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Original articles

Wronskian and lump wave solutions to an extended second KP equation

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Abstract

The letter aims to generalize the second KP equation to a new one which still possesses a trilinear form. We construct Wronskian determinant solutions, based on its associated trilinear equation, instead of a Hirota bilinear form. Multi-soliton solutions are generated from the presented Wronskian formulation with higher-order dispersion relations. It is then shown that the extended second KP equation possesses resonant N-wave solutions. Moreover, generic one-lump and two-lump waves are built via the improved long wave limit procedure.

Keywords: The extended second KP equation; Wronskian determinant; Lump solutions; Trilinear form

1. Introduction

Nonlinear partial differential equations including lower-dimensional integrable soliton equations and higher-dimensional nonlinear evolution equations arise in various areas of applied science, such as plasma physics, biology, chemistry and mathematical physics. Integrable soliton equations possess a few nice aspects: N-soliton solutions, Lax representations, infinitely many conservation laws, Bäcklund transformation [3,17,26–28], etc. The Kadomtsev–Petviashvili (KP) hierarchy is one of the most basic integrable models in soliton theory [4,13]. According to the physical and mathematical characteristics of the KP hierarchy, it is also significant to pay a great deal of attention to extensions and generalizations for the KP hierarchy. Some related generalizations have been investigated, such as the (2+1)-dimensional Date–Jimbo–Kashiwara–Miwa (DJKM) equation [3,8,11,35,39], the (2+1)-dimensional Bogoyavlenskii–Kadomtsev–Petviashvili (BKP) equation [33,34,38], the KP equation with a self-consistent source [23] and the second KP equation [15,20].

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The second equation in the commuting KP hierarchy (see [15,20]) has been introduced by applying τ-subalgebras and master subalgebras, which reads as

$$u_t = -72\alpha^4 u_x \partial_x^{-1} u_y - 36\alpha^6 \partial_x^{-2} u_{yyy} - 144\alpha^4 u u_y - 36\alpha^4 u_{xxy}. \tag{1.1}$$

If $\alpha = (\sqrt{6}/6)\,\mathrm{I}$, $\mathrm{I} = \sqrt{-1}$ and $\alpha = \sqrt{6}/6$, then Eq. (1.1) reduces to the so-called second KPI equation

$$u_t + 2u_x v_x + 4uu_y - \frac{1}{6} v_{yy} + u_{xxy} = 0, \qquad v_{xx} = u_y \tag{1.2}$$

and the second KPII equation

$$u_t + 2u_x v_x + 4uu_y + \frac{1}{6} v_{yy} + u_{xxy} = 0, \qquad v_{xx} = u_y, \tag{1.3}$$

respectively. In Ref. [20], it was shown that the second KPI equation (1.2) existed a general kind of lump solutions with higher-order rational dispersion relations whereas the second KPII equation (1.3) had no lumps. Hirota trilinear forms play a leading part in presenting such solutions. These motivate us to explore an extended form for the second KP equation (1.1), expressed as

$$u_{xxy} + 2a\, u_x v_x + 4a\, uu_y + b\, v_{yy} + c\, u_t = 0, \qquad v_{xx} = u_y, \tag{1.4}$$

where a, b and c are all arbitrary non-zero real constants. By the potential $u = \psi_x$, the nonlinear equation (1.4) is written as

$$\psi_{xxxxy} + 4a\psi_{xxy}\psi_x + 2a\,\psi_{xxx}\psi_y + 6a\,\psi_{xx}\psi_{xy} + b\,\psi_{yyy} + c\,\psi_{xxt} = 0. \tag{1.5}$$

Taking the choice $a = 1, b = \pm 1, c = -2$ and $a = 2, b = -1, c = 1$, the potential equation (1.5) is transformed to the (2+1)-dimensional DJKM equation

$$u_{xxxxy} + 4u_{xxy}u_x + 2u_{xxx}u_y + 6u_{xy}u_{xx} \pm u_{yyy} - 2u_{xxt} = 0, \tag{1.6}$$

and the (2+1)-dimensional BKP equation

$$u_{xxt} + u_{xxxxy} + 12u_{xx}u_{xy} + 8u_x u_{xxy} + 4u_{xxx}u_y = u_{yyy}, \tag{1.7}$$

respectively. For all we know, there are a large number of studies on Eqs. (1.6) and (1.7) [3,8,11,33–35,38,39]. For example, Wronskian and Grammian determinant solutions have been derived for the DJKM equation through the Hirota bilinear formulation [39].

The goal of this paper is to investigate exact solutions to the extended second KP equation (1.4), including a Wronskian formulation, two-lump and resonant N-wave solutions. Eq. (1.4) possesses a trilinear form, and so, we would like to construct a Wronskian formulation, based on its associated trilinear equation, instead of a Hirota bilinear form in Section 2. Besides, N-soliton solutions will also be generated from the presented Wronskian structure. In Section 3, using the long wave limit approach, generic one-lump, two-lump and resonant N-wave solutions will be built from the corresponding multi-soliton solutions. Section 4 gives our conclusions and remarks.

2. Wronskian and N-soliton solutions

We first show that the extended second KP equation (1.4) has a trilinear form. Under the following transformations

$$u = \frac{2}{a}(\ln f)_{xx}, \quad v = \frac{2}{a}(\ln f)_y, \tag{2.1}$$

Eq. (1.4) is mapped into the trilinear form

$$\begin{aligned} &f^2 f_{xxxxy} + b\, f^2 f_{yyy} + c\, f^2 f_{xxt} - f f_{xxxx} f_y + 2 f f_{xx} f_{xxy} \\ &\quad - c\, f f_{xx} f_t - 2c\, f f_x f_{xt} - 4 f f_x f_{xxxy} - 3b\, f f_y f_{yy} + 4 f_x f_{xxx} f_y \\ &\quad - 2 f_{xx}^2 f_y + 2b\, f_y^3 - 4 f_{xx} f_x f_{xy} + 2c\, f_x^2 f_t + 4 f_x^2 f_{xxy} = 0. \end{aligned} \tag{2.2}$$

Next, we introduce the shorthand notation [5,9,10,18,24] to replace Wronskians as follows:

$$W(\phi_1,\phi_2,\ldots,\phi_N)=\begin{vmatrix}\phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)}\\ \phi_2 & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)}\\ \vdots & \vdots & \vdots & \vdots\\ \phi_N & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)}\end{vmatrix}=|0,1,\ldots,N-1|=|\widehat{N-1}|, \tag{2.3}$$

where $\phi_i^{(j)}=\dfrac{\partial^j\phi_i}{\partial x^j}$. Let us present a set of sufficient conditions such that the Wronskian determinant (2.3) is a solution to the trilinear equation (2.2),

Theorem 2.1. *Suppose that a group of functions $\phi_i=\phi_i(x,y,t)$ $(1\le i\le N)$ satisfy the following conditions:*

$$\phi_{i,y}=\beta_1\phi_{i,xx},\qquad \phi_{i,t}=\beta_2\phi_{i,xxxx}, \tag{2.4}$$

with

$$\beta_1^2=\frac{1}{b},\ \beta_2=\frac{-2}{c}\beta_1. \tag{2.5}$$

Then the Wronskian determinant $f=|\widehat{N-1}|$ defined by (2.3) solves the trilinear Eq. (2.2).

According to the existing research results [38], Theorem 2.1 might be proved (see the Appendix for details).

As an application case of Theorem 2.1, a set of sufficient conditions for the second KPI equation (1.2) is expressed as follows:

$$\phi_{i,y}=\pm\sqrt{6}\,\mathrm{I}\phi_{i,xx},\qquad \phi_{i,t}=\mp2\sqrt{6}\,\mathrm{I}\phi_{i,xxxx}, \tag{2.6}$$

where $\mathrm{I}=\sqrt{-1}$.

The Wronskian solutions to the trilinear Eq. (2.2) can be written as:

$$f_N=W(\phi_1,\phi_2,\ldots,\phi_N), \tag{2.7}$$

where

$$\phi_i=e^{\xi_i}+e^{\widehat{\xi_i}},\quad \xi_i=p_ix+\beta_1p_i^2y+\beta_2p_i^4t+\text{constant},\quad \widehat{\xi_i}=q_ix+\beta_1q_i^2y+\beta_2q_i^4t+\text{constant}, \tag{2.8}$$

in which p_i,q_i are free parameters and β_1,β_2 are defined by (2.5). Let $N=2$, and then f_N is given as

$$\begin{aligned} f_2=W(\phi_1,\phi_2)&=\begin{vmatrix}\phi_1 & \phi_{1x}\\ \phi_2 & \phi_{2x}\end{vmatrix}\\ &=\begin{vmatrix}e^{\xi_1}+e^{\widehat{\xi_1}} & p_1e^{\xi_1}+q_1e^{\widehat{\xi_1}}\\ e^{\xi_2}+e^{\widehat{\xi_2}} & p_2e^{\xi_2}+q_2e^{\widehat{\xi_2}}\end{vmatrix}=e^{\widehat{\xi_1}+\widehat{\xi_2}}\begin{vmatrix}e^{\xi_1-\widehat{\xi_1}}+1 & p_1e^{\xi_1-\widehat{\xi_1}}+q_1\\ e^{\xi_2-\widehat{\xi_2}}+1 & p_2e^{\xi_2-\widehat{\xi_2}}+q_2\end{vmatrix}\\ &=e^{\widehat{\xi_1}+\widehat{\xi_2}}(q_2-q_1)\left[1+\frac{q_2-p_1}{q_2-q_1}e^{\xi_1-\widehat{\xi_1}}+\frac{p_2-q_1}{q_2-q_1}e^{\xi_2-\widehat{\xi_2}}+\frac{p_2-p_1}{q_2-q_1}e^{\xi_1-\widehat{\xi_1}+\xi_2-\widehat{\xi_2}}\right]. \end{aligned} \tag{2.9}$$

Since the exponential factor in f_2 makes no contribution to the solutions u and v given by the transformations (2.1), the factor $e^{\widehat{\xi_1}+\widehat{\xi_2}}(q_2-q_1)$ can be omitted. Assuming that two phase constants δ_i and $\widehat{\delta_i}$ satisfy the relations

$$e^{\delta_1-\widehat{\delta_1}}=\frac{q_2-p_1}{q_2-q_1},\ e^{\delta_2-\widehat{\delta_2}}=\frac{p_2-q_1}{q_2-q_1}, \tag{2.10}$$

and taking the following variables transformations [10]

$$\xi_i+\delta_i\to\xi_i,\ \widehat{\xi_i}+\widehat{\delta_i}\to\widehat{\xi_i}, \tag{2.11}$$

then the above $W(\phi_1,\phi_2)$ can be rewritten as

$$W(\phi_1,\phi_2)\propto 1+e^{\xi_1-\widehat{\xi_1}}+e^{\xi_2-\widehat{\xi_2}}+\frac{(p_1-p_2)(q_1-q_2)}{(p_1-q_2)(q_1-p_2)}e^{\xi_1+\xi_2-\widehat{\xi_1}-\widehat{\xi_2}}, \tag{2.12}$$

which is the two-soliton solution to the trilinear Eq. (2.2).

In the general Wronskian determinant (2.7) , we have

$$f_N = W(\phi_1,\phi_2,\ldots,\phi_N) = \begin{vmatrix} \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2 & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_N & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}$$

$$= e^{\widehat{\xi_1}+\widehat{\xi_2}+\cdots+\widehat{\xi_N}} \begin{vmatrix} e^{\xi_1-\widehat{\xi_1}}+1 & p_1 e^{\xi_1-\widehat{\xi_1}}+q_1 & \cdots & p_1^{N-1}e^{\xi_1-\widehat{\xi_1}}+q_1^{N-1} \\ e^{\xi_2-\widehat{\xi_2}}+1 & p_2 e^{\xi_2-\widehat{\xi_2}}+q_2 & \cdots & p_2^{N-1}e^{\xi_2-\widehat{\xi_2}}+q_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ e^{\xi_N-\widehat{\xi_N}}+1 & p_N e^{\xi_N-\widehat{\xi_N}}+q_N & \cdots & p_N^{N-1}e^{\xi_N-\widehat{\xi_N}}+q_N^{N-1} \end{vmatrix}. \tag{2.13}$$

Each element of the above determinant (2.13) is the sum of an exponential function and a constant; so it can be divided into the sum of 2^N determinants by rows. Next, extracting the common factors associated with exponential functions from each determinant and using the following Vandermonde determinant:

$$\begin{vmatrix} 1 & q_1 & \cdots & q_1^{N-1} \\ 1 & q_2 & \cdots & q_2^{N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & q_N & \cdots & q_N^{N-1} \end{vmatrix} = \prod_{1\le l<m}^{N} (q_m - q_l), \tag{2.14}$$

the above Wronskian solution f_N has the form

$$\begin{aligned} f_N = {}& e^{\widehat{\xi_1}+\widehat{\xi_2}+\cdots+\widehat{\xi_N}} \prod_{1\le l<m}^{N}(q_m-q_l)\Bigg(1+\sum_{i=1}^{N}\frac{\prod_{\substack{1\le r\le N\\ r\ne i}}(q_r-p_i)}{\prod_{\substack{1\le s\le N\\ s\ne i}}(q_s-q_i)}e^{\xi_i-\widehat{\xi_i}} \\ &+\sum_{1\le i<j}^{N}\frac{\prod_{\substack{1\le r\le N\\ r\ne i,j}}(q_r-p_i)(q_r-p_j)(p_i-p_j)}{\prod_{\substack{1\le s\le N\\ s\ne i,j}}(q_s-q_i)(q_s-q_j)(q_i-q_j)}e^{\xi_i-\widehat{\xi_i}+\xi_j-\widehat{\xi_j}} \\ &+\sum_{1\le i<j<k}^{N}\frac{\prod_{\substack{1\le r\le N\\ r\ne i,j,k}}(q_r-p_i)(q_r-p_j)(q_r-p_k)(p_i-p_j)(p_i-p_k)(p_j-p_k)}{\prod_{\substack{1\le s\le N\\ s\ne i,j,k}}(q_s-q_i)(q_s-q_j)(q_s-q_k)(q_i-q_j)(q_i-q_k)(q_j-q_k)} \\ &\times e^{\xi_i-\widehat{\xi_i}+\xi_j-\widehat{\xi_j}+\xi_k-\widehat{\xi_k}}+\cdots+\frac{\prod_{1\le l<m}^{N}(p_m-p_l)}{\prod_{1\le l<m}^{N}(q_m-q_l)}e^{\xi_1-\widehat{\xi_1}+\xi_2-\widehat{\xi_2}+\cdots+\xi_N-\widehat{\xi_N}}\Bigg). \end{aligned} \tag{2.15}$$

We omit the factor $e^{\widehat{\xi_1}+\widehat{\xi_2}+\cdots+\widehat{\xi_N}} \prod_{1\le l<m}^{N}(q_m-q_l)$ and make a series of variable transformations

$$e^{\delta_i-\widehat{\delta_i}} = \frac{\prod_{\substack{1\le r\le N\\ r\ne i}}(q_r-p_i)}{\prod_{\substack{1\le s\le N\\ s\ne i}}(q_s-q_i)},\ \xi_i+\delta_i\to\xi_i,\ \widehat{\xi_i}+\widehat{\delta_i}\to\widehat{\xi_i},\ i=1,\ldots,N, \tag{2.16}$$

and then f_N is equivalent to

$$\begin{aligned} f_N = {}& 1+\sum_{i=1}^{N} e^{\xi_i-\widehat{\xi_i}}+\sum_{1\le i<j}^{N}A_{ij}e^{\xi_i-\widehat{\xi_i}+\xi_j-\widehat{\xi_j}} \\ &+\sum_{1\le i<j<k}^{N}A_{ij}A_{jk}A_{ik}e^{\xi_i-\widehat{\xi_i}+\xi_j-\widehat{\xi_j}+\xi_k-\widehat{\xi_k}}+\cdots+\prod_{1\le i<j}^{N}A_{ij}e^{\sum_{i=1}^{N}\xi_i-\widehat{\xi_i}}, \end{aligned} \tag{2.17}$$

where

$$A_{ij} = \frac{(p_i-p_j)(q_i-q_j)}{(p_i-q_j)(q_i-p_j)},\ 1\le i<j\le N, \tag{2.18}$$

and ξ_i and $\widehat{\xi_i}$ are defined by (2.8).

Furthermore, taking the following new parameters:

$$k_i = p_i - q_i,\ l_i = \beta_1(p_i^2-q_i^2),\ w_i = \beta_2(p_i^4-q_i^4), \tag{2.19}$$

and using the condition (2.5), the dispersion relation of the trilinear Eq. (2.2) reads as

$$w_i = -\frac{l_i(k_i^4 + bl_i^2)}{ck_i^2}. \tag{2.20}$$

Obviously, the corresponding two-soliton and four-soliton solutions to the trilinear form (2.2) become

$$f_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}, \tag{2.21}$$

and

$$\begin{aligned} f_4 = 1 &+ \sum_{i=1}^{4} e^{\eta_i} + \sum_{1\le i<j}^{4} A_{ij}e^{\eta_i+\eta_j} + \sum_{1\le i<j<k}^{4} A_{ij}A_{jk}A_{ik}e^{\eta_i+\eta_j+\eta_k} \\ &+ \prod_{1\le i<j}^{4} A_{ij}e^{\eta_1+\eta_2+\eta_3+\eta_4}, \end{aligned} \tag{2.22}$$

respectively, with

$$\eta_i = k_i x + l_i y + w_i t + \delta_i^0,$$

and

$$A_{ij} = \frac{-k_i^2k_j^2(k_i-k_j)^2 + b(k_il_j - k_jl_i)^2}{-k_i^2k_j^2(k_i+k_j)^2 + b(k_il_j - k_jl_i)^2},\ 1 \le i < j \le 4, \tag{2.23}$$

where w_i are defined by (2.20).

By analogy with the N-soliton solutions of the (2+1)-dimensional KP [10], the N-soliton solution to the extended second KP equation (1.4) can be expressed as follows:

$$u = \frac{2}{a}(\ln f)_{xx}, \quad v = \frac{2}{a}(\ln f)_y, \quad f \equiv f_N = \sum_{\mu=0,1} e^{\sum_{1\le i<j}^{N}\mu_i\mu_j \ln A_{ij} + \sum_{i=1}^{N}\mu_i\eta_i}, \tag{2.24}$$

where η_i, A_{ij} and w_i are defined by (2.20) and (2.23) respectively. Here $\sum_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_i = 0, 1, (i = 1, 2, \ldots, N)$ and $\sum_{1\le i<j}^{N}$ is the sum over all possible combinations of the N elements with the specific condition $1 \le i < j$.

3. Resonant and multi-lump solutions

The investigation of exact solutions has always been one of the main topics in soliton theory. It has been found there exist various kinds of exact solutions to soliton equations, such as resonant solutions and lump solutions. Resonant multiple wave solutions, which are linear combinations of exponential waves, can be construct for many soliton equations by applying the linear superposition principle. A few cases with N-wave satisfying and not satisfying the dispersion relation were presented [6,19,22]. In contrast to resonant solutions, lumps are a type of analytic rational solutions which tend to zero in all directions in the space [1,2,7,14,16,20,21,29–31]. In what follows, we will focus on resonant multiple waves and lump solutions for the extended second KP equation (1.4), based on the above N-soliton solutions.

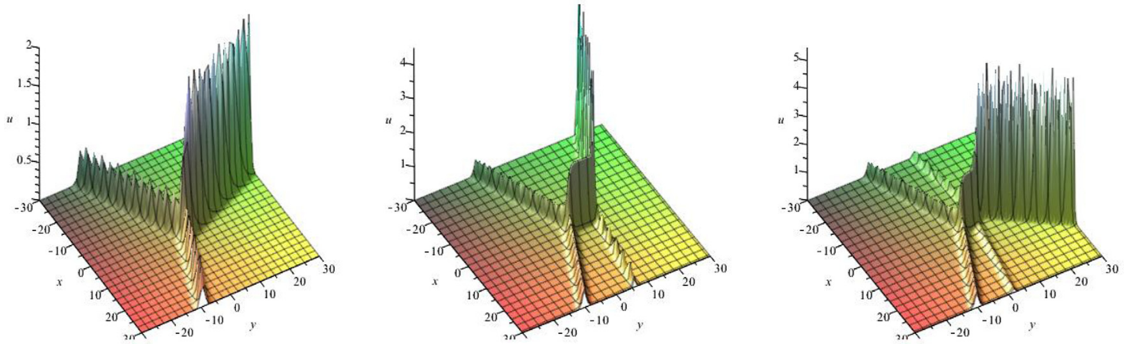
3.1. Resonant solutions

In the two-soliton solution (2.21), the resonant condition corresponds to $A_{12} = 0$ or ∞. Choosing the phase shift $A_{12} = 0$ and using the transformations (2.1), a direct calculation yields the following resonant two-wave solutions to the trilinear Eq. (2.2):

$$f = 1 + e^{\eta_1} + e^{\eta_2},\ \eta_i = k_i x + l_i y + w_i t + \delta_i^0,\ i = 1, 2, \tag{3.1}$$

where

$$l_1 = \frac{k_1(l_2\sqrt{b} + k_1k_2 - k_2^2)}{\sqrt{b}k_2},\ w_2 = -\frac{l_2(k_2^4 + bl_2^2)}{ck_2^2},$$



(a) (b) (c)

Fig. 1. (a) Profile of u in the two-wave solution (3.1) with (3.2) and parameters: $k_1 = 1, k_2 = -1, l_2 = 2, t = 1$. (b) Profile of u in the three-wave solution (3.5) with $N = 3, \varepsilon_1 = 1, \varepsilon_2 = 1, \varepsilon_3 = 1, k_1 = 1, k_2 = 2, k_3 = -1, t = 1$. (c) Profile of u in the four-wave solution (3.5) with $N = 4, \varepsilon_1 = 1, \varepsilon_2 = 1, \varepsilon_3 = 1, \varepsilon_4 = 1, k_1 = 1, k_2 = -1.8, k_3 = -1, k_4 = 1.5, t = 1$.

$$w_1 = \frac{-k_1(\sqrt{b}l_2 + k_1k_2 - k_2^2)(2k_1^2k_2^2 - 2k_1k_2^3 + 2\sqrt{b}k_1k_2l_2 + k_2^4 - 2\sqrt{b}k_2^2l_2 + bl_2^2)}{\sqrt{b}ck_2^3}, \tag{3.2}$$

and

$$\begin{aligned} l_1 &= \frac{k_1(l_2\sqrt{b} - k_1k_2 + k_2^2)}{\sqrt{b}k_2}, \;\; w_2 = -\frac{l_2(k_2^4 + bl_2^2)}{ck_2^2}, \\ w_1 &= \frac{k_1(\sqrt{b}l_2 - k_1k_2 + k_2^2)(2k_1k_2^3 - 2k_1^2k_2^2 + 2\sqrt{b}k_1k_2l_2 - k_2^4 - 2\sqrt{b}k_2^2l_2 - bl_2^2)}{\sqrt{b}ck_2^3}. \end{aligned} \tag{3.3}$$

Moreover, under the choices

$$l_i = \pm\frac{1}{\sqrt{b}}k_i^2, \; w_i = \mp\frac{2}{c\sqrt{b}}k_i^4, \; i = 1, 2, \ldots, N, \tag{3.4}$$

the phase shifts (2.23) can be expressed as $A_{ij} = 0, 1 \le i < j \le N$, which generate a kind of resonant N-wave solutions to the extended second KP equation (1.4):

$$u = \frac{2}{a}(\ln f)_{xx}, \quad v = \frac{2}{a}(\ln f)_y, \quad f = 1 + \sum_{i=1}^{N} \varepsilon_i e^{k_i x + \frac{1}{\sqrt{b}}k_i^2 y - \frac{2}{c\sqrt{b}}k_i^4 t}, \tag{3.5}$$

or

$$u = \frac{2}{a}(\ln f)_{xx}, \quad v = \frac{2}{a}(\ln f)_y, \quad f = 1 + \sum_{i=1}^{N} \varepsilon_i e^{k_i x - \frac{1}{\sqrt{b}}k_i^2 y + \frac{2}{c\sqrt{b}}k_i^4 t}, \tag{3.6}$$

where ε_i's and k_i's are arbitrary constants. Figs. 1 and 2 exhibit three-dimensional plots of the resonant wave solutions (3.1) and (3.5) to the second KPII equation (1.3) with $a = 1, b = \frac{1}{6}, c = 1$ for three subsequent time instances. In the resonant two-wave solution (3.1), we can observe that two line solitons merge into a big soliton after a resonant interaction and the amplitude of the resonant soliton will increase as shown in Fig. 1(a). Fig. 1(b) and (c) describe that such soliton resonance behaviors can also occur in the resonant three- and four-wave solutions. Fig. 2 similarly shows the resonant behaviors of the traveling two-kink , three-kink and four-kink waves in the (x, y) plan. It is not difficult to see that the propagation of kink wave fusion or fission tends to be more complicated with the increase of the number of solitons.

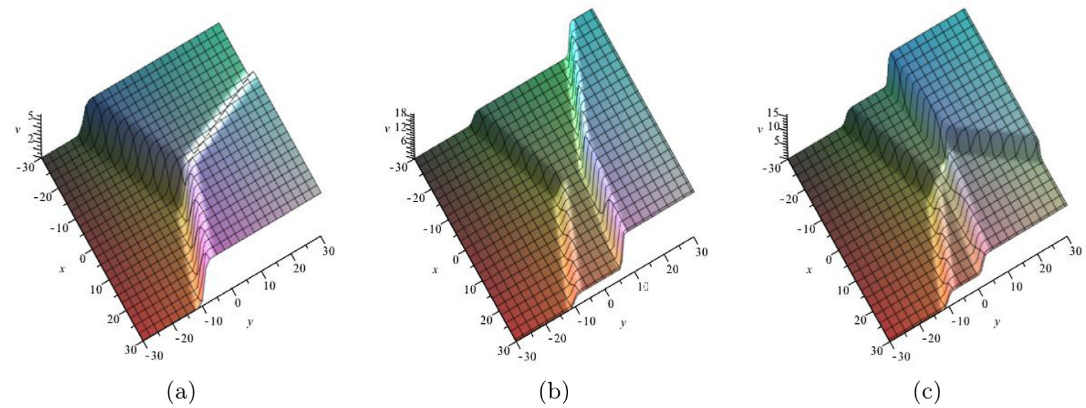


Fig. 2. Profiles of v in the two-,three- and four-wave solutions. The related expressions and parameters are the same as in Fig. 1.

3.2. Multi-lump solutions

We now discuss the long wave limit of the two-soliton solution to obtain generic lump solutions. Let us set the following variable substitution:

$$\eta_i \to \gamma_i(k_i x + l_i y + w_i t + \zeta_i) + \delta_i^0, \quad i \geq 1, \tag{3.7}$$

and then the two-soliton solution (2.21) can be equivalently rewritten as

$$f_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2}, \tag{3.8}$$

where

$$\begin{aligned} \eta_i &= \gamma_i\Big[k_i x + l_i y - \frac{l_i(\gamma_i^2 k_i^4 + b l_i^2)}{c k_i^2} t + \zeta_i\Big] + \delta_i^0, i = 1, 2, \\ A_{12} &= \frac{-k_1^2 k_2^2(\gamma_1 k_1 - \gamma_2 k_2)^2 + b(k_1 l_2 - k_2 l_1)^2}{-k_1^2 k_2^2(\gamma_1 k_1 + \gamma_2 k_2)^2 + b(k_1 l_2 - k_2 l_1)^2}. \end{aligned} \tag{3.9}$$

Let us put $e^{\delta_i^0} = -1$, $\gamma_1/\gamma_2 = O(1)$, $k_i = O(1)$, $l_i = O(1)$, $i = 1, 2$ and take the limit as $\gamma_i \to 0$. We get

$$f_2 = \gamma_1\gamma_2\Big[\theta_1\theta_2 + \frac{4k_1^3 k_2^3}{b(k_1 l_2 - k_2 l_1)^2} + O(\gamma)\Big], \tag{3.10}$$

with

$$\theta_i = k_i x + l_i y - \frac{b l_i^3}{c k_i^2} t + \zeta_i, i = 1, 2, \tag{3.11}$$

owing to

$$A_{12} = 1 + \frac{4\gamma_1\gamma_2 k_1^3 k_2^3}{b(k_1 l_2 - k_2 l_1)^2} + O(\gamma^3). \tag{3.12}$$

By means of the transformation (2.1), the factor $\gamma_1\gamma_2$ can be ignored in the expression (3.10). We still denote the above expression as f_2, then f_2 has the following form:

$$f_2 = \theta_1\theta_2 + B_{12}, \quad B_{12} = \frac{4k_1^3 k_2^3}{b(k_1 l_2 - k_2 l_1)^2}, \tag{3.13}$$

where $\theta_i, i = 1, 2$ are defined by (3.11). To construct nonsingular rational solutions, we choose

$$\begin{aligned} &k_1 = r_1 + d_1\mathrm{I},\ l_1 = r_2 + d_2\mathrm{I},\ \zeta_1 = h + m\mathrm{I},\ \mathrm{I} = \sqrt{-1}, \\ &k_2 = k_1^*,\ l_2 = l_1^*,\ \zeta_2 = \zeta_1^*, r_i, d_i, h, m \in \mathbb{R}, i = 1, 2, \end{aligned} \tag{3.14}$$

where the asterisk indicates the complex conjugate. Substituting these expressions into (3.13) yields a class of quadratic function solutions as follows:

$$\begin{aligned} f_2 = &\left(r_1x + r_2y - \frac{b\big[(r_1^2 - d_1^2)(r_2^3 - 3r_2d_2^2) - 2r_1d_1(d_2^3 - 3r_2^2d_2)\big]}{c(r_1^2 + d_1^2)^2}t + h\right)^2 \\ &+ \left(d_1x + d_2y - \frac{b\big[(r_1^2 - d_1^2)(3r_2^2d_2 - d_2^3) - 2r_1d_1(r_2^3 - 3r_2d_2^2)\big]}{c(r_1^2 + d_1^2)^2}t + m\right)^2 \\ &- \frac{(r_1^2 + d_1^2)^3}{b(r_1d_2 - r_2d_1)^2}, \end{aligned} \tag{3.15}$$

where $r_1d_2 - r_2d_1 \neq 0$ and the other parameters involved are arbitrary. Obviously, this kind of quadratic function solutions is always positive when the coefficient $b < 0$. Therefore, under the conditions $r_1d_2 - r_2d_1 \neq 0$ and $b < 0$, one-lump waves of Eq. (1.4) can be expressed as

$$u = \frac{2}{a}(\ln f_2)_{xx} = \frac{2(f_{2,xx}f_2 - f_{2,x}^2)}{af_2^2}, \quad v = \frac{2}{a}(\ln f_2)_y = \frac{2f_{2,y}}{af_2}, \tag{3.16}$$

which are localized along all space directions. It is also demonstrated that the above polynomial solutions never generate any lump solution to the extended KP equation (1.4) when the coefficient $b > 0$.

By making a long wave limit on the four-soliton solutions (2.22), the resulting polynomial function solution to Eq. (1.4) can be written as [31]

$$\begin{aligned} f_4 = &\ \theta_1\theta_2\theta_3\theta_4 + B_{12}\theta_3\theta_4 + B_{13}\theta_2\theta_4 + B_{14}\theta_2\theta_3 + B_{23}\theta_1\theta_4 \\ &+ B_{24}\theta_1\theta_3 + B_{34}\theta_1\theta_2 + B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}, \end{aligned} \tag{3.17}$$

where

$$\theta_i = k_ix + l_iy - \frac{bl_i^3}{ck_i^2}t + \zeta_i, \ \ i = 1, 2, 3, 4, \tag{3.18}$$

and

$$B_{ij} = \frac{4k_i^3k_j^3}{b(k_il_j - k_jl_i)^2}, \ \ 1 \le i < j \le 4. \tag{3.19}$$

Taking $b < 0, k_{2+i} = k_i^*, l_{2+i} = l_i^*, \zeta_{2+i} = \zeta_i^*, i = 1, 2$, and applying the results presented in Ref. [31], we can get two-lump wave solutions to Eq. (1.4) via the transformation (2.1). In what follows, let us present an illustrative example to shed light on a multiple collision of two lumps.

If we choose

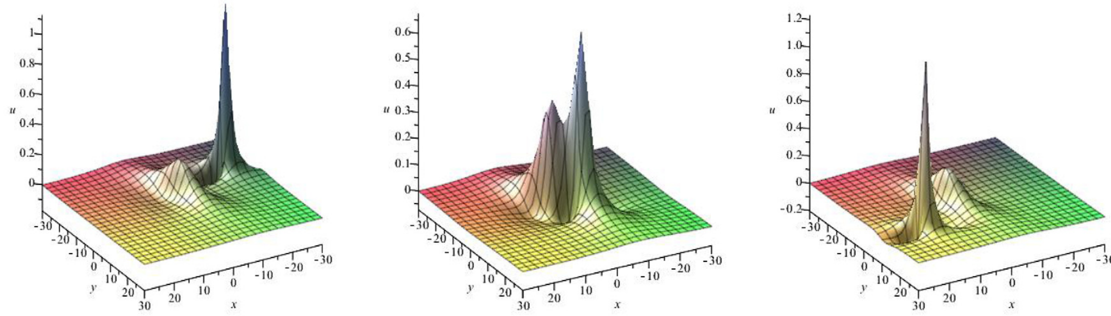
$$\begin{aligned} &k_1 = 1 + \mathrm{I}, l_1 = -1 + 2\mathrm{I}, k_2 = 2 - \mathrm{I}, l_2 = 1 + \mathrm{I}, k_3 = k_1^*, k_4 = k_2^* \\ &l_3 = l_1^*, l_4 = l_2^*, \zeta_1 = 2 + \mathrm{I}, \zeta_2 = 4 + 3\mathrm{I}, \zeta_3 = \zeta_1^*, \zeta_4 = \zeta_2^*, \end{aligned} \tag{3.20}$$

the second KPI equation (1.2) possesses a special two-lump solution as follows:

$$u = 2(\ln f_4)_{xx} = \frac{2(f_{4,xx}f_4 - f_{4,x}^2)}{f_4^2}, \quad v = 2(\ln f_4)_y = \frac{2f_{4,y}}{f_4},$$

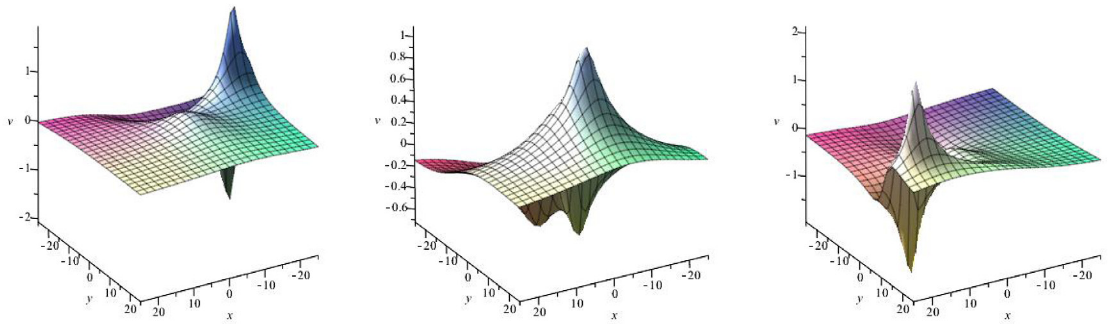
where

$$\begin{aligned} f_4 = &\ 10x^4 + 10y^4 + \frac{5}{648}t^4 + 14x^3y + 14xy^3 + 33x^2y^2 - \frac{1729}{150}x^3t - \frac{553}{25}x^2yt \\ &+ \frac{6131}{1200}x^2t^2 - \frac{329}{25}xy^2t + \frac{2023}{600}xyt^2 - \frac{1729}{5400}xt^3 - \frac{116}{15}y^3t + \frac{299}{120}y^2t^2 - \frac{29}{135}t^3y \\ &+ 50x^3 + 70y^3 - \frac{799}{1080}t^3 + 60x^2y - \frac{379}{10}x^2t + 90xy^2 + \frac{1367}{120}xt^2 - \frac{358}{5}xyt \\ &- \frac{279}{5}y^2t + \frac{1853}{120}yt^2 + \frac{4287}{5}x^2 + \frac{2577}{5}y^2 + \frac{133841}{1200}t^2 + \frac{34764}{25}xy - \frac{889183}{1250}xt \\ &- \frac{65297}{125}yt + \frac{7444}{3}x + \frac{11002}{15}y - \frac{1809107}{2250}t + \frac{30985}{3}. \end{aligned} \tag{3.21}$$



(a) (b) (c)

Fig. 3. The propagation of u in the two-lump solution (3.21) when (a) $t = -50$, (b) $t = 0$ and (c) $t = 50$.



(a) (b) (c)

Fig. 4. The propagation of v in the two-lump solution (3.21) when (a) $t = -50$, (b) $t = 0$ and (c) $t = 50$.

Figs. 3 and 4 show three three-dimensional plots of the two-lump solutions u and v at several time steps. We see that the two-lump solution u possesses two distinct peaks in the propagation process, and decays algebraically in all directions in the (x, y)-plane. The lump wave with high amplitude is behind the lump wave with low amplitude. Because the lump wave with high amplitude moves faster than the other, after a period of time, the lump wave with high amplitude catches up with the other to run ahead.

4. Concluding remarks

In summary, we furnished a Wronskian formulation for the extended second KP equation (1.4), using its Hirota trilinear form. The resulting Wronskian determinant structure generated N-soliton solutions and resonant wave solutions. Moreover, generic one-lump and two-lump solutions were obtained via the long wave limit technique.

We also remark that in the N-soliton solution (2.24) with (2.23), the parameters k_i, l_i and w_i, defining the wave numbers and frequency, satisfy the following dispersion relation

$$k_i^4 l_i + b l_i^3 + c k_i^2 w_i = 0, \tag{4.1}$$

which involves higher-order structure compared with the dispersion relation of the KPI equation [31]. Thus the numerators of higher-order rational dispersion relations are of degree 5 in the one-lump solution (3.15). More research problems need to be investigated, including how to apply trilinear forms to explore interaction solutions and integrability [12,25,32,36,37] for the extended second KP equation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

The proof of Theorem 2.1 is as follows:

Proof. By the condition (2.4), various derivatives of the Wronskian determinant can be computed as follows:

$$
\begin{aligned}
&f_x = |\widehat{N-2}, N|, \quad f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|,\\
&f_{xxx} = |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|,\\
&f_{xxxx} = |\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N+1|\\
&\qquad + 2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|,\\
&f_y = \beta_1\big(-|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|\big),\\
&f_{xy} = \beta_1\big(-|\widehat{N-4}, N-2, N-1, N| + |\widehat{N-2}, N+2|\big),\\
&f_{xxy} = \beta_1\big(-|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-4}, N-2, N-1, N+1|\\
&\qquad + |\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|\big),\\
&f_{xxxy} = \beta_1\big(-|\widehat{N-6}, N-4, N-3, N-2, N-1, N| - |\widehat{N-4}, N-2, N, N+1|\\
&\qquad - 2|\widehat{N-5}, N-3, N-2, N-1, N+1| + |\widehat{N-3}, N, N+2|\\
&\qquad + 2|\widehat{N-3}, N-1, N+3| + |\widehat{N-2}, N+4|\big),\\
&f_{xxxxy} = \beta_1\big(-|\widehat{N-7}, N-5, N-4, N-3, N-2, N-1, N| - 3|\widehat{N-5}, N-3, N-2, N, N+1|\\
&\qquad - 2|\widehat{N-5}, N-3, N-2, N-1, N+2| - 3|\widehat{N-6}, N-4, N-3, N-2, N-1, N+1|\\
&\qquad - |\widehat{N-4}, N-1, N, N+1| + 2|\widehat{N-4}, N-2, N-1, N+3| + |\widehat{N-2}, N+5|\\
&\qquad + |\widehat{N-3}, N+1, N+2| + 3|\widehat{N-3}, N, N+3| + 3|\widehat{N-3}, N-1, N+4|\big),\\
&f_{yy} = \beta_1^2\big(|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-4}, N-2, N-1, N+1|\\
&\qquad + 2|\widehat{N-3}, N, N+1| - |\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|\big),\\
&f_{yyy} = \beta_1^3\big(-|\widehat{N-7}, N-5, N-4, N-3, N-2, N-1, N| + |\widehat{N-6}, N-4, N-3, N-2, N-1, N+1|\\
&\qquad - 3|\widehat{N-5}, N-3, N-2, N, N+1| + 2|\widehat{N-5}, N-3, N-2, N-1, N+2|\\
&\qquad + 3|\widehat{N-4}, N-1, N, N+1| - 2|\widehat{N-4}, N-2, N-1, N+3| - 3|\widehat{N-3}, N+1, N+2|\\
&\qquad + 3|\widehat{N-3}, N, N+3| - |\widehat{N-3}, N-1, N+4| + |\widehat{N-2}, N+5|\big),\\
&f_t = \beta_2\big(-|\widehat{N-5}, N-3, N-2, N-1, N| + |\widehat{N-4}, N-2, N-1, N+1|\\
&\qquad - |\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|\big),\\
&f_{xt} = \beta_2\big(-|\widehat{N-6}, N-4, N-3, N-2, N-1, N| + |\widehat{N-4}, N-2, N, N+1|\\
&\qquad - |\widehat{N-3}, N, N+2| + |\widehat{N-2}, N+4|\big),\\
&f_{xxt} = \beta_2\big(-|\widehat{N-7}, N-5, N-4, N-3, N-2, N-1, N| + |\widehat{N-5}, N-3, N-2, N, N+1|\\
&\qquad - |\widehat{N-6}, N-4, N-3, N-2, N-1, N+1| + |\widehat{N-4}, N-1, N, N+1|\\
&\qquad - |\widehat{N-3}, N+1, N+2| - |\widehat{N-3}, N, N+3| + |\widehat{N-3}, N-1, N+4| + |\widehat{N-2}, N+5|\big).
\end{aligned}
$$

Substituting these derivatives into Eq. (2.2) and using the condition (2.5), we can now calculate that

$$\begin{aligned}
&f^2 f_{xxxxy} + b f^2 f_{yyy} + c f^2 f_{xxt} \\
&\quad = -8\beta_1 |\widehat{N-1}|^2 \Big[|\widehat{N-5}, N-3, N-2, N, N+1| - |\widehat{N-3}, N, N+3| \Big],
\end{aligned}$$

$$\begin{aligned}
&- f f_{xxxx} f_y + 2 f f_{xx} f_{xxy} - c f f_{xx} f_t - 2c f f_x f_{xt} - 4 f f_x f_{xxxy} - 3b f f_y f_{yy} \\
&\quad = 8\beta_1 |\widehat{N-1}| \Big[\big(|\widehat{N-3}, N-1, N| - |\widehat{N-2}, N+1|\big) |\widehat{N-3}, N, N+1| \\
&\qquad + |\widehat{N-2}, N| \big(|\widehat{N-5}, N-3, N-2, N-1, N+1| + |\widehat{N-4}, N-2, N, N+1| \\
&\qquad - |\widehat{N-3}, N, N+2| - |\widehat{N-3}, N-1, N+3|\big) + |\widehat{N-2}, N+3| |\widehat{N-3}, N-1, N| \\
&\qquad - |\widehat{N-2}, N+1| |\widehat{N-5}, N-3, N-2, N-1, N| \Big],
\end{aligned}$$

$$\begin{aligned}
&4 f_x f_{xxx} f_y - 2 f_{xx}^2 f_y + 2b f_y^3 - 4 f_{xx} f_x f_{xy} + 2c f_x^2 f_t + 4 f_x^2 f_{xxy} \\
&\quad = 8\beta_1 \Big[\big(|\widehat{N-3}, N-1, N| - |\widehat{N-2}, N+1|\big) |\widehat{N-2}, N+1| |\widehat{N-3}, N-1, N| \\
&\qquad - \big(|\widehat{N-3}, N-1, N| - |\widehat{N-2}, N+1|\big) |\widehat{N-2}, N| |\widehat{N-3}, N-1, N+1| \\
&\qquad + |\widehat{N-2}, N| \big(|\widehat{N-2}, N| |\widehat{N-3}, N-1, N+2| - |\widehat{N-2}, N+2| |\widehat{N-3}, N-1, N| \\
&\qquad + |\widehat{N-2}, N+1| |\widehat{N-4}, N-2, N-1, N| - |\widehat{N-2}, N| |\widehat{N-4}, N-2, N-1, N+1|\big) \Big].
\end{aligned}$$

Therefore, the left hand side of Eq. (2.2) yields

$$\begin{aligned}
&8\beta_1 \Big[|\widehat{N-1}| \Big(|\widehat{N-1}| |\widehat{N-3}, N, N+3| - |\widehat{N-2}, N| |\widehat{N-3}, N-1, N+3| \\
&\quad + |\widehat{N-2}, N+3| |\widehat{N-3}, N-1, N| \Big) - |\widehat{N-1}| \Big(|\widehat{N-1}| |\widehat{N-5}, N-3, N-2, N, N+1| \\
&\quad - |\widehat{N-2}, N| |\widehat{N-5}, N-3, N-2, N-1, N+1| + |\widehat{N-2}, N+1| |\widehat{N-5}, N-3, N-2, N-1, N| \Big) \\
&\quad - |\widehat{N-2}, N| \Big(|\widehat{N-1}| |\widehat{N-3}, N, N+2| - |\widehat{N-2}, N| |\widehat{N-3}, N-1, N+2| \\
&\quad + |\widehat{N-2}, N+2| |\widehat{N-3}, N-1, N| \Big) + |\widehat{N-2}, N| \Big(|\widehat{N-1}| |\widehat{N-4}, N-2, N, N+1| \\
&\quad - |\widehat{N-2}, N| |\widehat{N-4}, N-2, N-1, N+1| + |\widehat{N-2}, N+1| |\widehat{N-4}, N-2, N-1, N| \Big) \\
&\quad + \Big(|\widehat{N-3}, N-1, N| - |\widehat{N-2}, N+1| \Big) \Big(|\widehat{N-1}| |\widehat{N-3}, N, N+1| \\
&\quad - |\widehat{N-2}, N| |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+1| |\widehat{N-3}, N-1, N| \Big) \Big].
\end{aligned} \tag{A.1}$$

It is obvious that the sums of three terms in parenthesis in the above expression (A.1) are all Plücker relations [10]; so the left hand side of Eq. (2.2) is equal to zero and $f = W = |\widehat{N-1}|$ solves the trilinear Eq. (2.2).

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