Pramana – J. Phys. (2019) 93:4
https://doi.org/10.1007/s12043-019-1752-0

Pfaffians of B-type Kadomtsev–Petviashvili equation and complexitons to a class of nonlinear partial differential equations in (3+1) dimensions

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MS received 21 February 2018; revised 4 November 2018; accepted 16 November 2018; published online 16 April 2019

Abstract. The aim of this paper is to investigate a class of generalised Kadomtsev–Petviashvili (KP) and B-type Kadomtsev–Petviashvili (BKP) equations, which include many important nonlinear evolution equations as its special cases. By applying the fundamental Pfaffian identity, a general Pfaffian formulation is established and all the involved generating functions for Pfaffian entries need to satisfy a system of combined linear partial differential equations. The illustrative examples of the presented Pfaffian solutions are given for the (3+1)-dimensional generalised KP, Jimbo–Miwa and BKP equations. Moreover, we use the linear superposition principle to generate exponential travelling wave solutions and mixed resonant solutions of the considered equations.

Keywords. Generalised Kadomtsev–Petviashvili and B-type Kadomtsev–Petviashvili equations; Pfaffian formulation; sufficient conditions; N-wave solutions; complexitons.

PACS Nos **02.30.Ik; 05.45.Yv**

1. Introduction

The study on exact solutions to nonlinear partial differential equations, especially soliton solutions or multiple wave solutions, has attracted more and more attention from physicists as well as mathematicians in applied mathematics, fluid mechanics, plasma physics, chemistry, biology and other fields. It is well known that the Hirota bilinear form plays a crucial role in the soliton theory. Once a nonlinear equation is written in bilinear form, one can systematically derive particular solutions including determinant-type solutions with a Wronskian or Grammian structure [1–5]. Nevertheless, not all nonlinear equations have determinant-type solutions, and some soliton equations possess Pfaffian solutions complementing Wronskian and Grammian determinant solutions [6–9]. By using the formulation in terms of determinants or Pfaffians, various classes of interesting exact solutions, such as solitons solutions, rational solutions, positons and complexiton solutions can be derived [3,4,7,10,11].

In this paper, with the help of specific mathematical techniques, a new and generalised Hirota bilinear equation is proposed and studied, which reads as

$$\begin{aligned}(D_x^3D_y &+ a_{12}D_xD_y + a_{13}D_xD_z + a_{14}D_xD_t \\ &+ a_{23}D_yD_z + a_{24}D_yD_t + a_{34}D_zD_t + a_{11}D_x^2 \\ &+ a_{22}D_y^2 + a_{33}D_z^2) f\cdot f = 0, \qquad (1)\end{aligned}$$

where D_x, D_y, D_z and D_t are the Hirota bilinear differential operators [12] and the coefficients a_{ij}'s ($1\le i, j\le 4$) are the real parameters. Equivalently, we have

$$\begin{aligned}(f_{xxxy} &+ a_{12}f_{xy} + a_{13}f_{xz} + a_{14}f_{xt} + a_{23}f_{yz} + a_{24}f_{yt} \\ &+ a_{34}f_{zt} + a_{11}f_{xx} + a_{22}f_{yy} + a_{33}f_{zz})f \\ &- f_{xxx}f_y + 3f_{xx}f_{xy} - 3f_xf_{xxy}\end{aligned}$$

$$- a_{12} f_x f_y - a_{13} f_x f_z - a_{14} f_x f_t$$
$$- a_{23} f_y f_z - a_{24} f_y f_t - a_{34} f_z f_t$$
$$- a_{11} f_x^2 - a_{22} f_y^2 - a_{33} f_z^2 = 0. \quad (2)$$

Under the transformation

$$u = \frac{6}{\alpha}(\ln f)_x, \quad (3)$$

this equation is mapped into

$$u_{xxxy} + \alpha(u_x u_y)_x + a_{12}u_{xy} + a_{13}u_{xz}$$
$$+a_{14}u_{xt} + a_{23}u_{yz} + a_{24}u_{yt} + a_{34}u_{zt}$$
$$+a_{11}u_{xx} + a_{22}u_{yy} + a_{33}u_{zz} = 0, \quad (4)$$

where α is a non-zero parameter.

In the study of nonlinear physical phenomena, many nonlinear equations of mathematical physics, such as the (3+1)-dimensional Jimbo–Miwa equation [13–15], the (3+1)-dimensional generalised Kadomtsev–Petviashvili (KP) equation [3] and the (3 + 1)-dimensional generalised B-type Kadomtsev–Petviashvili (BKP) equation [6,16] are special cases of eq. (4). We list the following two interesting special examples in (3+1) dimensions which provide a physical background of the generalised nonlinear equation (4).

- The second equation in the well-known KP hierarchy is the Jimbo–Miwa equation, which is used for describing the propagation of three-dimensional nonlinear waves in physics. This equation does not pass the well-known integrability tests [13]. If $\alpha = 3, a_{12} = a_{14} = a_{23} = a_{34} = a_{11} = a_{22} = a_{33} = 0, a_{24} = -1, a_{13} = -3$ and $\alpha = 3, a_{12} = a_{13} = a_{14} = a_{23} = a_{34} = a_{11} = a_{22} = 0, a_{24} = 2, a_{33} = -3$, then eq. (4) reduces to the (3 + 1)-dimensional equations of Jimbo–Miwa type presented in [14]:

$$u_{yt} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz} = 0 \quad (5)$$

and

$$u_{xxxy} + 3(u_x u_y)_x + 2u_{yt} - 3u_{zz} = 0 \quad (6)$$

respectively. Through the logarithmical derivative transformation $u = 2(\ln f)_x$, the nonlinear equations (5) and (6) are written as

$$(D_x^3 D_y - D_t D_y - 3D_x D_z) f \cdot f = 0 \quad (7)$$

and

$$(D_x^3 D_y + 2D_t D_y - 3D_z^2) f \cdot f = 0, \quad (8)$$

respectively.

- If $\alpha = 3, a_{12} = a_{13} = a_{14} = a_{23} = a_{24} = a_{22} = a_{33} = 0, a_{34} = -1$ and $a_{11} = -3$, then eq. (4) becomes the (3+1)-dimensional nonlinear Ma–Fan equation [14,17]:

$$u_{zt} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} = 0, \quad (9)$$

which belongs to a class of generalised BKP equations. The BKP equation is a physically important equation that exhibits the evolution of nonlinear waves in areas such as plasma physics, fluid mechanics, solid state physics and optical fibres. Thus, the (3 + 1)-dimensional nonlinear Ma–Fan equation can also have good physical characteristics, as a higher-dimensional extension of the BKP equation. Under the dependent variable transformation $u = 2(\ln f)_x$, eq. (9) is mapped into

$$(D_x^3 D_y - D_t D_z - 3D_x^2) f \cdot f = 0. \quad (10)$$

This paper is structured as follows. In §2, by applying the fundamental Pfaffian identity, a Pfaffian formulation is established for eq. (1), with all generating functions for Pfaffian entries satisfying a system of combined linear partial differential equations. The fundamental Pfaffian identity is the key for constructing the Pfaffian formulation. In §3, based on the presented theory, illustrative examples will be presented, including the (3 + 1)-dimensional generalised KP, BKP and Jimbo–Miwa equations. Additionally, the linear superposition principles of exponential, hyperbolic and trigonometric function solutions are also proposed for eq. (1) in §4. Finally, our conclusions and remarks are given in §5.

2. Pfaffian formulation

We first list some of the basic results in terms of Pfaffians. By introducing an $2N \times 2N$ skew-symmetric matrix $A = (a_{ij})_{1\le i,j\le 2N}$, the Pfaffian $\mathrm{Pf}(A)$ of A is defined conventionally as follows [12]:

$$\mathrm{Pf}(A) = (\alpha_1, \alpha_2, \ldots, \alpha_{2N})$$
$$= \left| \begin{matrix} a_{12} & a_{13} & \cdots & a_{1,2N} \\ & a_{23} & \cdots & a_{2,2N} \\ & & \ddots & \vdots \\ & & & a_{2N-1,2N} \end{matrix} \right|$$
$$= \sum_{\sigma} \mathrm{sgn}(\sigma) a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2N-1} i_{2N}},$$

where the summation is taken over all permutations

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & 2N \\ i_1 & i_2 & \cdots & i_{2N} \end{pmatrix}$$

with

$$i_1 < i_2, i_3 < i_4, \ldots, i_{2N-1} < i_{2N}, i_1 < i_3$$
$$< \cdots < i_{2N-1}$$

and $\mathrm{sgn}(\sigma) = \pm 1$ means the parity of the permutations σ. For instance, when $N = 1, 2$, the Pfaffians can be expressed as

$$\begin{aligned}\mathrm{Pf}(A) &= (\alpha_1, \alpha_2) = a_{12},\\ \mathrm{Pf}(A) &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\\ &= a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.\end{aligned}$$

Moreover, there exist Pfaffian identities similar to the Jacobi identity for determinants. The simplest Pfaffian identity used is [12]

$$\begin{aligned}&(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 1, 2, \ldots, 2N)(1, 2, \ldots, 2N)\\ &\quad = (\alpha_1, \alpha_2, 1, 2, \ldots, 2N)(\alpha_3, \alpha_4, 1, 2, \ldots, 2N)\\ &\quad\quad -(\alpha_1, \alpha_3, 1, 2, \ldots, 2N)(\alpha_2, \alpha_4, 1, 2, \ldots, 2N)\\ &\quad\quad +(\alpha_1, \alpha_4, 1, 2, \ldots, 2N)(\alpha_2, \alpha_3, 1, 2, \ldots, 2N).\end{aligned} \tag{11}$$

Here we to focus on Pfaffian solutions and establish a set of sufficient conditions that make the Pfaffian to solve the bilinear equation (1).

Theorem 1. *Bilinear equation (1) has the Pfaffian form solution*

$$\begin{aligned}&f_N = \mathrm{Pf}(a_{ij})_{1\le i\le j\le 2N},\\ &a_{ij} = c_{ij} + \int^x D_x\psi_i\psi_j\,\mathrm{d}x, \quad i, j = 1, 2, \ldots, 2N,\end{aligned} \tag{12}$$

where all $\psi_i = \psi_i(x, y, z, t)$ satisfy the following linear differential equations:

$$\begin{aligned}&\frac{\partial\psi_i}{\partial y} = b_1\frac{\partial\psi_i}{\partial x_{-1}}, \quad \frac{\partial\psi_i}{\partial z} = b_2\frac{\partial\psi_i}{\partial x} + b_5\frac{\partial\psi_i}{\partial x_{-1}},\\ &\frac{\partial\psi_i}{\partial t} = b_3\frac{\partial^3\psi_i}{\partial x^3} + b_4\frac{\partial\psi_i}{\partial x_{-1}},\end{aligned} \tag{13}$$

where the constants $c_{ij} = -c_{ji}$, each ψ_i has the boundary condition $\psi_i \to 0$ as $x \to -\infty$ for $i = 1, 2, \ldots, 2N$, and $\partial\psi_i/\partial x_{-1}$ is defined by

$$\frac{\partial\psi_i}{\partial x_{-1}} = \int^x \psi_i\,\mathrm{d}x$$

and b_i, $1 \le i \le 5$, need to satisfy

$$\begin{cases} a_{14}b_3 + a_{34}b_2b_3 = 0,\\ b_1 + a_{24}b_1b_3 + a_{34}b_3b_5 = 0,\\ a_{23}b_1b_5 + a_{24}b_1b_4 + a_{34}b_4b_5 + a_{22}b_1^2\\ \quad +a_{33}b_5^2 = 0,\\ a_{12}b_1 + a_{13}b_5 + a_{14}b_4 + a_{23}b_1b_2 + a_{34}b_2b_4\\ \quad +2a_{33}b_2b_5 = 0,\\ 2b_1 + a_{13}b_2 - a_{24}b_1b_3 - a_{34}b_3b_5 + a_{11}\\ \quad +a_{33}b_2^2 = 0.\end{cases} \tag{14}$$

Proof. Let us consider the Pfaffian f_N as

$$f_N = (1, 2, \ldots, 2N) = (\bullet), \tag{15}$$

where $(i, j) = a_{ij}$ and $(d_m, d_n) = 0$ for all integers m and n. If we introduce the new Pfaffian entries

$$(d_n, i) = \frac{\partial^n\psi_i}{\partial x^n}, \quad (d_{-n}, i) = \frac{\partial^n\psi_i}{\partial x_{-1}^n} \quad \text{for } n \ge 0, \tag{16}$$

where

$$\frac{\partial^n\psi_i}{\partial x_{-1}^n} = \int^x\int^x\ldots\int^x \psi_i\,\mathrm{d}x\,\mathrm{d}x\ldots\mathrm{d}x.$$

Then we have the following derivatives of the elements (i, j) $(1 \le i \le j \le 2N)$:

$$\begin{aligned}\frac{\partial}{\partial x}(i, j) &= \psi_j\frac{\partial\psi_i}{\partial x} - \psi_i\frac{\partial\psi_j}{\partial x} = (d_0, d_1, i, j),\\ \frac{\partial}{\partial y}(i, j) &= b_1\frac{\partial}{\partial x_{-1}}\int^x\left[\psi_j\frac{\partial\psi_i}{\partial x} - \psi_i\frac{\partial\psi_j}{\partial x}\right]\mathrm{d}x\\ &= b_1\left[\psi_i\frac{\partial\psi_j}{\partial x_{-1}} - \psi_j\frac{\partial\psi_i}{\partial x_{-1}}\right]\\ &= b_1(d_{-1}, d_0, i, j),\\ \frac{\partial}{\partial z}(i, j) &= b_2(d_0, d_1, i, j) + b_5(d_{-1}, d_0, i, j),\\ \frac{\partial}{\partial t}(i, j) &= b_3\left[\psi_j\frac{\partial^3\psi_i}{\partial x^3} - \psi_i\frac{\partial^3\psi_j}{\partial x^3}\right.\\ &\quad\left. - 2\left(\frac{\partial\psi_j}{\partial x}\frac{\partial^2\psi_i}{\partial x^2} - \frac{\partial\psi_i}{\partial x}\frac{\partial^2\psi_j}{\partial x^2}\right)\right]\\ &\quad + b_4\left[\psi_i\frac{\partial\psi_j}{\partial x_{-1}} - \psi_j\frac{\partial\psi_i}{\partial x_{-1}}\right]\\ &= b_3\left[(d_0, d_3, i, j) - 2(d_1, d_2, i, j)\right]\\ &\quad + b_4(d_{-1}, d_0, i, j).\end{aligned}$$

By using the differential rules for Pfaffians introduced in [12], we can compute various derivatives for f_N with respect to the variables x, y, z, t as follows:

$$\begin{aligned}&f_{N,x} = (d_0, d_1, \bullet), \quad f_{N,y} = b_1(d_{-1}, d_0, \bullet),\\ &f_{N,z} = b_2(d_0, d_1, \bullet) + b_5(d_{-1}, d_0, \bullet),\\ &f_{N,t} = b_3[(d_0, d_3, \bullet) - 2(d_1, d_2, \bullet)] + b_4(d_{-1}, d_0, \bullet),\\ &f_{N,xx} = (d_0, d_2, \bullet),\\ &f_{N,xxx} = (d_1, d_2, \bullet) + (d_0, d_3, \bullet),\\ &f_{N,xy} = b_1(d_{-1}, d_1, \bullet), \quad f_{N,yy} = b_1^2(d_{-2}, d_0, \bullet),\\ &f_{N,yz} = b_1b_5(d_{-2}, d_0, \bullet) + b_1b_2(d_{-1}, d_1, \bullet),\\ &f_{N,xxy} = b_1[(d_{-1}, d_2, \bullet) + (d_0, d_1, \bullet)],\\ &f_{N,xxxy} = b_1[(d_{-1}, d_3, \bullet) + 2(d_0, d_2, \bullet)\\ &\qquad + (d_{-1}, d_0, d_1, d_2, \bullet)],\\ &f_{N,zz} = b_2^2(d_0, d_2, \bullet) + 2b_2b_5(d_{-1}, d_1, \bullet)\\ &\qquad + b_5^2(d_{-2}, d_0, \bullet),\end{aligned}$$

$$f_{N,xz} = b_2(d_0, d_2, \bullet) + b_5(d_{-1}, d_1, \bullet),$$
$$f_{N,xt} = b_3[(d_0, d_4, \bullet) - (d_1, d_3, \bullet)] + b_4(d_{-1}, d_1, \bullet),$$
$$f_{N,yt} = b_1b_3[(d_{-1}, d_3, \bullet) - (d_0, d_2, \bullet) - 2(d_{-1}, d_0, d_1, d_2, \bullet)] + b_1b_4(d_{-2}, d_0, \bullet),$$
$$f_{N,zt} = b_3b_5[(d_{-1}, d_3, \bullet) - (d_0, d_2, \bullet) - 2(d_{-1}, d_0, d_1, d_2, \bullet)] + b_4b_5(d_{-2}, d_0, \bullet) + b_2b_4(d_{-1}, d_1, \bullet) + b_2b_3[(d_0, d_4, \bullet) - (d_1, d_3, \bullet)],$$

where the abbreviated notation $\bullet$ denotes the list of indices $1, 2, \ldots, 2N$ common to each Pfaffian. Applying system (14), we can further obtain

$$\begin{aligned}&(f_{xxxy} + a_{12}f_{xy} + a_{13}f_{xz} + a_{14}f_{xt} + a_{23}f_{yz} \\ &\quad + a_{24}f_{yt} + a_{34}f_{zt} + a_{11}f_{xx} + a_{22}f_{yy} + a_{33}f_{zz})f \\ &= (b_1 - 2a_{24}b_1b_3 - 2a_{34}b_3b_5)(\bullet)(d_{-1}, d_0, d_1, d_2, \bullet) \\ &= 3b_1(\bullet)(d_{-1}, d_0, d_1, d_2, \bullet),\end{aligned} \tag{17}$$

$$\begin{aligned}&-f_{xxx}f_y + 3f_{xx}f_{xy} - 3f_xf_{xxy} - a_{12}f_xf_y \\ &\quad - a_{13}f_xf_z - a_{14}f_xf_t - a_{23}f_yf_z - a_{24}f_yf_t \\ &\quad - a_{34}f_zf_t - a_{11}f_x^2 - a_{22}f_y^2 - a_{33}f_z^2 \\ &= -(b_1 - 2a_{24}b_1b_3 - 2a_{34}b_3b_5)(d_{-1}, d_0, \bullet) \\ &\quad \times (d_1, d_2, \bullet) + 3b_1[(d_0, d_2, \bullet)(d_{-1}, d_1, \bullet) \\ &\quad - (d_0, d_1, \bullet)(d_{-1}, d_2, \bullet)] \\ &= 3b_1[(d_0, d_2, \bullet)(d_{-1}, d_1, \bullet) - (d_0, d_1, \bullet) \\ &\quad \times (d_{-1}, d_2, \bullet)] - 3b_1(d_{-1}, d_0, \bullet)(d_1, d_2, \bullet).\end{aligned} \tag{18}$$

Therefore, we have

$$\begin{aligned}&(f_{xxxy} + a_{12}f_{xy} + a_{13}f_{xz} + a_{14}f_{xt} \\ &\quad + a_{23}f_{yz} + a_{24}f_{yt} + a_{34}f_{zt} \\ &\quad + a_{11}f_{xx} + a_{22}f_{yy} + a_{33}f_{zz})f - f_{xxx}f_y \\ &\quad + 3f_{xx}f_{xy} - 3f_xf_{xxy} - a_{12}f_xf_y - a_{13}f_xf_z \\ &\quad - a_{14}f_xf_t - a_{23}f_yf_z - a_{24}f_yf_t - a_{34}f_zf_t \\ &\quad - a_{11}f_x^2 - a_{22}f_y^2 - a_{33}f_z^2 \\ &= 3b_1[(d_{-1}, d_0, d_1, d_2, \bullet)(\bullet) \\ &\quad - (d_{-1}, d_0, \bullet)(d_1, d_2, \bullet) + (d_0, d_2, \bullet)(d_{-1}, d_1, \bullet) \\ &\quad - (d_0, d_1, \bullet)(d_{-1}, d_2, \bullet)] = 0.\end{aligned} \tag{19}$$

It is easy to see that the final expression in (19) is nothing but the Pfaffian identity (11). As a result, we have shown that the Pfaffian $f_N = (1, 2, \ldots, 2N)$ with the conditions (12)–(14) solves eq. (1). □

3. Applications to generalised KP and BKP equations

As we stated in §1, the generalised nonlinear equation (4) includes many important nonlinear evolution equations as its special cases. In the following, we present a few examples in 3 + 1 dimensions to shed light on the applications of Theorem 1. According to system (14), there are some special cases to determine b_i, $1 \le i \le 5$, as follows.

3.1 *The case of $a_{24} \neq 0$ but $b_5 = 0$*

In this case, if $a_{14}a_{34} \neq 0$ but $a_{12} = a_{23} = 0$, then a direct computation yields

$$\begin{aligned}b_1 &= \frac{a_{13}a_{14}a_{34} - a_{11}a_{34}^2 - a_{33}a_{14}^2}{3a_{34}^2}, \quad b_2 = -\frac{a_{14}}{a_{34}}, \\ b_3 &= -\frac{1}{a_{24}}, \\ b_4 &= \frac{a_{11}a_{22}a_{34}^2 + a_{14}^2a_{22}a_{33} + a_{13}a_{14}a_{34}a_{22}}{3a_{24}a_{34}^2}\end{aligned} \tag{20}$$

to keep the non-triviality $b_1b_2 \neq 0$. Based on the above expression, two illustrative examples will be given.

The first example is the (3 + 1)-dimensional generalised KP equation

$$u_{xxxy} + 3(u_xu_y)_x + u_{tx} + u_{yt} + u_{zt} - u_{zz} = 0, \tag{21}$$

which has been presented by Wazwaz and El-Tantawy [18]. By the typical transformation $u = 2(\ln f)_x$, the corresponding Hirota bilinear form reads as

$$(D_x^3D_y + D_tD_x + D_tD_y + D_tD_z - D_z^2)f \cdot f = 0. \tag{22}$$

A set of sufficient conditions that make Pfaffian (12) a solution to the (3+1)-dimensional bilinear equation (22) can be expressed as follows:

$$\frac{\partial \psi_i}{\partial y} = \frac{1}{3}\frac{\partial \psi_i}{\partial x_{-1}}, \quad \frac{\partial \psi_i}{\partial z} = -\frac{\partial \psi_i}{\partial x}, \quad \frac{\partial \psi_i}{\partial t} = -\frac{\partial^3 \psi_i}{\partial x^3}. \tag{23}$$

The other example is the following extended (3 + 1)-dimensional Jimbo–Miwa equation

$$\begin{aligned}&u_{xxxy} + 3u_yu_{xx} + 3u_xu_{xy} \\ &\quad + 2(u_{xt} + u_{yt} + u_{zt}) - 3u_{xz} = 0,\end{aligned} \tag{24}$$

which has been introduced in [19] recently. Through the dependent variable transformation $u = 2(\ln f)_x$, this equation is written as

$$(D_x^3 D_y + 2D_x D_t + 2D_y D_t + 2D_z D_t - 3D_x D_z) f \cdot f = 0. \quad (25)$$

Following expression (20), the corresponding sufficient conditions on the Pfaffian form solution read as

$$\frac{\partial \psi_i}{\partial y} = -\frac{\partial \psi_i}{\partial x_{-1}}, \quad \frac{\partial \psi_i}{\partial z} = -\frac{\partial \psi_i}{\partial x}, \quad \frac{\partial \psi_i}{\partial t} = -\frac{1}{2}\frac{\partial^3 \psi_i}{\partial x^3}. \quad (26)$$

If $a_{14} = a_{34} = 0$ but $a_{23} \neq 0$, then a similar direct computation shows that

$$b_1 = \frac{a_{12}a_{13}a_{23} - a_{33}a_{12}^2 - a_{11}a_{23}^2}{3a_{23}^2}, \quad b_2 = -\frac{a_{12}}{a_{23}}, \quad b_3 = -\frac{1}{a_{24}}, \quad b_4 = -\frac{a_{22}(a_{12}a_{13}a_{23} - a_{33}a_{12}^2 - a_{11}a_{23}^2)}{3a_{24}a_{23}^2} \quad (27)$$

to keep the non-triviality $b_1 b_2 \neq 0$.

If $a_{14} = a_{34} = a_{23} = 0$, then we automatically have $a_{12} = 0$ to keep the non-triviality $b_1 \neq 0$. A direct computation shows that

$$b_1 = \frac{-a_{11} - a_{33}b_2^2 - a_{13}b_2}{3}, \quad b_3 = -\frac{1}{a_{24}}, \quad b_4 = \frac{a_{11}a_{22} + a_{22}a_{33}b_2^2 + a_{13}a_{22}b_2}{3a_{24}}, \quad (28)$$

where b_2 is arbitrary to keep the non-triviality $b_1 b_2 \neq 0$. For instance, we have the Pfaffian form solution of the Hirota bilinear equation (8) defined by (12) with

$$\frac{\partial \psi_i}{\partial y} = b_2^2 \frac{\partial \psi_i}{\partial x_{-1}}, \quad \frac{\partial \psi_i}{\partial z} = b_2 \frac{\partial \psi_i}{\partial x}, \quad \frac{\partial \psi_i}{\partial t} = -\frac{1}{2}\frac{\partial^3 \psi_i}{\partial x^3}, \quad (29)$$

where b_2 is a free parameter. We next discuss the (3+1)-dimensional generalised BKP equation

$$u_{yt} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} + 3u_{zz} = 0, \quad (30)$$

which has been proposed and studied in [6,15,16]. This corresponds to the (3+1)-dimensional bilinear equation

$$(D_x^3 D_y - D_t D_y - 3D_x^2 - 3D_z^2) f \cdot f = 0 \quad (31)$$

through the dependent variable transformation

$$u = 2(\ln f)_x.$$

Similarly, by inspection, a set of sufficient conditions to the Hirota bilinear equation (31) is

$$\frac{\partial \psi_i}{\partial y} = (1 + b_2^2)\frac{\partial \psi_i}{\partial x_{-1}}, \quad \frac{\partial \psi_i}{\partial z} = b_2 \frac{\partial \psi_i}{\partial x}, \quad \frac{\partial \psi_i}{\partial t} = \frac{\partial^3 \psi_i}{\partial x^3}, \quad (32)$$

recently presented by Asaad and Ma [6].

3.2 *The case of $a_{24}b_5 \neq 0$ but $a_{14} = a_{34} = 0$*

In this case, if $(a_{12}^2 + a_{23}^2)(a_{13}^2 + a_{33}^2) \neq 0$, then a similar direct computation leads to

$$b_1 = \frac{-a_{11} - a_{33}b_2^2 - a_{13}b_2}{3}, \quad b_3 = -\frac{1}{a_{24}}, \quad b_5 = \frac{(a_{11} + a_{33}b_2^2 + a_{13}b_2)(a_{12} + a_{23}b_2)}{3(a_{13} + 2a_{33}b_2)},$$
$$b_4 = (a_{11} + a_{33}b_2^2 + a_{13}b_2)\left(\frac{a_{22}}{3a_{24}} - \frac{a_{23}(a_{12} + a_{23}b_2)}{3a_{24}(a_{13} + 2a_{33}b_2)} + \frac{a_{33}(a_{12} + a_{23}b_2)^2}{3a_{24}(a_{13} + 2a_{33}b_2)^2}\right), \quad (33)$$

where b_2 is arbitrary and $a_{13} + 2a_{33}b_2 \neq 0$, to keep the non-triviality $b_1 b_5 \neq 0$. Next we consider the second extended (3 + 1)-dimensional Jimbo–Miwa equation [19]

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3(u_{xz} + u_{yz} + u_{zz}) = 0. \quad (34)$$

Under the dependent variable transformation $u = 2(\ln f)_x$, this equation can be expressed as

$$(D_x^3 D_y + 2D_y D_t - 3D_x D_z - 3D_y D_z - 3D_z^2) f \cdot f = 0. \quad (35)$$

Based on the sufficient conditions in Theorem 1, the extended (3 + 1)-dimensional Jimbo–Miwa equation (34) has the following sufficient conditions on the Pfaffian form solution:

$$\frac{\partial \psi_i}{\partial y} = (b_2 + b_2^2)\frac{\partial \psi_i}{\partial x_{-1}}, \quad \frac{\partial \psi_i}{\partial z} = b_2 \frac{\partial \psi_i}{\partial x} - \frac{b_2^2(1 + b_2)}{1 + 2b_2}\frac{\partial \psi_i}{\partial x_{-1}},$$
$$\frac{\partial \psi_i}{\partial t} = -\frac{1}{2}\frac{\partial^3 \psi_i}{\partial x^3} - \frac{3b_2^2(1 + b_2)^2}{2(1 + 2b_2)^2}\frac{\partial \psi_i}{\partial x_{-1}}, \quad (36)$$

where $b_2 \neq 0, -1, -\frac{1}{2}$, which is a free parameter.

If $a_{13} = a_{33} = 0$ and $a_{23} \neq 0$, then a straightforward calculation yields

$$b_1 = -\frac{a_{11}}{3}, \quad b_2 = -\frac{a_{12}}{a_{23}}, \quad b_3 = -\frac{1}{a_{24}}, \quad b_4 = \frac{a_{11}a_{22} - 3a_{23}b_5}{3a_{24}}, \quad (37)$$

where b_5 is arbitrary to keep the non-triviality $b_1 b_2 \neq 0$.

If $a_{12}=a_{23}=0$ and $a_{33}\neq 0$, then a similar direct computation shows that

$$b_1=\frac{a_{13}^2-4a_{11}a_{33}}{12a_{33}},\quad b_2=-\frac{a_{13}}{2a_{33}},\quad b_3=-\frac{1}{a_{24}},$$
$$b_4=\frac{4a_{11}a_{22}a_{33}-a_{13}^2a_{22}}{12a_{24}a_{33}}-\frac{12a_{33}^2}{a_{24}a_{13}^2-4a_{11}a_{33}a_{24}}b_5^2, \tag{38}$$

where b_5 is arbitrary and $a_{13}^2-4a_{11}a_{33}\neq 0$ to keep the non-triviality $b_1b_2\neq 0$. For example, we give a new Pfaffian form solution of the Hirota bilinear equation (31) defined by (12) with

$$\frac{\partial\psi_i}{\partial y}=\frac{\partial\psi_i}{\partial x_{-1}},\quad \frac{\partial\psi_i}{\partial z}=b_5\frac{\partial\psi_i}{\partial x_{-1}},$$
$$\frac{\partial\psi_i}{\partial t}=\frac{\partial^3\psi_i}{\partial x^3}-3b_5^2\frac{\partial\psi_i}{\partial x_{-1}}, \tag{39}$$

where b_5 is an arbitrary constant, not to be zero. It is clear that the corresponding sufficient conditions on Pfaffian formulation are different from the ones (32) presented by Asaad and Ma [6].

3.3 *The case of $a_{24}a_{34}\neq 0$ but $a_{14}=0$*

If $a_{34}=0$ and $a_{14}\neq 0$, then we automatically have $b_1=0$ in terms of the first and second equations in (14). We point out that when eq. (1) reduces to the generalised KP equation [3], given as

$$(D_x^3D_y+D_tD_x+D_tD_y-D_z^2)f\cdot f=0, \tag{40}$$

the Pfaffian form solution defined by (12) and (13) is not available.

Let us next consider the case of $a_{24}a_{34}\neq 0$ and $a_{14}=0$. In this case, we automatically have $b_2=0$ to keep the non-triviality $b_1\neq 0$. If we take $a_{13}\neq 0$, then a solution of system (14) is determined by

$$b_1=-\frac{a_{11}}{3},\quad b_2=0,\quad b_3=\frac{a_{13}}{a_{12}a_{34}-a_{13}a_{24}},$$
$$b_4=\frac{a_{11}(a_{12}a_{13}a_{23}-a_{12}^2a_{33}-a_{13}^2a_{22})}{3a_{13}(a_{12}a_{34}-a_{13}a_{24})},$$
$$b_5=\frac{a_{11}a_{12}}{3a_{13}}, \tag{41}$$

where $a_{12}a_{34}-a_{13}a_{24}\neq 0$ to keep the non-triviality $b_1b_5(b_3^2+b_4^2)\neq 0$. Moreover, if $a_{12}=a_{13}=0$, then a similar direct computation shows that

$$b_1=-\frac{a_{11}}{3},\quad b_2=0,\quad b_3=\frac{a_{11}}{-a_{11}a_{24}+3a_{34}b_5},$$
$$b_4=\frac{3a_{23}a_{11}b_5-a_{22}a_{11}^2-9a_{33}b_5^2}{9a_{34}b_5-3a_{11}a_{24}},$$

where b_5 is arbitrary and $9a_{34}b_5-3a_{11}a_{24}\neq 0$ to keep the non-triviality $b_1b_5(b_3^2+b_4^2)\neq 0$.

3.4 *The case of $a_{24}=0$*

In this case, the second equation in (14) determines $a_{34}\neq 0$ and $b_3b_5\neq 0$ to keep the non-triviality $b_1\neq 0$. If $a_{12}=a_{23}=0$, then the first and fourth equations in (14) lead to

$$b_2=-\frac{a_{14}}{a_{34}},\quad a_{13}a_{34}=2a_{33}a_{14}. \tag{42}$$

After straightforward calculations, we get

$$b_1=\frac{a_{13}a_{14}a_{34}-a_{11}a_{34}^2-a_{33}a_{14}^2}{3a_{34}^2},$$
$$b_4=\frac{(a_{13}a_{14}a_{34}-a_{11}a_{34}^2-a_{33}a_{14}^2)(a_{22}a_{34}^2b_3^2+a_{33})}{3a_{34}^4b_3},$$
$$b_5=-\frac{(a_{13}a_{14}a_{34}-a_{11}a_{34}^2-a_{33}a_{14}^2)}{3a_{34}^3b_3}, \tag{43}$$

where b_3 is an arbitrary constant, not to be zero, to keep the non-triviality $b_1(b_2^2+b_5^2)\neq 0$. A Pfaffian form solution defined by (12) and (13) for the Ma–Fan equation (9) is just one special example with

$$\frac{\partial\psi_i}{\partial y}=\frac{\partial\psi_i}{\partial x_{-1}},\quad \frac{\partial\psi_i}{\partial z}=\frac{1}{b_3}\frac{\partial\psi_i}{\partial x_{-1}},$$
$$\frac{\partial\psi_i}{\partial t}=b_3\frac{\partial^3\psi_i}{\partial x^3}, \tag{44}$$

where b_3 is an arbitrary constant, not to be zero.

4. Resonant solitons and complexitons

As we know, the Hirota bilinear equations may possess linear subspaces of solutions [14,20]. Ma and Fan [14] and Ma *et al* [20] have established a sufficient and necessary criterion for the existence of linear subspaces of exponential travelling wave solutions to Hirota bilinear equations. Some examples with N-waves satisfying and not satisfying the dispersion relation have been constructed in [14,20,21]. This means that it is not necessary to satisfy the dispersion relation. Complexiton solutions, which are combinations of trigonometric function waves and exponential function waves, were proposed by Ma [22,23]. In what follows, we first describe the linear superposition principle for constructing exponential wave function solutions, and then obtain resonant solutions in terms of hyperbolic and trigonometric functions [24,25].

Theorem 2. (Linear superposition principle). *Let N-wave variables*

$$\eta_i = k_{1,i}x_1 + k_{2,i}x_2 + \cdots + k_{M,i}x_M, \quad 1 \le i \le N, \tag{45}$$

where $k_{j,i}$ are constants and a Hirota bilinear equation, denoted by

$$P(D_{x_1}, D_{x_2}, \ldots, D_{x_M}) f \cdot f = 0, \tag{46}$$

where P is an even polynomial in the indicated variables satisfying

$$P(k_{1,i} - k_{1,j}, \ldots, k_{M,i} - k_{M,j}) = 0, \quad 1 \le i \ne j \le N, \tag{47}$$

then any linear combination of the exponential waves e^{η_i}, $1 \le i \le N$, *solves the bilinear equation* (46) *and f is called an N-wave solution to eq.* (46).

Theorem 2 has been proved in [14,20].

For bilinear equation (1), the corresponding polynomial reads as

$$P(x, y, z, t) = x^3 y + a_{12}xy + a_{13}xz + a_{14}xt + a_{23}yz + a_{24}yt + a_{34}zt + a_{11}x^2 + a_{22}y^2 + a_{33}z^2. \tag{48}$$

Assume that the wave variables are

$$\eta_i = k_i x + b_1 k_i^{-1} y + (b_2 k_i + b_5 k_i^{-1}) z + (b_3 k_i^3 + b_4 k_i^{-1}) t, \quad 1 \le i \le N,$$

where k_i, $1 \le i \le N$, are arbitrary constants, but b_i, $1 \le i \le 5$, are constants to be determined.

By the linear superposition principle in Theorem 2, we find that the Hirota bilinear equation (1) corresponding to polynomial (48) has the following N-wave solution:

$$f = \sum_{i=1}^{N} \varepsilon_i f_i = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + b_1 k_i^{-1} y + (b_2 k_i + b_5 k_i^{-1}) z + (b_3 k_i^3 + b_4 k_i^{-1}) t}, \tag{49}$$

where ε_i's and k_i's are arbitrary constants, but b_i, $1 \le i \le 5$, satisfy system (14). An analysis of the existence of real b_i, $1 \le i \le 5$, can be given similarly from the results described in §3. Each exponential wave f_i in this solution does not satisfy the corresponding nonlinear dispersion relation, i.e.,

$$P(k_i, b_1 k_i^{-1}, b_2 k_i + b_5 k_i^{-1}, b_3 k_i^3 + b_4 k_i^{-1}) \ne 0.$$

Let us assume that $N = 2K$ is even, and take the choices

$$\eta_{2i-1} = k_{2i-1}x + b_1 k_{2i-1}^{-1} y + (b_2 k_{2i-1} + b_5 k_{2i-1}^{-1}) z + (b_3 k_{2i-1}^3 + b_4 k_{2i-1}^{-1}) t,$$
$$\eta_{2i} = (-k_{2i-1})x + b_1(-k_{2i-1})^{-1} y + [b_2(-k_{2i-1}) + b_5(-k_{2i-1})^{-1}] z + [b_3(-k_{2i-1})^3 + b_4(-k_{2i-1})^{-1}] t, \quad i = 1, 2, \ldots, K, \tag{50}$$

which yield $\eta_{2i} = -\eta_{2i-1}$. Then the Hirota bilinear equation (1) corresponding to polynomial (48) possesses an N-wave solution

$$f = \sum_{i=1}^{K} \left[(\varepsilon_{2i-1} + \varepsilon_{2i}) \cosh(\eta_{2i-1}) + (\varepsilon_{2i-1} - \varepsilon_{2i}) \sinh(\eta_{2i-1}) \right]. \tag{51}$$

If we take $\varepsilon_{2i} = \varepsilon_{2i-1}$, $i = 1, \ldots, K$, then

$$f = 2 \sum_{i=1}^{K} \varepsilon_{2i-1} \cosh(\eta_{2i-1}). \tag{52}$$

If we take $\varepsilon_{2i} = -\varepsilon_{2i-1}$, $i = 1, \ldots, K$, then

$$f = 2 \sum_{i=1}^{K} \varepsilon_{2i-1} \sinh(\eta_{2i-1}). \tag{53}$$

Furthermore, under the construction $k_{2i-1} \to k_{2i-1} \mathrm{I}$, $\mathrm{I} = \sqrt{(-1)}$, we may get the following N wave solutions formed by linear combinations of trigonometric functions

$$f = 2 \sum_{i=1}^{K} \varepsilon_{2i-1} \cos[k_{2i-1}x - b_1 k_{2i-1}^{-1} y + (b_2 k_{2i-1} - b_5 k_{2i-1}^{-1}) z - (b_3 k_{2i-1}^3 + b_4 k_{2i-1}^{-1}) t] \tag{54}$$

and

$$f = 2 \sum_{i=1}^{K} \varepsilon_{2i-1} \mathrm{I} \sin[k_{2i-1}x - b_1 k_{2i-1}^{-1} y + (b_2 k_{2i-1} - b_5 k_{2i-1}^{-1}) z - (b_3 k_{2i-1}^3 + b_4 k_{2i-1}^{-1}) t], \tag{55}$$

where ε_{2i-1}'s and k_{2i-1}'s are arbitrary constants, and b_i, $1 \le i \le 5$, satisfy system (14). Besides, we obtain the mixed-type function solutions such as complexiton solutions as follows:

$$f = \sum_{i=1}^{N} \Big(\varepsilon_i \cosh\big[k_i x + b_1 k_i^{-1} y + (b_2 k_i + b_5 k_i^{-1}) z + (b_3 k_i^3 + b_4 k_i^{-1}) t\big] + \lambda_i \cos\big[k_i x - b_1 k_i^{-1} y + (b_2 k_i - b_5 k_i^{-1}) z - (b_3 k_i^3 + b_4 k_i^{-1}) t\big] \Big) \tag{56}$$

and

$$f=\sum_{i=1}^{N}\Big(\varepsilon_i \sinh\big[k_i x+b_1k_i^{-1}y+(b_2k_i+b_5k_i^{-1})z +(b_3k_i^3+b_4k_i^{-1})t\big]+\lambda_i \sin\big[k_i x-b_1k_i^{-1}y +(b_2k_i-b_5k_i^{-1})z-(b_3k_i^3+b_4k_i^{-1})t\big]\Big), \qquad (57)$$

where ε_i's, λ_i's and k_i's are arbitrary constants, and $b_i, 1\le i\le 5$, satisfy system (14).

Let us next take the (3 + 1)-dimensional Ma–Fan equation (9) as an illustrative example to investigate the propagations of resonant solutions and complexitons. Firstly, by expression (49), the (3+1)-dimensional Ma–Fan equation (9) has the following N-wave solution:

$$u=2(\ln f)_x,\quad f=\sum_{i=1}^{N}\varepsilon_i \mathrm{e}^{k_i x+k_i^{-1}y+\frac{1}{b}k_i^{-1}z+bk_i^3 t}, \qquad (58)$$

where b is an arbitrary constant, not to be zero. When $N=2$, the single-front wave solution to eq. (9) is given by

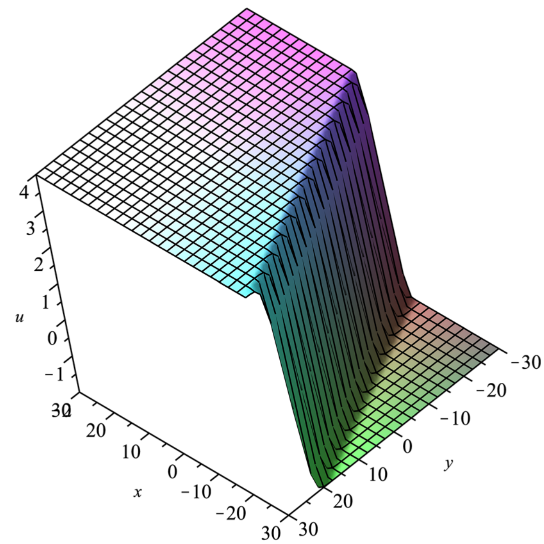

Figure 1. The plot of the single-front wave solution (59) with parameters: $\varepsilon_1=1, \varepsilon_2=1, k_1=-1, k_2=2, b=3, z=0, t=2$.

$$u=2\frac{k_1\varepsilon_1 \mathrm{e}^{k_1x+k_1^{-1}y+\frac{1}{b}k_1^{-1}z+bk_1^3t}+k_2\varepsilon_2 \mathrm{e}^{k_2x+k_2^{-1}y+\frac{1}{b}k_2^{-1}z+bk_2^3t}}{\varepsilon_1 \mathrm{e}^{k_1x+k_1^{-1}y+\frac{1}{b}k_1^{-1}z+bk_1^3t}+\varepsilon_2 \mathrm{e}^{k_2x+k_2^{-1}y+\frac{1}{b}k_2^{-1}z+bk_2^3t}}. \qquad (59)$$

Taking $N\ge 3$, we can obtain the resonant multiple wave solutions u to eq. (9). For instance, the resonant three-wave solution is presented by

$$u=2(\ln f)_x,$$

$$f=\varepsilon_1 \mathrm{e}^{k_1x+k_1^{-1}y+\frac{1}{b}k_1^{-1}z+bk_1^3t}+\varepsilon_2 \mathrm{e}^{k_2x+k_2^{-1}y+\frac{1}{b}k_2^{-1}z+bk_2^3t} +\varepsilon_3 \mathrm{e}^{k_3x+k_3^{-1}y+\frac{1}{b}k_3^{-1}z+bk_3^3t} +\varepsilon_4 \mathrm{e}^{k_4x+k_4^{-1}y+\frac{1}{b}k_4^{-1}z+bk_4^3t}. \qquad (60)$$

Figures 1 and 2 show three-dimensional graphs of the single-front wave and resonant two-wave solutions determined by expression (58) with special parameters, respectively. The resonant three- and four-wave solutions with special parameters are also plotted in figures 3 and 4, respectively. It is easy to see that the interaction will become more complicated with the increase of the positive integer N.

Secondly, we can obtain two types of complexiton solutions to the (3 + 1)-dimensional Ma–Fan equation (9) as

$$u=2(\ln f)_x,$$

$$f=\sum_{i=1}^{N}\left[\varepsilon_i \cosh\left(k_i x+k_i^{-1}y+\frac{1}{b}k_i^{-1}z+bk_i^3t\right) +\lambda_i \cos\left(k_i x-k_i^{-1}y-\frac{1}{b}k_i^{-1}z-bk_i^3t\right)\right] \qquad (61)$$

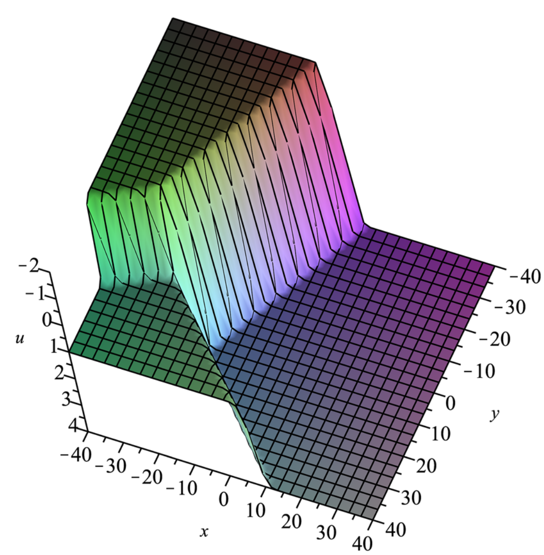

Figure 2. The plot of the two-wave solution (58) with parameters: $N=3, \varepsilon_1=1, \varepsilon_2=1, \varepsilon_3=1, k_1=-1, k_2=2, k_3=0.5, b=3, z=0, t=2$.

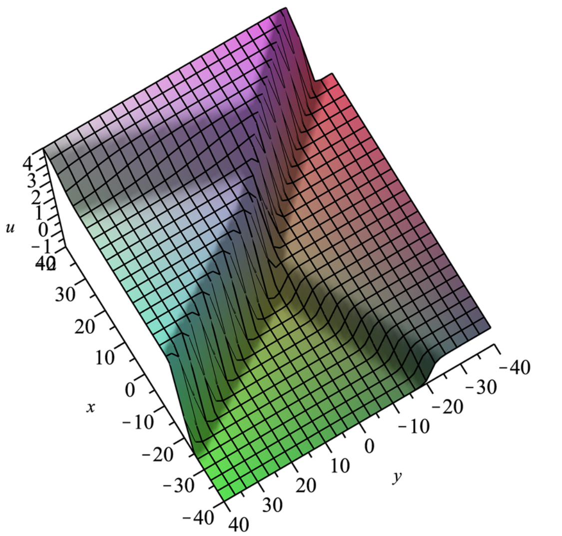

Figure 3. The plot of the three-wave solution (60) with parameters: $\varepsilon_1 = 1, \varepsilon_2 = 1, \varepsilon_3 = 1, \varepsilon_4 = 1, k_1 = -0.5, k_2 = 1.5, k_3 = 2, k_4 = -1, b = 2, z = 0, t = -1$.

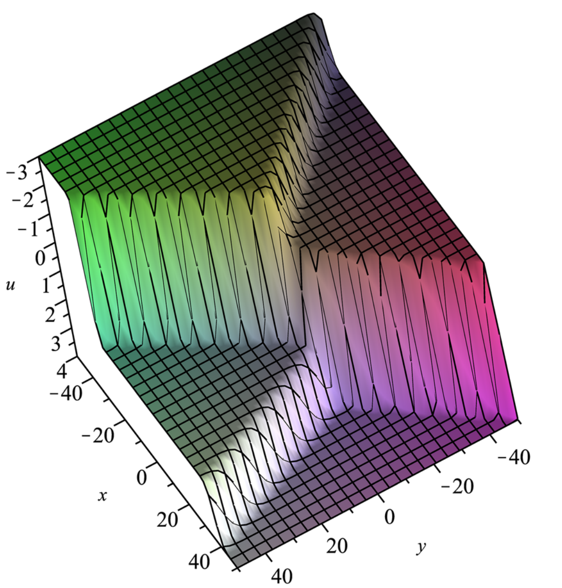

Figure 4. The plot of the four-wave solution (58) with parameters: $N = 5, \varepsilon_1 = 1, \varepsilon_2 = 1, \varepsilon_3 = 1, \varepsilon_4 = 1, \varepsilon_5 = 1, k_1 = -1.5, k_2 = -0.75, k_3 = -0.6, k_4 = 1, k_5 = 2, b = 2, z = 0, t = -1$.

and

$$u = 2(\ln f)_x,$$
$$f = \sum_{i=1}^{N} \left[\varepsilon_i \sinh\left(k_i x + k_i^{-1} y + \frac{1}{b} k_i^{-1} z + b k_i^3 t \right) + \lambda_i \sin\left(k_i x - k_i^{-1} y - \frac{1}{b} k_i^{-1} z - b k_i^3 t \right) \right], \quad (62)$$

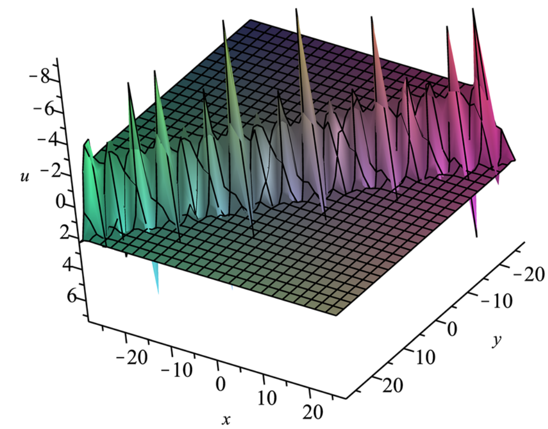

Figure 5. The plot of the complexiton solution (61) with parameters: $N = 1, \varepsilon_1 = 1, \lambda_1 = 1, k_1 = -1, b = 3, z = 0, t = 1$.

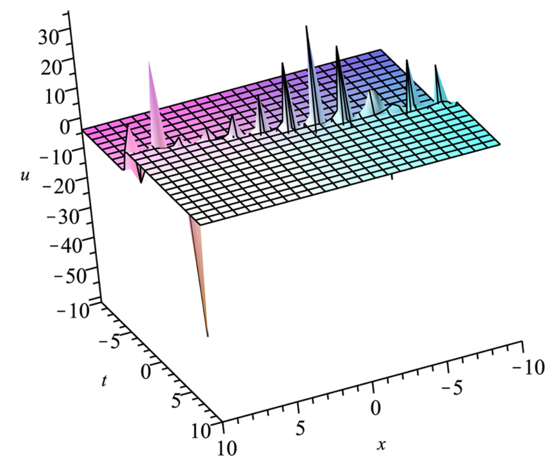

Figure 6. The plot of the complexiton solution (62) with parameters: $N = 1, \varepsilon_1 = 1, \lambda_1 = 1, k_1 = 2, b = 1, y = 0, z = 2$.

where ε_i's, λ_i's and k_i's are arbitrary constants, and b is an arbitrary constant, not to be zero. When $N = 1$, the graphs of the complexiton solutions with specific values being chosen for the parameters are presented in figures 5 and 6, which show some singularities of the solutions.

5. Conclusions and remarks

In summary, by means of specific mathematical techniques, we built the Pfaffian formulations and constructed the N-wave solutions for a general class of generalised KP and BKP equations. The obtained N-wave solutions were formulated by linear combinations of exponential travelling waves. The results presented in this paper include Pfaffian formulations for some important Hirota bilinear equations as their special cases. The Pfaffian formulation introduced by (12) with conditions (13) and (14) can generate types of Pfaffian solutions when $b_5 \neq 0$ to some generalised KP and BKP equations. For example, the set of sufficient conditions (39) in the Hirota bilinear equation (31) has not been revealed previously to the best of our knowledge.

Moreover, it is easy to see that the constants b_i, $1 \leq i \leq 5$, of the N wave solution defined by (49) also satisfy system (14). Thus, the solutions should motivate us to consider an open question: Can the N-wave solution (49) be derived by using the Pfaffian formulation defined by (12) and (13) to the introduced generalised KP and BKP equations?

In addition, there are many directions which need to be discussed, such as multiple wave solutions [26–29], breather-type periodic soliton solutions [17] and the bilinear Bäcklund transformation [30,31]. There is great interest to understand whether positive quadratic function solutions [32–44] to eq. (1) exist. Therefore, although difficult, it should be interesting to discover other particular solutions to eq. (4) via these existing methods.

Acknowledgements

The authors express their sincere thanks to the referees and editors for their valuable comments. This work was supported by the National Natural Science Foundation of China (No. 11371326).

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