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Wronskian N-soliton solutions to a generalized KdV equation in $(2+1)$-dimensions

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Abstract The aim of this paper is to construct Wronskian solutions to a generalized KdV equation in (2+1)-dimensions, which possesses a trilinear form. On the basis of two useful properties associated with Hirota differential operators, a general Wronskian formulation is established and the involved functions for Wronskian entries satisfy a system of combined linear partial differential equations. The key technique is to apply the Wronskian identity of the bilinear KP equation while presenting those sufficient conditions. Other illustrative examples of sufficient conditions are also given for the cKP3-4 equation, the $(2+1)$-dimensional DJKM equation, and the dissipative $(2+1)$-dimensional AKNS equation. Finally, N-soliton solutions and soliton molecules are worked out through the presented Wronskian formation.

Keywords Generalized KdV equation · Trilinear form · Wronskian formulation · N-soliton solution

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1 Introduction

It is of great significance in nonlinear science to investigate nonlinear integrable equations and discuss their integrable properties. There are many interesting characteristics, such as N-soliton solutions, bilinear Bäcklund transformations, Lax pairs, Painlevé test, and infinitely many conservation laws, which could represent integrability of nonlinear differential equations [1–15]. Among those integrable properties, the existence of N-soliton solutions usually implies the integrability [16] of the underlying nonlinear evolution equations. N-soliton solutions are a kind of exact multiple exponential wave solutions [17]. It is known that the Hirota bilinear method is widely used in constructing N-soliton solutions by perturbation [18]. Once a nonlinear differential equation is expressed as a Hirota bilinear form, one can always work out one-soliton and two-soliton solutions, based on the corresponding Hirota bilinear equation. However, when $N \geq 3$, a

Hirota bilinear equation possesses an N-soliton solution if and only if the Hirota N-soliton condition is satisfied [19,20]. More recently, one of the authors (Ma) has presented an improved algorithm to verify the Hirota conditions via comparing degrees of the polynomials on the basis of the Hirota functions in N wave vectors [21–23]. In addition, Wronskian formulas are a universal feature for nonlinear integrable equations and also become a powerful and standard way to establish N-soliton solutions in terms of Wronskian-type determinants [24–27].

In mathematical physics, the Korteweg de-Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1}$$

and its $(2+1)$-dimensional generalization, the Kadomtsev–Petviashvili-I (KPI) equation [28]

$$u_t = -6uu_x - u_{xxx} + 3\partial_x^{-1} u_{yy}, \tag{1.2}$$

are two well-known integrable models which possess N-soliton solutions that could be generated through their Hirota bilinear forms. Based on the Lax operator of the KP equation, the commuting KP hierarchy [29], also known as the integrable positive KP hierarchy [30], was presented in previous studies. The third member of the integrable positive KP hierarchy is just the KPI equation (1.2), and the fourth member can be written as

$$u_t = 12(2u_x\partial_x^{-1}u_y - \partial_x^{-2}u_{yyy} + u_{xxy} + 4uu_y), \tag{1.3}$$

via appropriate scaling and Galileo transformations. These two equations have lump solutions with different kinds of rational dispersion relations [31,32]. A novel $(2+1)$-dimensional system, the combination of the Eq. (1.2) and the Eq. (1.3) (cKP3-4)

$$\begin{aligned} u_t &= a(6uu_x + u_{xxx} - 3w_y) \\ &\quad + b(2wu_x - z_y + u_{xxy} + 4uu_y), \\ u_y &= w_x, \ u_{yy} = z_{xx}, \end{aligned} \tag{1.4}$$

has been systematically considered by Lou recently [33]. Lou named Eq. (1.4) as the cKP3-4 equation in Ref. [33]. Clearly, the cKP3-4 equation will be reduced back to an equivalent form of the standard KdV equation (1.1) when $y = x$, so the system (1.4) is also an extension of the KdV equation in $(2+1)$-dimensions. The Lax pair, the dual Lax pair and multiple soliton solutions were written down directly by Lou [33], upon observing that a linear combination in a soliton hierarchy is still integrable.

Besides, the KdV equation has some other generalized forms in $(2+1)$-dimensions, including the $(2+1)$-dimensional generalized breaking soliton equation

$$\begin{aligned} &u_t + au_{xxx} + bu_{xxy} + \frac{3ae}{b}uu_x \\ &\quad + 2euv_x + eu_xv = 0, \ u_y = v_x, \end{aligned} \tag{1.5}$$

which was first proposed by Xu [34]. This generalized equation passes the Painlevé test and possesses some integrable properties, such as N-soliton solutions, bilinear Bäcklund transformations, Lax pairs and infinitely many conservation laws [34]. Infinitely many nonlocal symmetries and similarity reductions were also discussed for the equivalent form of Eq. (1.5) in Ref. [35].

Motivated by the abovementioned results, we would like to extend the cKP3-4 equation (1.4) and the $(2+1)$-dimensional generalized breaking soliton Eq. (1.5) to a new one which still has many significant properties:

$$\begin{aligned} &a(6\alpha uu_x + u_{xxx} + 3\gamma v_{xy}) + b(u_{xxy} + 2\alpha u_xv_x \\ &\quad + 4\alpha uu_y + \gamma v_{yy}) + cu_t + du_x = 0, \ v_{xx} = u_y, \end{aligned} \tag{1.6}$$

where a, b, c, d, α and γ are real constant coefficients, which satisfy $c\alpha(a^2 + b^2) \neq 0$ to guarantee the nonlinearity of Eq. (1.6). Obviously, taking

$$\alpha = 1, \ \gamma = c = -1, \ d = 0, \ v_x = w \tag{1.7}$$

and

$$\alpha = \frac{e}{2b}, \gamma = 0, c = 1, d = 0, \tag{1.8}$$

in Eq. (1.6) gives Eqs. (1.4) and (1.5), respectively. Setting $a = \gamma = 0, \alpha = 2, b = 1, c = 4, d \neq 0$, and making use of the potential $u = \varphi_x$, we see that Eq. (1.6) reduces to the following dissipative $(2+1)$-dimensional Ablowitz–Kaup–Newell–Segur (AKNS) equation:

$$4\varphi_{xt} + \varphi_{xxxy} + 4\varphi_{xx}\varphi_y + 8\varphi_x\varphi_{xy} + d\varphi_{xx} = 0. \tag{1.9}$$

The bilinear Bäcklund transformation, Lax pair, infinitely many conservation laws and various exact solutions for Eq. (1.9) were investigated in Refs. [36,37].

To sum up, the system (1.6) includes a great number of significant integrable equations as its special cases. It is known that integrable equations play a major role in modeling complex physical phenomena due to their interesting integrability. For instance, if the coefficient $d = 0$, Eq. (1.9) is just the AKNS equation [38], which is one of the important integrable model equations for the propagation of long waves in nonlinear dispersive systems. Moreover, since the dissipation will inevitably occur when the shallow water waves come across damping, the dissipative (2 + 1)-dimensional AKNS equation is widely used to characterize the (2+1)-dimensional interaction of waves with dissipative effect in various branches of science especially in fluid mechanics [39]. According to the physical properties of the integrable equations mentioned above, the extended (2 + 1)-dimensional KdV equation (1.6) may be described as the interaction of two-dimensional shallow water waves in nonlinear dispersive systems. Therefore, we may trust that the discussion of Eq. (1.6) is beneficial to explain diverse physical phenomena in many areas of nonlinear science.

We consider the logarithmic transformations

$$u = \frac{2}{\alpha}(\ln f)_{xx} + \frac{u_0}{\alpha}, \; v = \frac{2}{\alpha}(\ln f)_y, \tag{1.10}$$

where u_0 is a real parameter. Under the transformations (1.10), the extended form (1.6) is translated into a trilinear form

$$\begin{aligned} &a(f^2 f_{xxxxx} + 2 f f_{xx} f_{xxx} - 5 f f_x f_{xxxx} \\ &\quad -6 f_{xx}^2 f_x + 8 f_{xxx} f_x^2 + 3\gamma f^2 f_{xyy} \\ &\quad -3\gamma f f_x f_{yy} - 6\gamma f f_y f_{xy} + 6\gamma f_x f_y^2 \\ &\quad +6u_0 f^2 f_{xxx} - 18u_0 f f_x f_{xx} + 12u_0 f_x^3) \\ &\quad +b(f^2 f_{xxxxy} - f f_{xxxx} f_y + 2 f f_{xx} f_{xxy} \\ &\quad -4 f f_x f_{xxxy} + 4 f_x f_{xxx} f_y \\ &\quad -2 f_{xx}^2 f_y - 4 f_{xx} f_x f_{xy} + 4 f_x^2 f_{xxy} + \gamma f^2 f_{yyy} \\ &\quad +2\gamma f_y^3 - 3\gamma f f_y f_{yy} \\ &\quad +4u_0 f^2 f_{xxy} - 4u_0 f f_{xx} f_y - 8u_0 f f_x f_{xy} \\ &\quad +8u_0 f_x^2 f_y) + c(f^2 f_{xxt} \\ &\quad - f f_{xx} f_t - 2 f f_x f_{xt} + 2 f_x^2 f_t) \\ &\quad +d(f^2 f_{xxx} - 3 f f_x f_{xx} + 2 f_x^3) = 0. \end{aligned} \tag{1.11}$$

It seems that the trilinear equation (1.11) cannot be directly bilinearized. But we have the following so-called quadrilinear representation of Eq. (1.11) [8,40, 41]:

$$\begin{aligned} &D_x\Big[\big(3aD_x^4 + 9a\gamma D_y^2 + 2bD_x^3 D_y \\ &\quad +3cD_x D_t + 3dD_x^2 + 18au_0 D_x^2\big) f \cdot f\Big] \cdot f^2 \\ &\quad +D_y\Big[\big(bD_x^4 + 3b\gamma D_y^2 \\ &\quad +12bu_0 D_x^2\big) f \cdot f\Big] \cdot f^2 = 0, \end{aligned} \tag{1.12}$$

where the D-operator is defined by

$$\begin{aligned} &D_{x_1}^{n_1} D_{x_2}^{n_2} f \cdot g \\ &\quad = (\partial_{x_1} - \partial_{x_1'})^{n_1} (\partial_{x_2} - \partial_{x_2'})^{n_2} \\ &\quad f(x_1, x_2) g(x_1', x_2')|_{x_1'=x_1, x_2'=x_2}, \end{aligned} \tag{1.13}$$

with n_1 and n_2 being arbitrary nonnegative integers [18].

The purpose of this paper is to furnish a Wronskian formulation to the extended (2 + 1)-dimensional KdV equation (1.6) by employing Wronskian identities of the bilinear KP hierarchy, thereby presenting its N-soliton solutions. This paper is structured as follows. In Sect. 2, by using two helpful properties associated with Hirota differential operators, a general Wronskian formulation will be constructed. Furthermore, N-soliton solutions and soliton molecules will be generated from the obtained Wronskian formulation in Sects. 3 and 4, respectively. Conclusions and remarks will be given in the last section.

2 A new Wronskian structure

The Wronskian technique is a valid method to establish exact solutions for Hirota bilinear forms. To this end, we adopt the following helpful notation defined by Freeman and Nimmo [24,25]:

$$W(\phi_1, \phi_2, \ldots, \phi_N) = \begin{vmatrix} \phi_1 & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2 & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_N & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \tag{2.1a}$$

where $\phi_i^{(j)} = \dfrac{\partial^j \phi_i}{\partial x^j}$, $i, j \geq 1$. It can often be written in the compact form as follows:

$$\begin{aligned} W &= |\phi, \phi^{(1)}, \ldots, \phi^{(N-1)}| \\ &= |0, 1, \ldots, N-1| = |\widehat{N-1}|, \end{aligned} \tag{2.1b}$$

where $\phi^{(l)}$ denotes the column vector $(\phi_1^{(l)}, \phi_2^{(l)}, \ldots, \phi_N^{(l)})^T$.

As we know, the first two equations of the KP hierarchy [42] can be written in Hirota bilinear form as

$$(D_1^4 - 4D_1 D_3 + 3D_2^2) f \cdot f = 0, \tag{2.2a}$$

$$[(D_1^3 + 2D_3) D_2 - 3D_1 D_4] f \cdot f = 0, \tag{2.2b}$$

where f is a function of the variables x_j, $j = 1, 2, 3, \ldots$, and $D_j \equiv D_{x_j}$. Note that the first member

(2.2a) just becomes the typical bilinear KP equation, upon setting $(x_1, x_2, x_3) = (x, y, t)$. If the Wronskian entries $\phi_i = \phi_i(x_1, x_2, x_3, \ldots)$ $(1 \le i \le N)$ satisfy that

$$\partial_{x_j}\phi_i = \frac{\partial^j \phi_i}{\partial x^j}, \quad j = 1, 2, 3, \ldots, \tag{2.3}$$

then $f = f_N = |\widehat{N-1}|$ defined by the Wronskian determinant (2.1) solves the bilinear equations (2.2a) and (2.2b) [18,43]. Sato et al. showed that the KP hierarchy equations in bilinear forms reduce to the Plücker relations for determinants when the function f is expressed by the Wronskian [18,44]. For example, under the conditions (2.3), the first member (2.2a) in the KP hierarchy yields the following identity

$$\begin{aligned}&(D_1^4 - 4D_1D_3 + 3D_2^2)|\widehat{N-1}| \cdot |\widehat{N-1}|\\ &= 24(|\widehat{N-1}||\widehat{N-3}, N, N+1|\\ &\quad -|\widehat{N-2}, N||\widehat{N-3}, N-1, N+1|\\ &\quad +|\widehat{N-2}, N+1||\widehat{N-3}, N-1, N|) \equiv 0,\end{aligned} \tag{2.4}$$

which is nothing but the Plücker relation.

To present Wronskian solutions, we first give two useful properties associated with Hirota differential operators.

Lemma 2.1 *If the Wronskian entries $\phi_i = \phi_i(x, y, t)$, $1 \le i \le N$, in the Wronskian determinant (2.1) satisfy the expressions (2.3) and the following conditions:*

$$\begin{aligned}\partial_y\phi_i &= (a_1\partial_x + a_2\partial_x^2 + \cdots + a_m\partial_x^m)\phi_i\\ &\equiv (a_1\partial_{x_1} + a_2\partial_{x_2} + \cdots + a_m\partial_{x_m})\phi_i,\end{aligned} \tag{2.5a}$$

$$\begin{aligned}\partial_t\phi_i &= (b_1\partial_x + b_2\partial_x^2 + \cdots + b_n\partial_x^n)\phi_i\\ &\equiv (b_1\partial_{x_1} + b_2\partial_{x_2} + \cdots + b_n\partial_{x_n})\phi_i,\end{aligned} \tag{2.5b}$$

where m, n are nonnegative integers, and $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ are arbitrary constants, then $f = f_N = |\widehat{N-1}|$ defined by the Wronskian determinant (2.1) possesses the following property:

$$\begin{aligned}&D_y^p D_t^q f \cdot f\\ &= (a_1D_1 + a_2D_2 + \cdots + a_mD_m)^p\\ &\quad (b_1D_1 + b_2D_2 + \cdots + b_nD_n)^q f \cdot f,\end{aligned} \tag{2.6}$$

with p, q are nonnegative integers and $D_j \equiv D_{x_j}$. Here the Hirota's bilinear operators D_{x_j}, $j = 1, 2, 3, \ldots$, are defined by the expression (1.13).

Lemma 2.2 *Suppose the Wronskian entries $\phi_i = \phi_i(x, y, t)$, $1 \le i \le N$, in the Wronskian determinant (2.1) satisfy the expressions (2.3) and*

$$\begin{aligned}\sum_{j=1}^{N}\lambda_{ij}\phi_j &= (c_1\partial_x + c_2\partial_x^2 + \cdots + c_m\partial_x^m)\phi_i\\ &\equiv (c_1\partial_{x_1} + c_2\partial_{x_2} + \cdots + c_m\partial_{x_m})\phi_i,\end{aligned} \tag{2.7}$$

where $c_1, c_2, \ldots, c_m$ are arbitrary constants and the coefficient matrix $A = (\lambda_{ij})_{1\le i,j\le N}$ is an arbitrary real constant matrix. Then we have

$$D_k(c_1D_1 + c_2D_2 + \cdots + c_mD_m)f \cdot f = 0, \tag{2.8}$$

where the function f is expressed by the Wronskian determinant (2.1) and $D_k = D_{x_k}$. Here the Hirota's bilinear operators D_{x_k}, $k = 1, 2, 3, \ldots$, are defined by the expression (1.13).

Lemmas 2.1 and 2.2 have been proved in Ref. [45]. We only make a supplement to the proof of Lemma 2.1 (see the "Appendix" for details).

In the following, we present a well-known example to shed light on applications of Lemma 2.2. It is known that the KdV equation (1.1) is mapped into the bilinear form

$$(D_xD_t + D_x^4)f \cdot f = 0, \tag{2.9}$$

by the dependent variable transformation $u = 2(\ln f)_{xx}$. For the convenience of later discussions, we can rescale the coordinates and rewrite Eq. (2.9) as follows:

$$(D_1^4 - 4D_1D_3)f \cdot f = 0, \tag{2.10}$$

where D_1 and D_3 stand for D_x and D_t, respectively. In the KP hierarchy, the bilinear KP equation (2.2a) is reduced to the bilinear KdV equation (2.10) if one takes $D_2 = 0$, which is the "2-reduction" of the KP equation [46]. Assume that the function $f = |\widehat{N-1}|$ is defined by the Wronskian determinant (2.1) and the Wronskian entries ϕ_i satisfy the following linear conditions

$$\phi_{i,xx} = \sum_{j=1}^{N}\lambda_{ij}\phi_j, \tag{2.11a}$$

$$\phi_{i,t} = \phi_{i,xxx}, \quad 1 \le i \le N, \tag{2.11b}$$

where the λ_{ij}'s are arbitrary constants. Using the condition (2.11a) and Lemma 2.2 gives $D_2^2 f \cdot f = 0$. Then applying the conditions in (2.11) and the Wronskian identity (2.4), we can deduce that

$$\begin{aligned}&(D_1^4 - 4D_1D_3)|\widehat{N-1}| \cdot |\widehat{N-1}|\\ &= \left[(D_1^4 - 4D_1D_3 + 3D_2^2) - 3D_2^2\right]|\widehat{N-1}| \cdot |\widehat{N-1}| = 0.\end{aligned} \tag{2.12}$$

This shows that the Wronskian determinant $f = |\widehat{N-1}|$ solves the bilinear KdV equation (2.10).

According to the above results, we now give a set of sufficient conditions, under which the Wronskian determinant solves the trilinear equation (1.11).

Theorem 2.1 *Assume that a set of functions* $\phi_i = \phi_i(x,y,t), 1 \le i \le N$, *satisfies the following combined linear conditions:*

$$\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j, \tag{2.13a}$$
$$\phi_{i,y} = r_1\phi_{i,x}, \tag{2.13b}$$
$$\phi_{i,t} = r_2\phi_{i,x} + r_3\phi_{i,xxx}, \tag{2.13c}$$

with

$$r_2 = -\frac{\gamma r_1^2(3a+br_1)}{c} - \frac{d+4bu_0r_1+6au_0}{c},$$
$$r_3 = -\frac{4}{c}(a+br_1), \tag{2.13d}$$

where $r_1 \neq 0$ *is a free parameter and the coefficient matrix* $A = (\lambda_{ij})_{1\le i,j\le N}$ *is an arbitrary real constant matrix. Then* $f = f_N = |\widehat{N-1}|$ *defined by the Wronskian determinant* (2.1) *solves the trilinear equation* (1.11).

Proof We introduce an auxiliary independent variable z, so that the quadrilinear representation (1.12) becomes

$$\begin{aligned}&D_x\Big[\big(3aD_x^4 + 9a\gamma D_y^2 + 2bD_x^3D_y + 3cD_xD_t + 3dD_x^2\\&\quad +18au_0D_x^2 - D_yD_z\big)f\cdot f\Big]\cdot f^2\\&\quad +D_y\Big[\big(bD_x^4 + 3b\gamma D_y^2 + 12bu_0D_x^2\\&\quad +D_xD_z\big)f\cdot f\Big]\cdot f^2 = 0.\end{aligned} \tag{2.14}$$

Suppose that the Wronskian entries $\phi_i,\ 1 \le i \le N$, satisfy

$$\begin{aligned}&\phi_{i,z} = r_4\phi_{i,x} + r_5\phi_{i,xxx},\\&r_4 = -3b\gamma r_1^2 - 12bu_0,\ r_5 = -4b.\end{aligned} \tag{2.15}$$

We utilize the conditions (2.13) and Lemma 2.1, and then the function $f = f_N = |\widehat{N-1}|$ defined by the Wronskian determinant (2.1) yields

$$\begin{aligned}&D_x^4 f\cdot f = D_1^4 f\cdot f,\ D_y^2 f\cdot f = r_1^2D_1^2 f\cdot f,\\&D_x^3D_y = r_1D_1^4 f\cdot f,\\&D_xD_t = D_1(r_2D_1 + r_3D_3)f\cdot f,\\&D_yD_z = r_1D_1(r_4D_1 + r_5D_3),\ D_xD_z\\&\quad = D_1(r_4D_1 + r_5D_3)f\cdot f.\end{aligned}$$

Now, we can compute that

$$\begin{aligned}&\big(3aD_x^4 + 9a\gamma D_y^2 + 2bD_x^3D_y + 3cD_xD_t\\&\quad +3dD_x^2 + 18au_0D_x^2 - D_yD_z\big)f\cdot f\\&= \big[3aD_1^4 + 9a\gamma r_1^2D_1^2 + 2br_1D_1^4\\&\quad +3cD_1(r_2D_1 + r_3D_3)\\&\quad +3dD_1^2 + 18au_0D_1^2 - r_1D_1(r_4D_1 + r_5D_3)\big]f\cdot f\\&= \big[(3a+2br_1)D_1^4 + (9a\gamma r_1^2\\&\quad +3cr_2 - r_1r_4 + 3d + 18au_0)D_1^2\\&\quad +(3cr_3 - r_1r_5)D_1D_3\big]f\cdot f\\&= (3a+2br_1)\big[D_1^4 - 4D_1D_3\big]f\cdot f = 0,\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}&\big(bD_x^4 + 3b\gamma D_y^2 + 12bu_0D_x^2 + D_xD_z\big)f\cdot f\\&= \big[bD_1^4 + 3b\gamma r_1^2D_1^2 + 12bu_0D_1^2\\&\quad +D_1(r_4D_1 + r_5D_3)\big]f\cdot f\\&= \big[bD_1^4 + (3b\gamma r_1^2 + 12bu_0 + r_4)D_1^2 + r_5D_1D_3\big]f\cdot f\\&= b\big[D_1^4 - 4D_1D_3\big]f\cdot f = 0,\end{aligned} \tag{2.17}$$

where the conditions (2.13), (2.15) and the Wronskian identity (2.12) of the KdV equation have been applied. It follows further that

$$\begin{aligned}&D_x\Big[\big(3aD_x^4 + 9a\gamma D_y^2 + 2bD_x^3D_y + 3cD_xD_t\\&\quad +3dD_x^2 + 18au_0D_x^2\big)|\widehat{N-1}|\cdot|\widehat{N-1}|\Big]\cdot|\widehat{N-1}|^2\\&\quad +D_y\Big[\big(bD_x^4 + 3b\gamma D_y^2\\&\quad +12bu_0D_x^2\big)|\widehat{N-1}|\cdot|\widehat{N-1}|\Big]\cdot|\widehat{N-1}|^2 = 0.\end{aligned} \tag{2.18}$$

Therefore, this shows that $f = |\widehat{N-1}|$ solves the bilinear trilinear equation (1.11).

Subsequently, three application examples will be given to illustrate the Wronskian sufficient conditions of Theorem 2.1.

Case 1 The cKP3-4 equation

The case of $a^2 + b^2 \neq 0, \alpha = 1, \gamma = c = -1$ and $d = u_0 = 0$ presents

$$\begin{aligned}&D_x\Big[\big(3aD_x^4 - 9aD_y^2 + 2bD_x^3D_y - 3D_xD_t\big)f\cdot f\Big]\cdot f^2\\&\quad +D_y\Big[\big(bD_x^4 - 3bD_y^2\big)f\cdot f\Big]\cdot f^2 = 0,\end{aligned} \tag{2.19}$$

which is mapped into the cKP3-4 equation (1.4) [33], under the logarithmic derivative transformations

$$u = 2(\ln f)_{xx},\ w = 2(\ln f)_{xy},\ z = 2(\ln f)_{yy}. \tag{2.20}$$

A set of sufficient conditions, which makes the Wronskian determinant a solution to the quadrilinear equation (2.19), can be expressed as

$$\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j, \tag{2.21a}$$

$$\phi_{i,y} = r_1\phi_{i,x}, \tag{2.21b}$$

$$\phi_{i,t} = -r_1^2(3a+br_1)\phi_{i,x}+4(a+br_1)\phi_{i,xxx}, \tag{2.21c}$$

where $r_1 \neq 0$ is a free parameter and the λ_{ij}'s are arbitrary constants.

Case 2 The (2 + 1)-dimensional DJKM equation

The case of $\alpha = b = 1, c = -2, \gamma = -1$ and $a = d = u_0 = 0$ yields

$$D_x\Big[\big(2D_x^3D_y - 6D_xD_t\big)f\cdot f\Big]\cdot f^2 + D_y\Big[\big(D_x^4 - 3D_y^2\big)f\cdot f\Big]\cdot f^2 = 0, \tag{2.22}$$

which is equivalent to the (2 + 1)-dimensional Date–Jimbo–Kashiwara–Miwa (DJKM) equation [5,47,48]:

$$\psi_{xxxxy} + 4\psi_{xxy}\psi_x + 2\psi_{xxx}\psi_y + 6\psi_{xy}\psi_{xx} - \psi_{yyy} - 2\psi_{xxt} = 0, \tag{2.23}$$

under the dependent variable transformation $\psi = 2(\ln f)_x$. Note that Eq. (1.3) can be transformed into an equivalent form of the (2 + 1)-dimensional DJKM equation (2.23) by the potential $u = \psi_x$. Based on the presented results in Theorem 2.1, the corresponding sufficient conditions for the Wronskian determinant $f = |\widehat{N-1}|$ read as

$$\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j, \tag{2.24a}$$

$$\phi_{i,y} = r_1\phi_{i,x}, \tag{2.24b}$$

$$\phi_{i,t} = -\frac{1}{2}r_1^3\phi_{i,x} + 2r_1\phi_{i,xxx}, \tag{2.24c}$$

where $r_1 \neq 0$ is a free parameter and the λ_{ij}'s are arbitrary constants. It is worth mentioning that the above sufficient conditions for the Wronskian determinant are different from the ones presented previous studies [49,50].

Case 3 The dissipative (2 + 1)-dimensional AKNS equation

The case of $\alpha = 2, b = 1, c = 4, d \neq 0$ and $a = \gamma = u_0 = 0$ gives

$$D_x\Big[\big(2D_x^3D_y + 12D_xD_t + 3dD_x^2\big)f\cdot f\Big]\cdot f^2 + D_y\big(D_x^4 f\cdot f\big)\cdot f^2 = 0. \tag{2.25}$$

which becomes the dissipative (2 + 1)-dimensional AKNS equation (1.9) [36,37], through the dependent variable transformation $\varphi = (\ln f)_x$. According to Theorem 2.1, for Eq. (1.9), a set of sufficient conditions for the Wronskian solution $f = |\widehat{N-1}|$ is

$$\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j, \tag{2.26a}$$

$$\phi_{i,y} = r_1\phi_{i,x}, \tag{2.26b}$$

$$\phi_{i,t} = -\frac{d}{4}\phi_{i,x} - r_1\phi_{i,xxx}, \tag{2.26c}$$

where r_1 is an arbitrary nonzero constant and the λ_{ij}'s are arbitrary constants.

3 N-soliton solutions

The Wronskian technique is a direct and comprehensive tool to construct N-soliton solutions to nonlinear integrable equations [26]. Assume that the coefficient matrix A in (2.13a) has the following form:

$$A = \begin{pmatrix} k_1^2 & & & 0 \\ & k_2^2 & & \\ & & \ddots & \\ 0 & & & k_N^2 \end{pmatrix}_{N\times N}, \tag{3.1}$$

where $0 < k_1 < k_2 < \cdots < k_N$ are arbitrary constants. Substituting the above special matrix into the Wronskian conditions (2.13)and (2.15), and solving the conditions (2.13) and (2.15), we obtain a general choice for ϕ_i, which is

$$\phi_i = e^{\xi_i} + (-1)^{i+1}e^{-\xi_i},$$
$$\xi_i = k_ix + r_1k_iy + (r_2k_i + r_3k_i^3)t + (r_4k_i + r_5k_i^3)z + \xi_i^0, \tag{3.2}$$

where ξ_i^0's are arbitrary constants, and $r_k,\ 2 \le k \le 5$, are defined by (2.13d) and (2.15). We only need to consider

$$\xi_i = k_ix + r_1k_iy + (r_2k_i + r_3k_i^3)t + \xi_i^0,\ 1 \le i \le N, \tag{3.3}$$

since the auxiliary parameter z can be absorbed in the the auxiliary parameters ξ_i^0. For the trilinear equation (1.11), N-soliton solutions associated with the Wronskian determinant can be written as:

$$f = f_N = W(\phi_1, \phi_2, \ldots, \phi_N),$$
$$= \Big|e^{\xi_i} + (-1)^{i+1}e^{-\xi_i}, k_i\big(e^{\xi_i} + (-1)^{i+2}e^{-\xi_i}\big), \ldots,$$

$$k_i^{N-1}\left(e^{\xi_i}+(-1)^{i+N}e^{-\xi_i}\right)\Big|, \tag{3.4}$$

where $\xi_i,\ 1\le i\le N$, are defined by the expression (3.3). Via the same way presented in [51], we get

$$f_N=(-1)^{\frac{N(N-1)}{2}}\prod_{1\le i<l}^{N}(k_i-k_l)e^{-\sum_{i=1}^{N}\xi_i}\sum_{\mu=0,1}\prod_{1\le i<l}^{N}\left(\frac{k_l-k_i}{k_l+k_i}\right)^{-\mu_i-\mu_l}e^{\sum_{i=1}^{N}2\mu_i\xi_i+\sum_{1\le i<l}^{N}\mu_i\mu_l A_{il}}, \tag{3.5}$$

where $\sum_{\mu=0,1}$ stands for the summation over all possible combinations of $\mu_i=0,1,\ i=1,2,\dots,N$, and the phase shifts are determined by

$$e^{A_{il}}=\left(\frac{k_l-k_i}{k_l+k_i}\right)^2. \tag{3.6}$$

When $0<k_1<k_2\cdots<k_N$, we have

$$\prod_{1\le i<l}^{N}\left(\frac{k_l-k_i}{k_l+k_i}\right)^{-\mu_i-\mu_l}=e^{-\frac{1}{2}\sum_{i=1}^{N}\sum_{l=1,l\ne i}^{N}\mu_i A_{il}}. \tag{3.7}$$

Thus, an N-soliton solution of Eq. (1.11) can be rewritten as

$$f_N=(-1)^{\frac{N(N-1)}{2}}\prod_{1\le i<l}^{N}(k_i-k_l)e^{-\sum_{i=1}^{N}\xi_i}\sum_{\mu=0,1}e^{\sum_{i=1}^{N}\mu_i\eta_i+\sum_{1\le i<l}^{N}\mu_i\mu_l A_{il}}, \tag{3.8}$$

if we put

$$2\xi_i-\frac{1}{2}\sum_{l=1,l\ne i}^{N}A_{il}\to\eta_i. \tag{3.9}$$

Omitting the factor $(-1)^{\frac{N(N-1)}{2}}\prod_{1\le i<l}^{N}(k_i-k_l)e^{-\sum_{i=1}^{N}\xi_i}$ and taking the logarithmic transformations (1.10), we can easily determine the following N-soliton solution to the extended KdV equation (1.6):

$$u=\frac{2}{\alpha}(\ln f)_{xx}+\frac{u_0}{\alpha},$$
$$v=\frac{2}{\alpha}(\ln f)_y,$$
$$f=f_N=\sum_{\mu=0,1}e^{\sum_{i=1}^{N}\mu_i\eta_i+\sum_{1\le i<l}^{N}\mu_i\mu_l A_{il}}, \tag{3.10}$$

where $\eta_i=2k_ix+2r_1k_iy+2(r_2k_i+r_3k_i^3)t+\xi_i^0$, and $r_k,\ 1\le k\le 3$, are defined by the condition (2.13d). Besides, if we introduce new parameters as follows:

$$K_i=2k_i,\ L_i=2r_1k_i,\ W_i=2(r_2k_i+r_3k_i^3), \tag{3.11}$$

then the dispersion relation of the trilinear equation (1.11) reads as

$$W_i=-\frac{a}{cK_i}(3\gamma L_i^2+K_i^4)-\frac{bL_i}{cK_i^2}(\gamma L_i^2+K_i^4)-\frac{(d+6au_0)K_i+4bu_0L_i}{c}, \tag{3.12}$$

which shows higher-order dispersion relations, compared with the KdV equation and the KP equation.

For Eq. (1.6), a special two-soliton framework of u generated by the expression (3.10) with the parameter selections

$$\alpha=b=1,\ c=-2,\ \gamma=-1,\ a=d=u_0=0,$$
$$N=2,\ k_1=1,\ k_2=3,\ r_1=0.5,\ \xi_1^0=1,\ \xi_2^0=3, \tag{3.13}$$

is plotted in Figs. 1 and 2. Figure 1 depicts the propagation of the two-soliton solution (3.10) for the field u with the parameter choices (3.13) at several time steps. We can see that two line soliton waves are parallel to each other due to $2k_1/2k_2=2r_1k_1/2r_1k_2$ in the (x,y)-plane, and they have different amplitudes and widths. The tall and thin wave is behind the short and fat wave at the beginning of the propagation. After a while, the tall wave catches up with the short wave and collides, since the tall wave with high amplitude moves faster than the other. Figure 2 shows the evolution of two solitary waves, one of which is a line soliton in the (x,t)-plane. From Figs. 1 and 2, it is easy to observe that the interaction between two solitons is elastic without changing their original shapes and velocities before and after the interaction.

Figures 3 and 4 display the propagation of a special three-soliton solution for the field u determined by the expression (3.10) with the parameter selections

$$\alpha=b=1,\ c=-2,\ \gamma=-1,$$
$$a=d=u_0=0,\ N=3,$$
$$k_1=1,\ k_2=3,\ k_3=-2,\ r_1=0.5,\ \xi_1^0=2,$$
$$\xi_2^0=3,\ \xi_3^0=1. \tag{3.14}$$

The development of three solitons is the same as above.

4 Soliton molecule solutions

Very recently, soliton molecules as bound states have attracted more and more attention from scholars in some fields such as Bose–Einstein condensates, wave

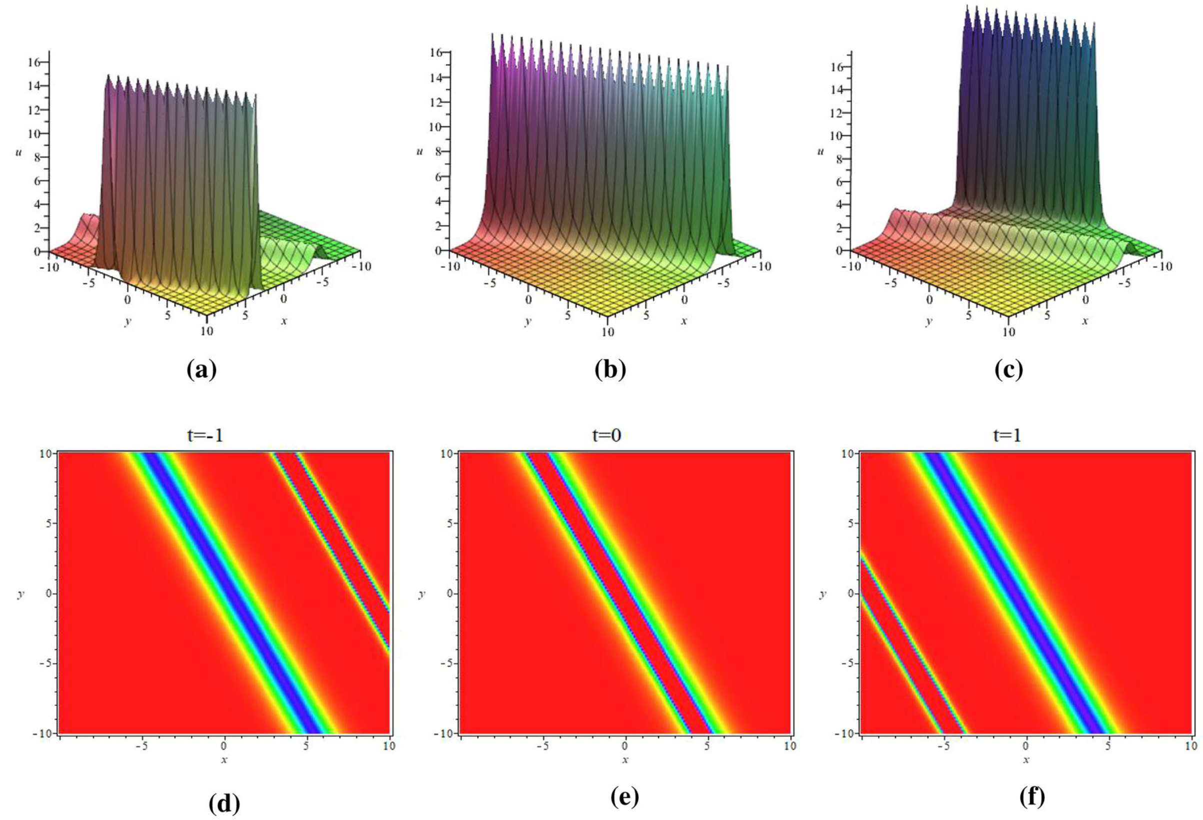

Fig. 1 Three-dimensional plots and density plots of u determined by the N-soliton solution (3.10) with (2.13d) and (3.13) when $t=-1$ in (**a**) and (**d**), $t=0$ in (**b**) and (**e**), and $t=1$ in (**c**) and (**f**), respectively

guides and nonlinear optics. Many kinds of theoretical or experimental approaches have been proposed to search for soliton molecules [52,53]. In particular, a velocity resonance mechanism was introduced to explore soliton molecules of (1 + 1)-dimensional nonlinear evolution equations by Lou [54,55]. It was further found that the velocity resonance mechanism can be widely used to find soliton molecules in higher-dimensional systems including the cKP3-4 equation [33]. In this section, we would like to study soliton molecules, which will be generated from the previous Wronskian N-soliton solutions to Eq. (1.6).

Based on the sufficient conditions (2.13), it is obvious that under a selection of $r_1=-a/b$, the N-soliton solution (3.10) reads as

$$u=\frac{2}{\alpha}(\ln f)_{xx}+\frac{u_0}{\alpha},$$
$$v=\frac{2}{\alpha}(\ln f)_y,\ f=f_N$$
$$=\sum_{\mu=0,1} e^{\sum_{i=1}^{N}\mu_i\eta_i+\sum_{1\le i<l}^{N}\mu_i\mu_l A_{il}}, \tag{4.1}$$

where

$$\eta_i=K_i x+L_i y+W_i t+\xi_i^0,$$
$$L_i=-\frac{a}{b}K_i,$$
$$W_i=-\left(\frac{2\gamma a^3}{cb^2}+\frac{2au_0+d}{c}\right)K_i,\ e^{A_{il}}$$
$$=\left(\frac{K_l-K_i}{K_l+K_i}\right)^2, \tag{4.2}$$

with K_i and ξ_i^0 being arbitrarily prescribed constants. We can further find that the following soliton resonant condition:

$$\frac{K_i}{K_j}=\frac{L_i}{L_j}=\frac{W_i}{W_j},\ K_i\ne\pm K_j, \tag{4.3}$$

is satisfied, based on the expression (4.2). Thus, within the velocity resonance mechanism formulation, the solution (4.1) with (4.2) just presents a kind of soliton molecule solutions.

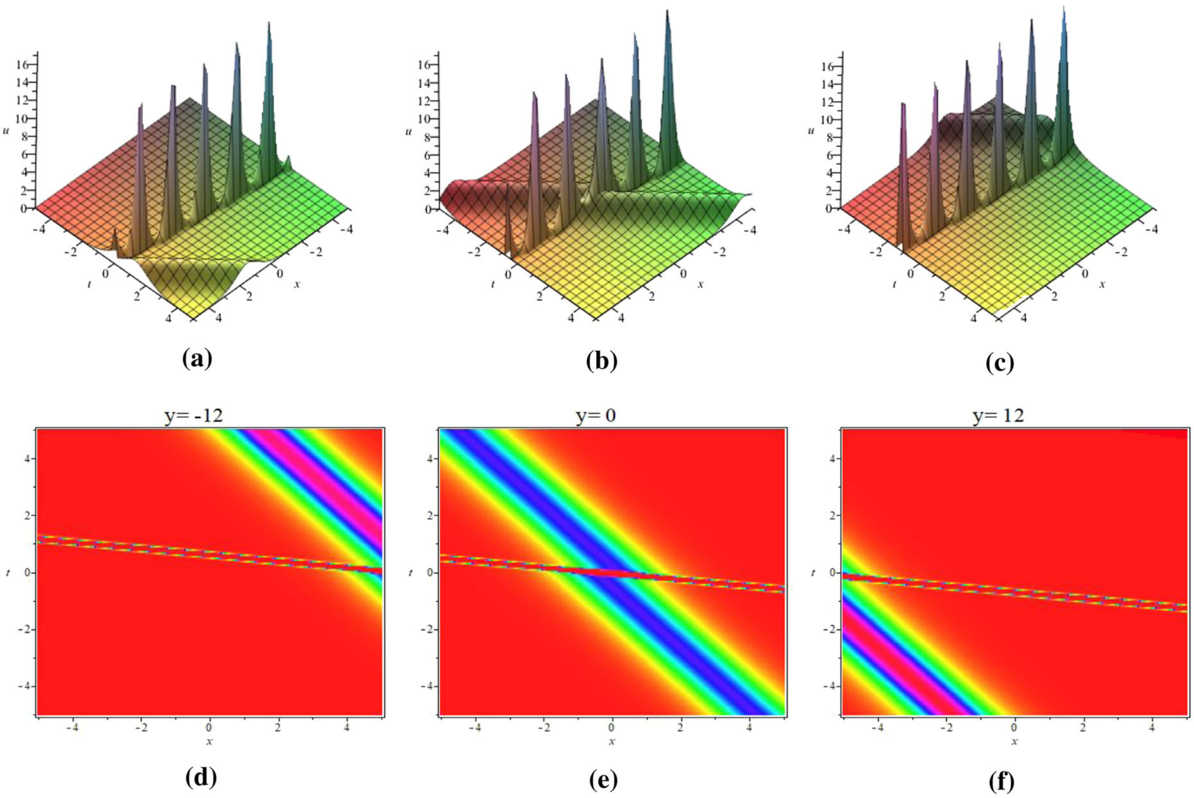

Fig. 2 Three-dimensional plots and density plots of u determined by the N-soliton solution (3.10) with (2.13d) and (3.13) when $y=-12$ in (**a**) and (**d**), $y=0$ in (**b**) and (**e**), and $y=12$ in (**c**) and (**f**), respectively

Particularly, setting

$$\alpha = 2, a = b = c = 1, \gamma = d = 0, u_0 = 4, \tag{4.4}$$

we obtain a generalized breaking soliton equation as follows:

$$u_t + u_{xxx} + u_{xxy} + 12uu_x + 8uu_y \\ +4u_x v_x = 0,\ v_{xx} = u_y, \tag{4.5}$$

which has a Hirota bilinear form

$$D_x\Big[\big(3D_x^4 + 2D_x^3 D_y + 3D_x D_t + 72D_x^2\big) f \cdot f\Big] \cdot f^2 \\ +D_y\Big[\big(D_x^4 + 48D_x^2\big) f \cdot f\Big] \cdot f^2 = 0, \tag{4.6}$$

under the logarithmic transformation $u = (\ln f)_{xx} + 2$, $v = (\ln f)_y$. Making use of the solution (4.1) with (4.2), we can present the following soliton molecule solution to Eq. (4.5):

$$u = (\ln f)_{xx} + 2, \\ v = (\ln f)_y, \\ f = \sum_{\mu=0,1} e^{\sum_{i=1}^N \mu_i\eta_i + \sum_{1\le i<l}^N \mu_i\mu_l A_{il}}, \tag{4.7}$$

where

$$\eta_i = K_i(x - y - 8t) + \xi_i^0,\ e^{A_{il}} = \left(\frac{K_l - K_i}{K_l + K_i}\right)^2,$$

with K_i and ξ_i^0 being arbitrary constants. To exhibit some characteristic properties of soliton molecule solutions, three density plots of the above solution (4.7) with specific parameters are made in Fig. 5.

In addition, we can present a special set of sufficient conditions, which guarantees that the Wronskian determinant solves the trilinear equation (1.11).

Theorem 4.1 *Suppose that a group of functions* $\phi_i = \phi_i(x, y, t)$, $1 \le i \le N$, *satisfies the two sets of conditions*

$$\phi_{i,y} = -\frac{a}{b}\phi_{i,x}, \tag{4.8a}$$

$$\phi_{i,t} = -\left(\frac{2\gamma a^3}{cb^2} + \frac{2au_0 + d}{c}\right)\phi_{i,x}, \tag{4.8b}$$

where the parameters a *and* b *are two nonzero constants. Then* $f = f_N = |\widehat{N-1}|$ *defined by the Wron-*

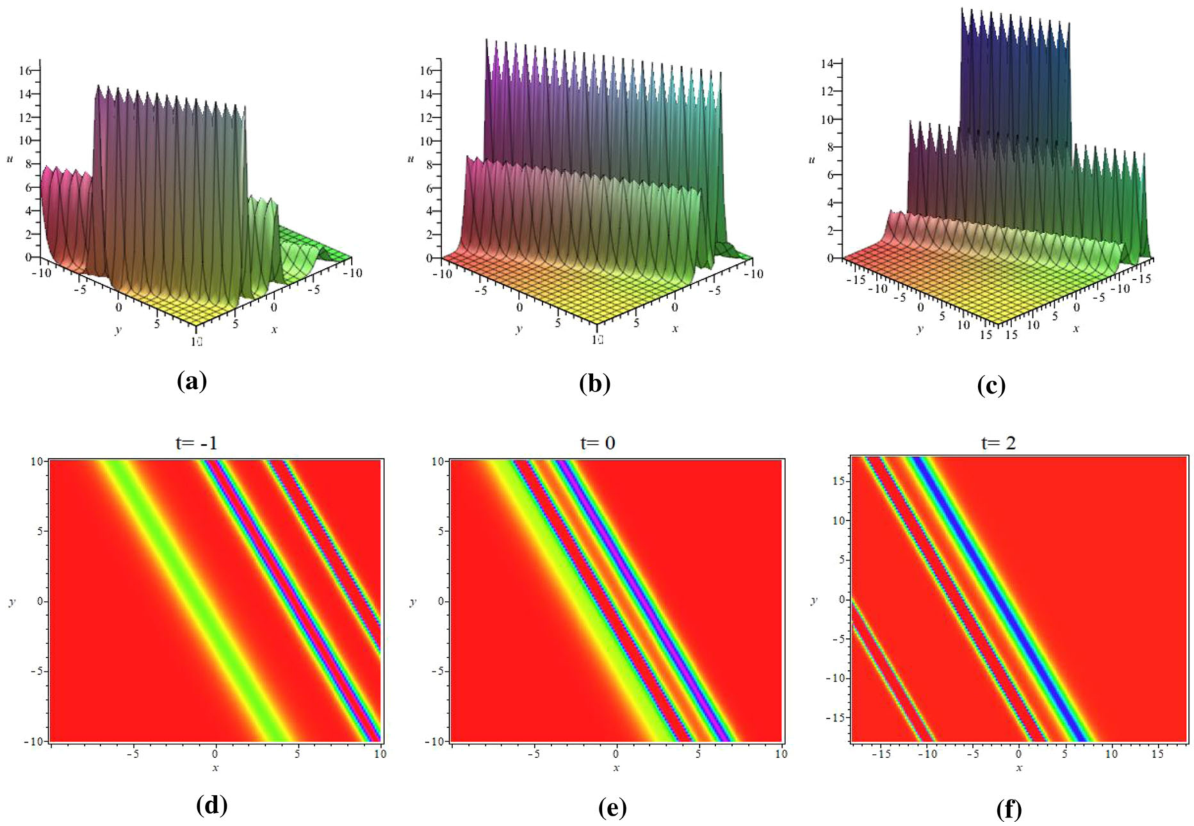

Fig. 3 Three-dimensional plots and density plots of u determined by the N-soliton solution (3.10) with (2.13d) and (3.14) when $t = -1$ in (**a**) and (**d**), $t = 0$ in (**b**) and (**e**), and $t = 2$ in (**c**) and (**f**), respectively

skian determinant (2.1) *solves the trilinear equation* (1.11).

Proof Under the conditions (4.8), we can obtain various derivatives of the Wronskian determinant $f = f_N = |\widehat{N-1}|$ with respect to the variables x, y, z as follows:

$$f_y = -\frac{a}{b} f_x,\ f_t = -\left(\frac{2\gamma a^3}{cb^2} + \frac{2au_0 + d}{c}\right) f_x,$$
$$f_{xy} = -\frac{a}{b} f_{xx},$$
$$f_{yy} = \frac{a^2}{b^2} f_{xx},\ f_{xyy} = \frac{a^2}{b^2} f_{xxx},$$
$$f_{xt} = -\left(\frac{2\gamma a^3}{cb^2} + \frac{2au_0 + d}{c}\right) f_{xx}, \ldots.$$

Substituting the above derivatives into the left side of the trilinear equation (1.11), we find that Eq. (1.11) holds following a direct observation. This shows that $f = f_N = |\widehat{N-1}|$ solves the trilinear equation (1.11).

5 Concluding remarks

The main goal of the paper is to investigate an extended (2 + 1)-dimensional KdV equation and its N-soliton solutions through the improved Wronskian technique. By introducing an auxiliary parameter, we have presented a set of sufficient conditions, consisting of systems of combined linear partial differential equations, which make the Wronskian determinant to be a solution to the resulting trilinear equation. The Wronskian identity of the bilinear KP equation and two universal properties of Hirota differential operators have played vital roles in exploring Wronskian solution structures. Other illustrative examples of sufficient conditions have also been presented for the cKP3-4 equation, the (2 + 1)-dimensional DJKM equation and the dissipative (2+1)-dimensional AKNS equation. Lastly, soliton molecules have been generated through the presented Wronskian N-soliton solutions when the coefficients a and b are two nonzero constants for Eq. (1.6). The obtained

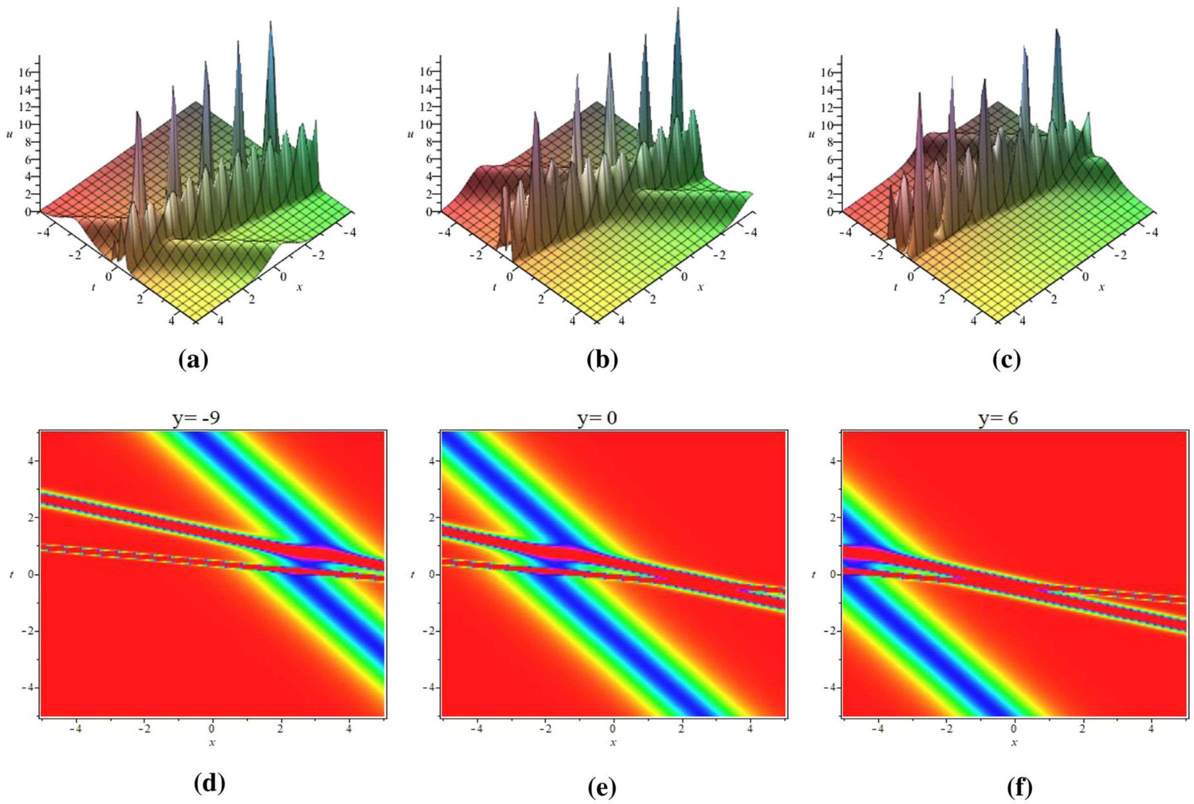

Fig. 4 Three-dimensional plots and density plots of u determined by the N-soliton solution (3.10) with (2.13d) and (3.14) when $y = -9$ in (**a**) and (**d**), $y = 0$ in (**b**) and (**e**), and $y = 6$ in (**c**) and (**f**), respectively

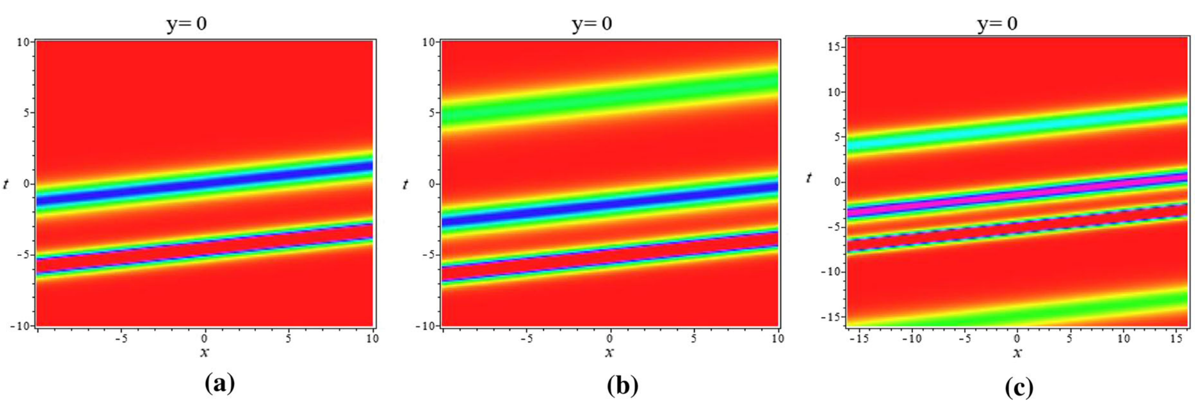

Fig. 5 **a** Two-soliton molecule profile of the solution u expressed by Eq. (4.7) with parameters: $N = 2, K_1 = \frac{1}{3}, K_2 = \frac{1}{2}, \xi_1^0 = 0, \xi_2^0 = -15, y = 0$. **b** Three-soliton molecule profile of the solution u expressed by Eq. (4.7) with parameters: $N = 3, K_1 = \frac{1}{3}, K_2 = \frac{1}{2}, K_3 = \frac{1}{4}, \xi_1^0 = 0, \xi_2^0 = -15, \xi_3^0 = 12, y = 0$. **c** Four-soliton molecule profile of the solution u expressed by Eq. (4.7) with parameters: $N = 4, K_1 = \frac{1}{3}, K_2 = \frac{1}{2}, K_3 = \frac{1}{4}, K_4 = \frac{1}{5}, \xi_1^0 = 0, \xi_2^0 = -15, \xi_3^0 = 12, \xi_4^0 = -15, y = 0$

results amend the existing studies on those equations and enrich solution structures of higher-dimensional integrable equations.

We point out that Eq. (1.6) has a class of traveling wave solutions with an arbitrary function:

$$u=\frac{2}{\alpha}(\ln h(\eta))_{xx}+\frac{u_0}{\alpha},$$
$$v=\frac{2}{\alpha}(\ln h(\eta))_y,\ \eta=x-\frac{a}{b}y-\left(\frac{2\gamma a^3}{cb^2}+\frac{2au_0+d}{c}\right)t+\eta^0,$$

where the coefficients a and b are two nonzero constants, h is an arbitrary function and η^0 is an arbitrary constant. Thus, even if the parameter $\gamma=0$, the extended equation (1.6) may possess more solution structures including the missing D'Alembert-type waves [33] and various lump-type solutions [56]. Such solutions are very helpful to the study of complex nonlinear phenomena in fluid dynamics, oceanic dynamics, nonlinear optics and other fields.

We also remark that there should exist other kinds of sufficient conditions for Wronskian solutions to the resulting trilinear equation with the coefficient $\gamma\neq 0$. The N-soliton solution generated from the Wronskian formulation is of $(1+1)$-dimension, owing to the dimensional reduction in the sufficient conditions. However, the considered generalized $(2+1)$-dimensional KdV equation should have general N-soliton solutions, since its several integrable properties have been explored. It would be particularly interesting to check if the Hirota N-soliton conditions hold for the introduced trilinear equation and if there exist Wronskian formulations for soliton solutions to variable-coefficient generalizations of integrable equations [57,58]. Furthermore, it has become an increasingly interesting research topic to investigate diverse hybrid solutions, including the mixed breather-soliton molecule solutions, the mixed lump-soliton molecule solutions [59,60] and the mixed lump-kink N-M-soliton solutions [15,61,62]. It is hoped that more novel hybrid-type solutions could be explored for Eq. (1.6) in future works.

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Data availability All data supporting the findings of this study are included in this published article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

The proof of Lemma 2.1 is as follows:

Proof By applying the conditions (2.5) and differential rules for determinants as defined in [18], we have the following derivatives of the Wronskian determinant $f=f_N=|\widehat{N-1}|$:

$$\frac{\partial f}{\partial y}=(a_1\partial_x+a_2\partial_x^2+\cdots+a_m\partial_x^m)f \tag{A.1a}$$
$$\equiv(a_1\partial_{x_1}+a_2\partial_{x_2}+\cdots+a_m\partial_{x_m})f, \tag{A.1b}$$
$$\frac{\partial f}{\partial t}=(b_1\partial_x+b_2\partial_x^2+\cdots+b_n\partial_x^n)f \tag{A.1c}$$
$$\equiv(b_1\partial_{x_1}+b_2\partial_{x_2}+\cdots+b_n\partial_{x_n})f. \tag{A.1d}$$

Let us next suppose that

$$(a_1+a_2+\cdots+a_m)^n=\sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_m=n\\ r=1,2,\ldots,\mathrm{C}^{\mathrm{n}}_{\mathrm{m+n-1}}\\ \lambda_i\in\{0,1,2,\ldots\}}}\delta_{nr}\prod_{i=1}^{m}a_i^{\lambda_i}, \tag{A.2}$$

where δ_{nr}, $r=1,2,\ldots,\mathrm{C}^{\mathrm{n}}_{\mathrm{m+n-1}}$, are expansion coefficients. Using the definition of D-operators and the above conditions (A.1), we further get

$$\begin{aligned}
&D_y^pD_t^q f\cdot f\\
&=(\partial_y-\partial_{y'})^p(\partial_t-\partial_{t'})^q\\
&\quad\times f(x,y,t)f(x,y',t')|_{y'=y,t'=t}\\
&=\big[a_1(\partial_{x_1}-\partial_{x_1'})+a_2(\partial_{x_2}-\partial_{x_2'})\\
&\quad+\cdots+a_m(\partial_{x_m}-\partial_{x_m'})\big]^p\\
&\quad\times\big[b_1(\partial_{x_1}-\partial_{x_1'})+b_2(\partial_{x_2}-\partial_{x_2'})+\cdots\\
&\quad+b_n(\partial_{x_n}-\partial_{x_n'})\big]^q f(x,y,t)f(x',y,t)|_{x'=x}\\
&=\sum_{\substack{r=1\\ \lambda_1+\lambda_2+\cdots+\lambda_m=p\\ \lambda_i\in\{0,1,2,\ldots,p\}}}^{\mathrm{C}^p_{m+p-1}}\delta_{pr}\prod_{i=1}^{m}[a_i(\partial_{x_i}-\partial_{x_i'})]^{\lambda_i}\\
&\times\sum_{\substack{s=1\\ \omega_1+\omega_2+\cdots+\omega_n=q\\ \omega_j\in\{0,1,2,\ldots,q\}}}^{\mathrm{C}^q_{n+q-1}}\delta_{qs}\prod_{j=1}^{n}[b_j(\partial_{x_j}-\partial_{x_j'})]^{\omega_j}
\end{aligned}$$

$$\times f(x,y,t)f(x',y,t)|_{x'=x}$$
$$= \sum_{\substack{r=1 \\ \lambda_1+\lambda_2+\cdots+\lambda_m=p \\ \lambda_i\in\{0,1,2,\ldots,p\}}}^{C_{m+p-1}^{p}} \delta_{pr}\prod_{i=1}^{m}(a_iD_i)^{\lambda_i}$$
$$\times \sum_{\substack{s=1 \\ \omega_1+\omega_2+\cdots+\omega_n=q \\ \omega_j\in\{0,1,2,\ldots,q\}}}^{C_{n+q-1}^{q}} \delta_{qs}\prod_{j=1}^{n}(b_jD_j)^{\omega_j}$$
$$\times f(x,y,t)f(x',y,t)|_{x'=x}$$
$$= (a_1D_1+a_2D_2+\cdots+a_mD_m)^p(b_1D_1 + b_2D_2+\cdots+b_nD_n)^q f\cdot f. \tag{A.3}$$

It means that Lemma 2.1 holds.

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