{1}{40\delta}x^5 y^2 - \frac{b}{10\tilde{\delta}c}x^5 t \\ &\quad + \frac{1}{20\delta}x^5 y^2 - \frac{1}{4\delta^2}xy^4 - \frac{2b}{\tilde{\delta}\delta c}xy^2 t - \frac{4b^2}{\delta c^2}xt^2, \\ \psi &= \frac{2}{\gamma}\partial_x \ln W(\phi_1, \phi_2, \phi_3) = \frac{2p}{\gamma xq}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} p &= 9\delta^3 c^2 x^8 - 112\tilde{\delta}\delta^2 c^2 x^6 y - 280\delta^2 c^2 x^4 y^2 - 1120\tilde{\delta}\delta^2 bctx^4 \\ &\quad + 560\delta^2 c^2 x^4 y^2 - 560\delta c^2 y^4 - 4480\tilde{\delta}\delta bcy^2 t - 8960\delta^2 b^2 t^2, \\ q &= \delta^3 c^2 x^8 - 16\tilde{\delta}\delta^2 c^2 x^6 y - 56\delta^2 c^2 x^4 y^2 - 224\tilde{\delta}\delta^2 bcx^4 t \\ &\quad + 112\delta^2 c^2 x^4 y^2 - 560\delta c^2 y^4 - 4480\tilde{\delta}\delta bcy^2 t - 8960\delta^2 b^2 t^2. \end{aligned}$$

Case 4 (1,1)-order: Let us now assume $\phi_{1,1} = 1, \phi_{1,2} = \frac{1}{6}x^3 - \frac{1}{\tilde{\delta}}xy$ and $\phi_{2,1} = x, \phi_{2,2} = \frac{1}{24}x^4 - \frac{1}{2\tilde{\delta}}x^2 y + \frac{1}{2\delta}y^2 + \frac{2b}{\tilde{\delta}c}t$ in this case. Then, the associated Wronskian determinant and rational Wronskian solution of (1,1)-order are presented by

$$f = W(\phi_{1,1}, \phi_{1,2}, \phi_{2,1}, \phi_{2,2}) = -\frac{1}{2}x^2 - \frac{1}{\tilde{\delta}}y,$$
$$\psi = \frac{2}{\gamma}\partial_x \ln W(\phi_{1,1}, \phi_{1,2}, \phi_{2,1}, \phi_{2,2}) = \frac{4\tilde{\delta}x}{\gamma(\tilde{\delta}x^2 + 2y)}. \tag{3.9}$$

If we take $\phi_{1,1} = 1, \phi_{1,2} = \frac{1}{6}x^3 - \frac{1}{\tilde{\delta}}xy$ and $\phi_{2,1} = \frac{1}{2}x^2 - \frac{1}{\tilde{\delta}}y, \phi_{2,2} = \frac{1}{120}x^5 - \frac{1}{6\tilde{\delta}}x^3 y + \frac{1}{2\delta}xy^2 + \frac{2b}{\tilde{\delta}c}xt$, the associated Wronskian determinant and rational Wronskian solution of (1,1)-order become

$$\begin{aligned} f &= W(\phi_{1,1}, \phi_{1,2}, \phi_{2,1}, \phi_{2,2}) \\ &= -\frac{1}{8}x^4 - \frac{1}{2\tilde{\delta}}x^2 y + \frac{1}{2\delta}y^2 - \frac{2b}{\tilde{\delta}c}t, \\ \psi &= \frac{2}{\gamma}\partial_x \ln W(\phi_{1,1}, \phi_{1,2}, \phi_{2,1}, \phi_{2,2}) \\ &= \frac{8\delta cx(\tilde{\delta}x^2 + 2y)}{\gamma(\tilde{\delta}\delta cx^4 + 4\delta cx^2 y - 4\tilde{\delta}cy^2 + 16\delta bt)}. \end{aligned} \tag{3.10}$$

Let us choose $\phi_{1,1} = x, \phi_{1,2} = \frac{1}{24}x^4 - \frac{1}{2\tilde{\delta}}x^2 y + \frac{1}{2\delta}y^2 + \frac{2b}{\tilde{\delta}c}t$ and $\phi_{2,1} = \frac{1}{2}x^2 - \frac{1}{\tilde{\delta}}y, \phi_{2,2} = \frac{1}{120}x^5 - \frac{1}{6\tilde{\delta}}x^3 y + \frac{1}{2\delta}xy^2 + \frac{2b}{\tilde{\delta}c}xt$. A direct computation shows that the associated Wronskian determinant and rational Wronskian solution of (1,1)-order are

$$f = W(\phi_{1,1}, \phi_{1,2}, \phi_{2,1}, \phi_{2,2}) = -\frac{1}{80}x^6 - \frac{1}{8\tilde{\delta}}x^4 y - \frac{1}{4\delta}x^2y^2 - \frac{b}{\tilde{\delta}c}x^2 t - \frac{1}{2\tilde{\delta}\delta}y^3 + \frac{2b}{\delta c}yt,$$

$$\psi = \frac{2}{\gamma}\partial_x \ln W(\phi_{1,1}, \phi_{1,2}, \phi_{2,1}, \phi_{2,2}) = \frac{12\tilde{\delta}\delta c x^5 + 80\delta c x^3 y + 80\tilde{\delta}cxy^2 + 320b\delta xt}{\gamma(\tilde{\delta}\delta c x^6 + 10\delta c x^4 y + 20\tilde{\delta}cx^2y^2 + 80\delta b x^2 t + 40cy^3 - 160\tilde{\delta}byt)}. \tag{3.11}$$

Remark 1. When $b = \gamma = 1, \delta = -1, c = -2$, the above-mentioned rational Wronskian solutions reduce to the solutions of the $(2+1)$-dimensional DJKM Eq. (1.3). Note that these rational solutions only yield complex-valued solutions due to $\tilde{\delta} = \pm \mathrm{i}$.

Remark 2. According to Theorem 2.1, the above-mentioned Wronskian determinants lead to rational Wronskian solutions of the extended $(2+1)$-dimensional KdV Eq. (1.11) through the transformation (1.12).

IV. CONCLUDING REMARKS

In summary, based on the bilinear Bäcklund transformation, we have constructed a new Wronskian formulation for the trilinear Eqs. (1.9) and (1.13), which involves fourth-order linear partial differential equations. By means of two useful lemmas associated with Wronskian identities of the bilinear KP hierarchy, the Wronskian determinant solution has been verified by a direct calculation. The newly presented Wronskian formulation provides us with a powerful way to construct all kinds of exact solutions, particularly rational solutions. Therefore, for the generalized DJKM Eq. (1.6) and the extended KdV Eq. (1.11) in $(2+1)$-dimensions, a few general rational Wronskian solutions have been determined by choosing the special coefficient matrix in the resulting Wronskian formulation. The presented results in this paper not only enrich the solution space of the considered equations, but also provide a direct and valuable method for verifying Wronskian sufficient conditions. The obtained solutions are expected to be widely applied in the field of science.

We point out that the extended $(2+1)$-dimensional KdV Eq. (1.11) can also be generalized to the following form:[39]

$$a(6\gamma u u_x + u_{xxx} + 3\delta v_{xy}) + b(u_{xxy} + 2\gamma u_x v_x + 4\gamma u u_y + \delta v_{yy}) + cu_t + du_x + hu_y = 0, \quad v_{xx} = u_y, \tag{4.1}$$

where the constants a, b, c, γ, and δ satisfy $\gamma\delta c(a^2+b^2) \neq 0$, but the constants d and h are arbitrary. Under the logarithmic derivative transformation

$$u = \frac{2}{\gamma}(\ln f)_{xx}, \quad v = \frac{2}{\gamma}(\ln f)_y, \tag{4.2}$$

this equation possesses a trilinear form,

$$D_x\Big[\Big(3aD_x^4 + 9a\delta D_y^2 + 2bD_x^3 D_y + 3cD_xD_t + 3dD_x^2 + 3hD_xD_y\Big)f\cdot f\Big]\cdot f^2 + D_y\Big[\Big(bD_x^4 + 3b\delta D_y^2\Big)f\cdot f\Big]\cdot f^2 = 0. \tag{4.3}$$

We can similarly determine that a set of functions $\phi_i = \phi_i(x_1, x_2, x_3, \ldots)$ $(1 \le i \le N)$ satisfies the following conditions:

$$\phi_{i,xxx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j, \tag{4.4a}$$

$$\phi_{i,y} = -\frac{1}{\tilde{\delta}}\phi_{i,xx}, \tag{4.4b}$$

$$\phi_{i,t} = \frac{2b}{\tilde{\delta}c}\phi_{i,xxxx} + \frac{h}{\tilde{\delta}c}\phi_{i,xx} - \frac{d}{c}\phi_{i,x}, \tag{4.4c}$$

where $\tilde{\delta}^2 = \delta$ and the coefficient matrix $A = (\lambda_{ij})_{1\le i,j\le N}$ is an arbitrary real constant matrix. Then, the Wronskian determinant $f = f_N = |\widehat{N-1}|$ defined in (2.5) is a solution to the trilinear Eq. (4.3). The proof of the Wronskian conditions (4.4) is similar to Theorem 2.1. In addition, recent studies show that abundant exact solutions, such as lump waves, lump–soliton interaction solutions, and analytical solutions, exist in nonlinear $(2+1)$-dimensional equations[46–49] and $(3+1)$-dimension equations.[50] Interestingly, it is expected that such interaction solutions could be discussed to Eqs. (1.6) and (1.11) through the Wronskian technique.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Li Cheng: Conceptualization (lead); Data curation (lead); Formal analysis (lead); Investigation (lead); Methodology (lead); Project administration (lead); Software (lead); Supervision (lead); Validation (lead); Writing – original draft (lead). **Yi Zhang:** Resources (equal); Software (equal); Supervision (equal). **Wen-Xiu Ma:** Resources (equal); Software (equal); Supervision (equal). **Ying-Wu Hu:** Conceptualization (supporting); Investigation (supporting); Software (supporting).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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