Communications in Theoretical Physics

PAPER

An extended (2+1)-dimensional modified Korteweg–de Vries–Calogero–Bogoyavlenskii–Schiff equation: Lax pair and Darboux transformation

Communications in Theoretical Physics
Commun. Theor. Phys. 77 (2025) 035002 (9pp) https://doi.org/10.1088/1572-9494/ad84d3

An extended (2+1)-dimensional modified Korteweg–de Vries–Calogero–Bogoyavlenskii–Schiff equation: Lax pair and Darboux transformation

Li Cheng[1,2,*], **Yi Zhang**[3] **and Wen-Xiu Ma**[3,4,5,6,*]

[1] Normal College, Jinhua University of Vocational Technology, Jinhua 321007, Zhejiang, China
[2] Key Laboratory of Crop Harvesting Equipment Technology, Jinhua University of Vocational Technology, Jinhua 321007, Zhejiang, China
[3] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China
[4] Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[5] Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620-5700, United States of America
[6] Material Science Innovation and Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa

E-mail: mawx@cas.usf.edu and jhchengli@126.com

Received 11 June 2024, revised 6 October 2024
Accepted for publication 9 October 2024
Published 4 December 2024

Abstract

The aim of this paper is to study an extended modified Korteweg–de Vries–Calogero–Bogoyavlenskii–Schiff (mKdV-CBS) equation and present its Lax pair with a spectral parameter. Meanwhile, a Miura transformation is explored, which reveals the relationship between solutions of the extended mKdV-CBS equation and the extended (2+1)-dimensional Korteweg–de Vries (KdV) equation. On the basis of the obtained Lax pair and the existing research results, the Darboux transformation is derived, which plays a crucial role in presenting soliton solutions. In addition, soliton molecules are given by the velocity resonance mechanism.

Keywords: extended mKdV-CBS equation, Lax pair, Darboux transformation, soliton solution

(Some figures may appear in colour only in the online journal)

1. Introduction

Nonlinear phenomena are general problems in every field of engineering technology, science research, the natural world and human society activities. Nonlinear integrable equations play a crucial role in revealing nonlinear phenomena in various fields due to their fascinating features, such as N-soliton solutions [1–6], Bäcklund transformations [7–9], Lax pairs and the Painlevé test [10–13]. Among these integrable features, the Lax pair is a wonderful representation of integrable systems involving two linear operators, which can be differential operators or matrices [14]. A pair of linear operators L and A related to a given nonlinear partial differential equation may pave a way for solving the equation. It is difficult to find L and A corresponding to a given equation, so assuming that L and A are given and determining which partial differential equation they correspond to is actually simpler. The Painlevé test is widely and successfully used to study the integrability of nonlinear partial differential equations through analyzing the singularity structure of the solution. There are abundant successful examples of the method [14–16].

For integrable equations, in addition to investigating their integrable properties, the study of exact solutions has always been an important foundational topic in nonlinear science. There are many types of effective approaches to solve integrable

* Authors to whom any correspondence should be addressed.

equations, such as the inverse scattering method [15, 16], the Darboux transformation [17–20], Painlevé series expansion method [21] and Hirota direct method [22]. Among the existing techniques, the Darboux transformation is one of the most important ways for discussing compatibility equations of spectral problems. It is worth mentioning that the Darboux transformation is extremely useful in finding soliton solutions for nonlinear integrable equations from a trivial seed solution. In fact, through iteration, N-soliton solutions represented by special determinants, such as the Wronskian or Grammian, can be generated. Such N-soliton solutions have certain research value and practical significance in various scientific fields. Moreover, a Lax pair is very helpful for constructing Darboux transformations of integrable systems [23–25].

The Korteweg–de Vries (KdV) equation was first derived analytically by Johannes Korteweg together with his student, Gustav de Vries, in 1895 when they developed a theory for nonlinear waves [26]. The standard KdV equation is written as

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1}$$

which describes the disturbance of long, one-dimensional water waves on shallow-water surfaces with small amplitude [27]. By using the recursion operator of the KdV equation [28]

$$\Phi(u) \equiv \partial_x^2 + 4u + 2u_x\partial_x^{-1}, \tag{1.2}$$

the KdV equation (1.1) can also be referred to as

$$u_t + \Phi(u)u_x = 0,$$

where $\partial_x^{-1}f = \int f\,\mathrm{d}x$. Moreover, applying the same form of the KdV recursion operator (1.2), the (2+1)-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) equation is obtained by

$$u_t + \Phi(u)u_y = 0, \tag{1.3}$$

which is equivalent to

$$u_t + u_{xxy} + 4uu_y + 2u_x\partial_x^{-1}u_y = 0. \tag{1.4}$$

Taking the potential $u = \varphi_x$, equation (1.4) becomes

$$\varphi_{xt} + \varphi_{xxxy} + 4\varphi_x\varphi_{xy} + 2\varphi_{xx}\varphi_y = 0. \tag{1.5}$$

The CBS equation, also known as the breaking soliton equation, was proposed by Calogero and Degasperis [29], and also constructed by Bogoyavlenskii and Schiff via different techniques [30, 31]. This equation is widely used to describe the (2+1)-dimensional interaction between Riemann waves and long waves in shallow water [30, 31]. The KdV equation and the CBS equation are two well-known integrable models in (1+1) and (2+1)-dimensions, respectively, both possessing N-soliton solutions [22, 32, 33], Painlevé properties and infinitely many conservation laws [14, 34].

As a modified form of the standard KdV equation (1.1) in the nonlinear term, the modified Korteweg–de Vries (mKdV) equation

$$v_t - 6v^2v_x + v_{xxx} = 0, \tag{1.6}$$

is also one of the well-known nonlinear integrable equations. Miura transformations exist between the KdV equation and the mKdV equation. The following Miura transformation

$$u = -v^2 \pm v_x, \tag{1.7}$$

connects the solution u of the KdV equation (1.1) with the solution v of the mKdV equation (1.6) [35]. Similarly, employing the Miura transformation (1.7), the modified Calogero–Bogoyavlenskii–Schiff (mCBS) equation

$$v_t - 4v^2v_y - 2v_x\partial_x^{-1}(v^2)_y + v_{xxy} = 0, \tag{1.8}$$

can be derived from the CBS equation (1.4) [32]. It is easy to see that equation (1.8) is reduced to the modified KdV equation (1.6) in the case of $y = x$. The N-soliton solutions for the mCBS equation (1.8) can be generated through the Hirota direct method [32, 36].

Latterly, a novel (2+1)-dimensional mKdV system

$$u_t - 4u^2u_y - 2u_x\partial_x^{-1}(u^2)_y + u_{xxy} - 6u^2u_x + u_{xxx} = 0, \tag{1.9}$$

was introduced by combining the mKdV equation (1.6) and the mCBS equation (1.8) by Wang and Wazwaz *et al* [37–39]. This equation is known as the (2+1)-dimensional modified Korteweg–de Vries–Calogero–Bogoyavlenskii–Schiff (mKdV-CBS) equation, which has attracted the attention of many scholars. The nonlocal symmetry and soliton-cnoidal wave interaction solutions for equation (1.9) were established by virtue of the truncated Painlevé expansion and consistent Riccati expansion approach, respectively [37]. A large number of solutions with various physical characteristics, including multiple soliton solutions, kink solutions and singular solutions [38], were given by applying the simplified Hirota's method and other ways. By means of a direct symbolic computation, three classes of rational solutions for equation (1.9) were presented [39]. To the best of our knowledge, the Lax pair with a spectral parameter and Darboux transformation for equation (1.9) have not been revealed in previous articles.

In our previous work, a (2+1)-dimensional generalized KdV equation was investigated in the form [40]

$$a(6\gamma uu_x + u_{xxx}) + b(u_{xxy} + 2\gamma u_x\partial_x^{-1}u_y + 4\gamma uu_y) + cu_t = 0, \tag{1.10}$$

where the constants a, b, c and γ satisfy $\gamma c(a^2 + b^2) \neq 0$. Obviously, taking $a = b = c = \gamma = 1$, equation (1.10) is a combination of the KdV equation (1.1) and the CBS equation (1.4), which is also referred to as the (2+1)-dimensional KdV-like model [8]. The bilinear Bäcklund transformation and Lax pair for equation (1.10) were obtained [40], implying that a linear combination in a soliton hierarchy is still integrable. The extended form (1.10) may potentially describe the propagation of long, two-dimensional solitary waves in the branches of physics, including plasma physics, condensed matter, nonlinear optics and fluid dynamics.

Motivated by the recent studies mentioned above, we will consider an extension of the mKdV-CBS equation (1.9) by connecting equation (1.10), written as

$$a(v_{xxx} - 6\gamma^2v^2v_x) + b(v_{xxy} - 2\gamma^2v_x\partial_x^{-1}(v^2)_y - 4\gamma^2v^2v_y) + cv_t = 0, \tag{1.11}$$

where the constants a, b, c and γ are arbitrary, which satisfy $\gamma c(a^2+b^2)\neq 0$. It is obvious that for $a=b=c=1$ and $\gamma=\pm 1$, equation (1.11) reduces to the mKdV-CBS equation (1.9). Taking $a=b=c=1$ and $\gamma=\pm\sqrt{-1}$, equation (1.11) yields the following mKdV-CBS equation:

$$(v_{xxx}+6v^2v_x)+(v_{xxy}+2v_x\partial_x^{-1}(v^2)_y+4v^2v_y)+v_t=0. \tag{1.12}$$

We can clearly see that the extended form (1.11) contains significant integrable equations as its special cases, such as the mKdV equation and the mCBS equation. The mKdV equation and the mCBS equation possess wide applications in the fields of nonlinear science [41, 42]. In particular, the mKdV equation is used to describe the propagation of solitons in lattices, the motion of nonlinear Alfvén waves in plasma and fluid mechanics [41]. Therefore, we believe that equation (1.11) can be effectively applied in practical systems.

The paper is structured as follows. In section 2, based on the Lax pair of equation (1.10), the extended mKdV-CBS equation (1.11) will be derived, thereby presenting its Lax pair with a spectral parameter. Furthermore, the Darboux transformation will be furnished with the help of the obtained Lax pair. In section 3, one-soliton, two-soliton solutions and soliton molecules will be explored for equation (1.11). Some concluding remarks will be given in the final section.

2. Lax pair and Darboux transformation

In this section, we first present the Lax pair of the generalized KdV equation (1.10) obtained through the bilinear Bäcklund transformation [40]. The Lax pair of equation (1.10) can be expressed as

$$L_1=\partial_x^2+\gamma u-\lambda, \tag{2.1a}$$

$$\begin{aligned}L_2=c\partial_t+a(4\partial_x^3+6\gamma u\partial_x+3\gamma u_x)\\+b(2\gamma(\partial_x^{-1}u_y)\partial_x+4\lambda\partial_y-\gamma u_y),\end{aligned} \tag{2.1b}$$

where λ is an arbitrary constant. We see that equation (1.10) arises from the compatibility condition $[L_1, L_2]=0$ of the above system. The system (2.1) is equivalent to the following representation:

$$\phi_{xx}=\lambda\phi-\gamma u\phi, \tag{2.2a}$$

$$\begin{aligned}\phi_t=-\tfrac{a}{c}(4\phi_{xxx}+6\gamma u\phi_x+3\gamma u_x\phi)\\-\tfrac{b}{c}(2\gamma(\partial_x^{-1}u_y)\phi_x+4\lambda\phi_y-\gamma u_y\phi),\end{aligned} \tag{2.2b}$$

where ϕ is an eigenfunction and λ is a spectral parameter. The compatibility condition $\phi_{xxt}=\phi_{txx}$ is nothing but the potential u is a solution of equation (1.10). We next consider another second-order spectral problem corresponding to (2.2a) as follows:

$$\psi_{xx}=\lambda\psi-2\gamma v\psi_x, \tag{2.3}$$

where ψ is an eigenfunction and λ is a spectral parameter. Substituting the transformation [22]

$$\psi=\mathrm{e}^{-\gamma\partial^{-1}v}\phi \tag{2.4}$$

between the eigenfunctions ϕ and ψ into (2.3), then (2.3) becomes

$$\phi_{xx}+(-\gamma v_x-\gamma^2v^2)\phi=\lambda\phi. \tag{2.5}$$

Setting

$$u=-v_x-\gamma v^2, \tag{2.6}$$

then we see that (2.5) is the spectral problem (2.2a). Moreover, substituting (2.4) and (2.6) into (2.2b), a direct calculation yields

$$\begin{aligned}\psi_t=\tfrac{a}{c}[(2\gamma^2v^2+2\gamma v_x-4\lambda)\psi_x-4\lambda\gamma v\psi]\\+\tfrac{b}{c}[(2\gamma^2\partial_x^{-1}(v^2)_y+2\gamma v_y)\psi_x\\-4\lambda\psi_y-4\lambda\gamma(\partial_x^{-1}v_y)\psi],\end{aligned} \tag{2.7}$$

where (2.2a) and the following constraint

$$\frac{a}{c}[2\gamma^2v^3-v_{xx}]+\frac{b}{c}[2\gamma^2v\partial_x^{-1}(v^2)_y-v_{xy}]-\partial_x^{-1}v_t=0 \tag{2.8}$$

have been applied. It is easy to observe that condition (2.8) is just equation (1.11) by taking the derivative with respect to x at both ends of (2.8). By using (2.3) and (2.7), it can directly verify that the compatibility condition $\psi_{xxt}=\psi_{txx}$ is nothing but the potential v is a solution of the extended mKdV-CBS equation (1.11). Namely, setting

$$L_1'=\partial_x^2+2\gamma v\partial_x-\lambda, \tag{2.9a}$$

$$\begin{aligned}L_2'=c\partial_t-a[(2\gamma^2v^2+2\gamma v_x-4\lambda)\partial_x-4\lambda\gamma v]\\-b[(2\gamma^2\partial_x^{-1}(v^2)_y+2\gamma v_y)\partial_x-4\lambda\partial_y-(4\lambda\gamma\partial_x^{-1}v_y)],\end{aligned} \tag{2.9b}$$

equation (1.11) is generated from the compatibility condition $[L_1', L_2']=0$ of these two operators. This shows the system (2.9) is a Lax pair of equation (1.11). Further, we also found that formula (2.6) is a Miura transformation, which provides a relation between solutions of equation (1.10) and equation (1.11).

In the following, according to the spectral problem (2.3), we derived the Darboux transformation for the extended mKdV-CBS equation (1.11) via the same method presented in the existing literature [14, 19, 22]. We consider a linear transformation $\psi\to\bar{\psi}$

$$\bar{\psi}=\sigma\psi_x+\psi, \tag{2.10}$$

which transforms (2.3) into the following spectral problem with a potential $\bar{v}$

$$\bar{\psi}_{xx}=\lambda\bar{\psi}-2\gamma\bar{v}\bar{\psi}_x. \tag{2.11}$$

Applying the spectral problem (2.3), the above transformation (2.10) becomes

$$\begin{aligned}\bar{\psi}_{xx}=(-2\gamma v-4\gamma v\sigma_x+4\gamma^2\sigma v^2+\sigma_{xx}-2\gamma\sigma v_x+\lambda\sigma)\psi_x\\+\lambda(1+2\sigma_x-2\gamma\sigma v)\psi.\end{aligned} \tag{2.12}$$

In addition, substituting (2.10) and its first-order derivative into (2.11) gives

$$\begin{aligned}\bar{\psi}_{xx}=(4\gamma^2\sigma v\bar{v}-2\gamma\sigma_x\bar{v}-2\gamma\bar{v}+\lambda\sigma)\psi_x\\+(-2\gamma\lambda\sigma\bar{v}+\lambda)\psi.\end{aligned} \tag{2.13}$$

Comparing the coefficients of ψ_x and ψ in (2.12) and (2.13), we have

$$\bar{v} = v - \frac{\sigma_x}{\gamma\sigma}, \tag{2.14}$$

$$\sigma_{xx} + 4\sigma\gamma^2 v^2 - 4\gamma\sigma_x v - 2a\gamma v_x - 2\gamma v = 4\sigma\gamma^2 v\bar{v} - 2\gamma\bar{v} - 2\gamma\sigma_x\bar{v}. \tag{2.15}$$

Substituting (2.14) into (2.15) and integrating the resulting form, we get

$$\sigma_x\sigma^{-2} - 2\gamma\sigma^{-1}v + \sigma^{-2} = \lambda_1, \tag{2.16}$$

where λ_1 is a constant. Furthermore, assuming $\sigma = -ff_x^{-1}$ and substituting it into (2.16) yields

$$f_{xx} + 2\gamma v f_x = \lambda_1 f. \tag{2.17}$$

Consequently, the above results can be summarized as the following theorem:

Theorem 2.1. *Suppose that the linear problems (2.3) and (2.11) of equation (1.11) have potentials* v *and* $\bar{v}$*, respectively. Then the linear transformation*

$$\bar{\psi} = \psi + \sigma\psi_x, \quad \sigma = -\frac{f(x, \lambda_1)}{f_x(x, \lambda_1)} \tag{2.18}$$

converts (2.3) into (2.11), where

$$\bar{v} = v + \frac{1}{\gamma}\left[\ln\frac{f_x(x, \lambda_1)}{f(x, \lambda_1)}\right]_x, \tag{2.19}$$

and f *is the fixed solution of (2.3) with* $\lambda = \lambda_1$.

As we know, transformation (2.18) with (2.19), which transforms the linear problem (2.3) into the linear problem (2.11) with the same form, is defined as the Darboux transformation [14, 22]. It is worth pointing out that the Darboux transformation is very powerful in constructing soliton solutions. Before applying the above theorem to soliton theory, we also need to consider the following proposition:

Proposition 2.1. *Suppose that* f *satisfies (2.17) and the following temporal part*

$$\begin{aligned} f_t = &\tfrac{a}{c}[(2\gamma^2 v^2 + 2\gamma v_x - 4\lambda_1)f_x - 4\lambda_1\gamma v f] \\ &+ \tfrac{b}{c}[(2\gamma^2\partial_x^{-1}(v^2)_y + 2\gamma v_y)f_x - 4\lambda_1 f_y \\ &- (4\lambda_1\gamma\partial_x^{-1}v_y)f]. \end{aligned} \tag{2.20}$$

Under the Darboux transformation (2.18) with (2.19), then $\bar{\psi}$ *satisfies*

$$\begin{aligned} \bar{\psi}_t = &\tfrac{a}{c}[(2\gamma^2\bar{v}^2 + 2\gamma\bar{v}_x - 4\lambda)\bar{\psi}_x - 4\lambda\gamma\bar{v}\bar{\psi}] \\ &+ \tfrac{b}{c}[(2\gamma^2\partial_x^{-1}(\bar{v}_y^2) + 2\gamma\bar{v}_y)\bar{\psi}_x \\ &- 4\lambda\bar{\psi}_y - (4\lambda\gamma\partial_x^{-1}\bar{v}_y)\bar{\psi}]. \end{aligned} \tag{2.21}$$

Using the same calculation as described in the existing literature [14, 19], the proof of proposition 2.1 will be given in appendix.

3. Soliton solutions and soliton molecules

From the presented results in section 2, it can be seen that if v is a solution to the extended mKdV-CBS equation (1.11), then $\bar{v}$ determined by (2.19) is also a solution of equation (1.11). We now construct soliton solutions for equation (1.11) by utilizing the Darboux transformation. Taking $v = 0$ as the seed solution and choosing $\lambda_1 = k_1^2$ in (2.17) and (2.20), we obtain the following linear partial differential equations:

$$f_{xx} = k_1^2 f, \tag{3.1a}$$

$$f_t = \frac{a}{c}(-4k_1^2 f_x) + \frac{b}{c}(2\gamma^2 d_1 f_x - 4k_1^2 f_y - 4\gamma k_1^2 d_2 f), \tag{3.1b}$$

where d_1, d_2 are arbitrary integral constants, and k_1 is a nonzero constant. Solving the above system (3.1), we have

$$f = C_1 e^{\xi_1} + C_2 e^{-\xi_1}, \tag{3.2}$$

with

$$\xi_1 = k_1 x + l_1 y - \left[\tfrac{4a}{c}k_1^3 - \tfrac{b}{c}(2\gamma^2 d_1 k_1 - 4k_1^2 l_1)\right]t + \xi_1^0, \; d_2 = 0. \tag{3.3}$$

Here C_1, C_2, l_1, d_1 and ξ_1^0 are arbitrary given that every term in the solution makes sense. Through transformation (2.19), an exact one-soliton solution of equation (1.11) can be given as

$$\bar{v} = \frac{4C_1C_2k_1}{\gamma(C_1^2 e^{2\xi_1} - C_2^2 e^{-2\xi_1})}. \tag{3.4}$$

In particular, taking $C_2 = \mathrm{i}C_1 e^{-\delta}$, $\mathrm{i} = \sqrt{-1}$, the above one-soliton solution (3.4) can be expressed as

$$\bar{v} = \frac{2\mathrm{i}k_1}{\gamma}\operatorname{sech}(2\xi_1 + \delta), \tag{3.5}$$

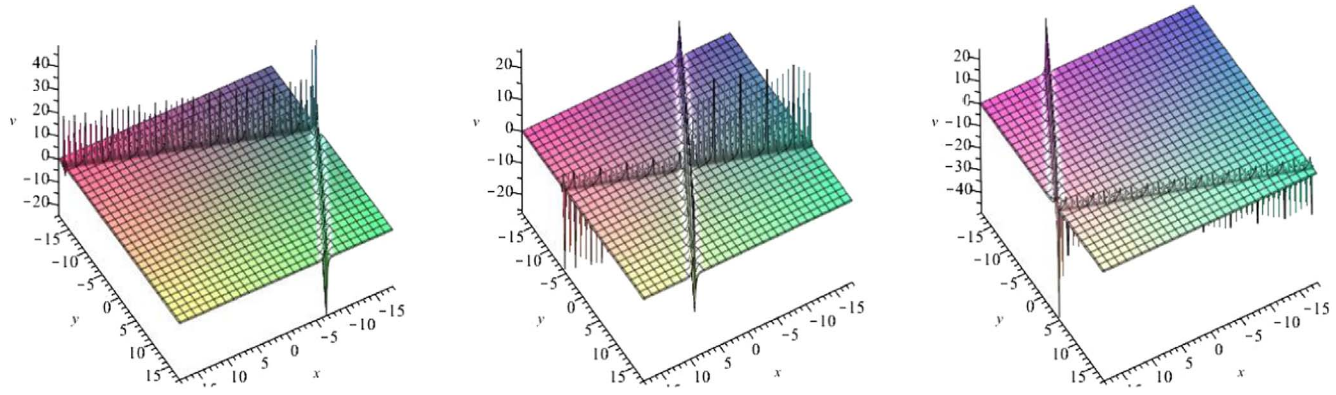
where ξ_1 is defined by (3.3) and δ is an arbitrary constant.

We iterate the above Darboux transformation, a direct computation yields:

$$\begin{aligned} \bar{\bar{v}} &= \bar{v} + \tfrac{1}{\gamma}\left[\ln\tfrac{\bar{\psi}_x(x, \lambda_2)}{\bar{\psi}(x, \lambda_2)}\right]_x \\ &= v + \tfrac{1}{\gamma}\left[\ln\tfrac{W_r(f_x(x, \lambda_1), f_x(x, \lambda_2))}{W_r(f(x, \lambda_1), f(x, \lambda_2))}\right]_x, \end{aligned} \tag{3.6}$$

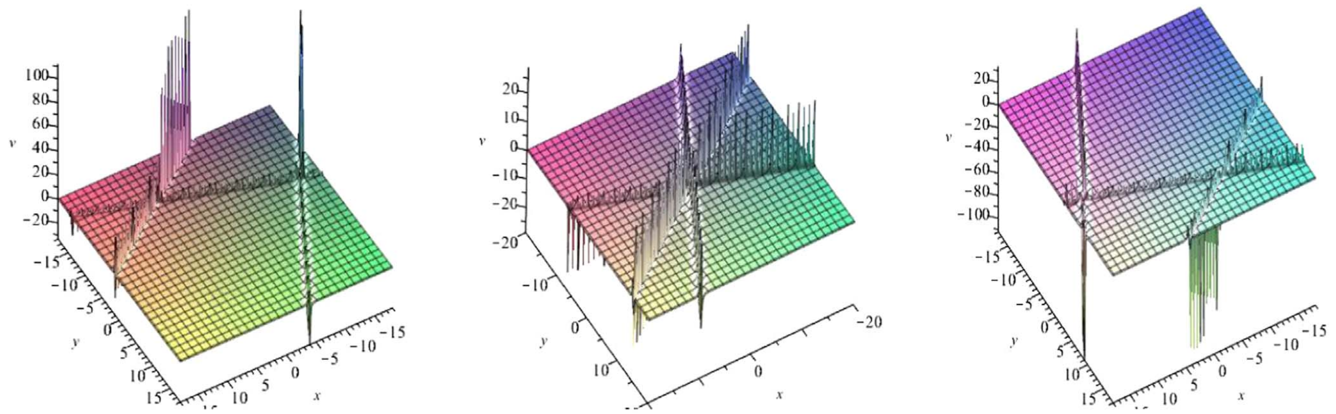
where $W_r(f(x, \lambda_1), f(x, \lambda_2)) = f(x, \lambda_1)f_x(x, \lambda_2) - f(x, \lambda_2)f_x(x, \lambda_1)$ is the standard Wronskian determinant. Let us take

$$\begin{aligned} &v = 0, f(x, \lambda_1) = \cosh\xi_1, \quad f(x, \lambda_2) = \sinh\xi_2, \\ &\xi_i = k_i x + l_i y - \left[\tfrac{4a}{c}k_i^3 - \tfrac{b}{c}(2\gamma^2 d_1 k_i - 4k_i^2 l_i)\right]t \\ &\quad + \xi_i^0, \; i = 1, 2. \end{aligned} \tag{3.7}$$



(a) (b) (c)

Figure 1. Three-dimensional plots of $\bar{\bar{v}}$ determined by (3.8) with (3.11) when $t = -2$ in (a), $t = 0$ in (b) and $t = 2$ in (c).



(a) (b) (c)

Figure 2. Three-dimensional plots of $v[3]$ determined by (3.9) with (3.10) and (3.12) when $t = -1.6$ in (a), $t = 0$ in (b) and $t = 1.6$ in (c).

A two-soliton solution of equation (1.11) possesses the form

$$\bar{\bar{v}} = \frac{2(k_1^2 - k_2^2)(k_2 \sinh 2\xi_1 + k_1 \sinh 2\xi_2)}{\gamma[(k_1 + k_2)^2 \cosh^2(\xi_1 - \xi_2) - (k_2 - k_1)^2 \cosh^2(\xi_1 + \xi_2)]}, \tag{3.8}$$

with ξ_i, $i = 1, 2$ being given by (3.7) and the parameters involved make the solution meaningful. The N-times iterated or N-fold Darboux transformation leads to

$$v[N] = v + \frac{1}{\gamma}\left[\ln \frac{W_r(f_x(x, \lambda_1), f_x(x, \lambda_2), \cdots, f_x(x, \lambda_N))}{W_r(f(x, \lambda_1), f(x, \lambda_2), \cdots, f(x, \lambda_N))}\right]_x, \tag{3.9}$$

where

$$W_r(f(x, \lambda_1), f(x, \lambda_2), \cdots f(x, \lambda_N)) = \det(\partial_x^{i-1} f(x, \lambda_j))_{1 \leqslant i \leqslant N, 1 \leqslant j \leqslant N}.$$

Furthermore, we choose

$$v = 0, f(x, \lambda_{2l+1}) = \cosh \xi_{2l+1}, f(x, \lambda_{2l}) = \sinh \xi_{2l},$$
$$\xi_i = k_i x + l_i y - \left[\frac{4a}{c} k_i^3 - \frac{b}{c}(2\gamma^2 d_1 k_i - 4k_i^2 l_i)\right] t + \xi_i^0, 1 \leqslant i \leqslant N, \tag{3.10}$$

then the function $v[N]$ defined by (3.9) and (3.10) is an N-soliton solution of equation (1.11). Figure 1 shows the evolution of a special two-soliton solution $\bar{\bar{v}}$ determined by (3.8) with the parameter selections

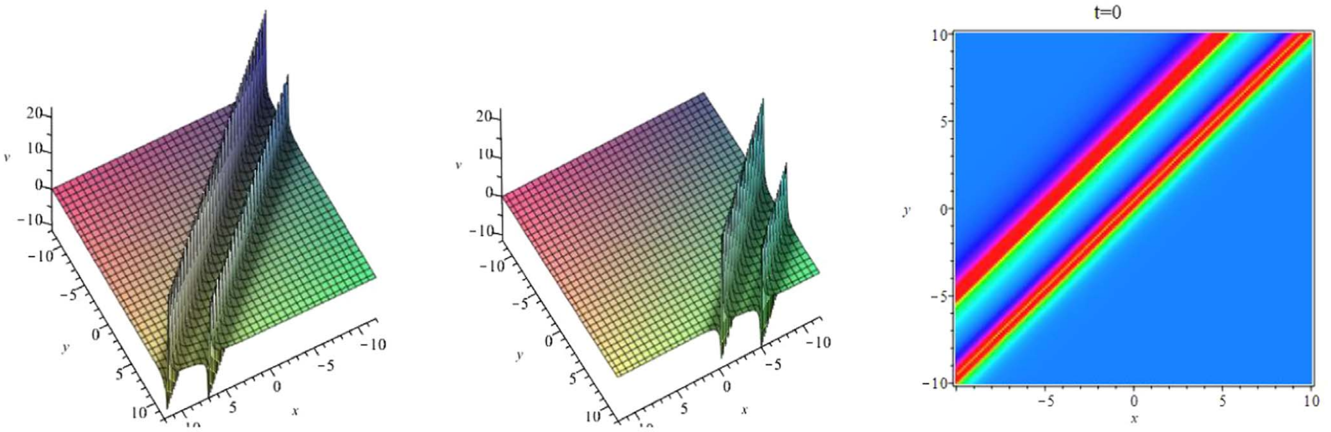
$$a = b = c = \gamma = 1, k_1 = 1, l_1 = 3, k_2 = 2, l_2 = -1, d_1 = \xi_1^0 = \xi_2^0 = 0. \tag{3.11}$$

Figure 2 displays the propagation of a special three-soliton solution determined by (3.9) with (3.10) under the parameter selections

$$a = b = c = \gamma = 1, k_1 = 1, l_1 = 3, k_2 = 2, l_2 = -1, k_3 = 3, l_3 = -4, d_1 = \xi_1^0 = \xi_2^0 = \xi_3^0 = 0. \tag{3.12}$$

It can be observed that two or three solitary waves propagate at a certain speed, collide after a while, and then continue to propagate in their original shapes.

Soliton molecules are bound states of solitons, which have been observed in systems from optical fibers to mode-locked lasers. At present, soliton molecules have increasingly become an interesting topic in various areas, including



(a) (b) (c)

Figure 3. Three-dimensional plots of $\bar{\bar{v}}$ determined by (3.15) with (3.16) when $t = 0$ in (a), $t = 2$ in (b) and density plot in (c).

nonlinear optics, fluid mechanics and Bose–Einstein condensates, since they provide important insights into the fundamental interactions between solitons and the potential dynamics in complex nonlinear systems. There are different solution methods for finding soliton molecules, for instance, the velocity resonance mechanism proposed by Lou [43, 44]. A new mechanism for discovering soliton molecules, called velocity resonance, was introduced by employing velocity resonance $k_i/k_j = \omega_i/\omega_j$, where the parameters k_i, k_j and ω_i, ω_j are wave numbers and frequencies, respectively. Under the above resonant condition, the ith and jth solitons are bounded and develop a soliton molecule by choosing appropriate solution parameters. This method can not only be extensively applied to (1+1)-dimensional systems [45], but also to higher-dimensional nonlinear systems [44]. In this section, we would like to investigate soliton molecules of equation (1.11).

For the (2+1) dimensional equation (1.11), the velocity resonant conditions become

$$\frac{k_i}{k_j} = \frac{l_i}{l_j} = \frac{-\frac{4a}{c}k_i^3 + \frac{b}{c}(2\gamma^2 d_1 k_i - 4k_i^2 l_i)}{-\frac{4a}{c}k_j^3 + \frac{b}{c}(2\gamma^2 d_1 k_j - 4k_j^2 l_j)},\ k_i \neq \pm k_j. \tag{3.13}$$

The solution of equation (3.13) can be obtained as follows:

$$l_i = -\frac{ak_i}{b},\ l_j = -\frac{ak_j}{b}. \tag{3.14}$$

Therefore, via the resonance condition (3.14), the two-soliton solution (3.8) of equation (1.11) yields a two-soliton molecule

$$\bar{\bar{v}} = \frac{2(k_1^2 - k_2^2)(k_2 \sinh 2\xi_1 + k_1 \sinh 2\xi_2)}{\gamma[(k_1 + k_2)^2 \cosh^2(\xi_1 - \xi_2) - (k_2 - k_1)^2 \cosh^2(\xi_1 + \xi_2)]},$$

$$\xi_i = k_i x - \frac{a}{b}k_i y + \frac{2b\gamma^2 d_1}{c}k_i t + \xi_i^0,\ i = 1, 2, \tag{3.15}$$

with arbitrary non-zero constants k_i and d_1.

From solution (3.15), a special two-soliton molecule profile of $\bar{\bar{v}}$ with the parameters

$$a = b = c = \gamma = 1,\ k_1 = 1,\ k_2 = 0.5,$$
$$d_1 = 3,\ \xi_1^0 = 1,\ \xi_2^0 = 2, \tag{3.16}$$

is plotted in figure 3. It can be observed that two line soliton waves are parallel to each other in the (x, y)-plane, and they carry different widths and amplitudes due to $k_1 \neq k_2$, $l_1 \neq l_2$. However, the velocities of the two solitons in the molecule are the same. It is also worth explicitly noting that the selection of the parameters ξ_1^0 and ξ_2^0 will cause a change in the distance between two solitons in the molecule.

4. Concluding remarks

In summary, an extended mKdV-CBS equation (1.11) has been explored by means of the existing results, thereby presenting its Lax pair with a spectral parameter. Meanwhile, a Miura transformation has been found, which provides a relation between solutions of the extended mKdV-CBS equation (1.11) and the extended KdV equation (1.10). Then, associated with the resulting Lax pair, the Darboux transformation has been derived to the introduced equation in detail. The resultant Darboux transformation has been applied to soliton solutions. Furthermore, the soliton molecules have been given by the velocity resonance mechanism. Our results indicate that equation (1.11) is integrable and they provide good supplements to the existing literature. The present study is believed to contribute to a general understanding of the complex dynamical phenomena in areas such as fluids, ocean dynamics and plasmas. In particular, the investigation of soliton solutions would be helpful in describing the behaviors of wave propagations in dispersive wave theories.

We also point out that equation (1.11) can be written as a Hirota bilinear form:

$$D_x^2 f \cdot f' = 0, \tag{4.1a}$$

$$(aD_x^3 + bD_x^2 D_y + cD_t) f \cdot f' = 0, \tag{4.1b}$$

under the logarithmic transformations

$$v = \frac{1}{\gamma}\left[\ln\left(\frac{f}{f'}\right)\right]_x,\ \rho = \ln(ff'), \tag{4.2}$$

where the auxiliary function ρ satisfies $\rho_{xx}+\gamma^2v^2=0$ and D is the Hirota's bilinear differential operator [1]. There are any potential extensions or future research directions that could be built upon our work. A large number of interesting solutions generated by the Darboux transformation and the Hirota bilinear form, including lump solutions [46–49], Hirota N-soliton solutions, breathers and Wronskian solutions [4, 7, 50–52], will be discussed in the future.

Acknowledgments

The authors express their sincere thanks to the referees and editors for their valuable comments. This work was supported by the National Natural Science Foundation of China (Grant No. 12271488).

Conflict of interest

The authors declare that they have no conflict of interest.

Appendix

We give a proof of proposition 2.1.

Proof. We differentiate the transformation (2.18) with respect to t, and by using (2.7), we obtain

$$\begin{aligned}\bar{\psi}_t=&\Big[\tfrac{a}{c}(\sigma A_1+B_1+\sigma B_{1,x}-2\gamma\sigma v B_1)\\&+\tfrac{b}{c}(\sigma A_2+B_2+\sigma B_{2,x}-2\gamma\sigma v B_2)+\sigma_t\Big]\psi_x\\&+\Big[\tfrac{a}{c}(A_1+\sigma A_{1,x}+\lambda\sigma B_1)\\&+\tfrac{b}{c}(A_2+\sigma A_{2,x}+\lambda\sigma B_2)\Big]\psi-\tfrac{b}{c}(4\lambda\psi_y+4\lambda\sigma\psi_{xy}),\end{aligned}\tag{A1}$$

where

$$\begin{aligned}&A_1=-4\lambda\gamma v,\ B_1=-4\lambda+2\gamma v_x+2\gamma^2v^2,\\&A_2=-4\lambda\gamma\partial_x^{-1}v_y,\ B_2=2\gamma^2\partial_x^{-1}(v^2)_y+2\gamma v_y.\end{aligned}\tag{A2}$$

In addition, substituting (2.18) into (2.21) yields

$$\begin{aligned}\bar{\psi}_t=&\Big\{\tfrac{a}{c}[\sigma\bar{A}_1+\bar{B}_1(1+\sigma_x-2\gamma\sigma v)]\\&+\tfrac{b}{c}[\sigma\bar{A}_2+\bar{B}_2(1+\sigma_x-2\gamma\sigma v)-4\lambda\sigma_y]\Big\}\psi_x\\&+\Big\{\tfrac{a}{c}(\bar{A}_1+\lambda\sigma\bar{B}_1)+\tfrac{b}{c}(\bar{A}_2+\lambda\sigma\bar{B}_2)\Big\}\psi\\&-\tfrac{b}{c}(4\lambda\psi_y+4\lambda\sigma\psi_{xy}),\end{aligned}\tag{A3}$$

where

$$\begin{aligned}&\bar{A}_1=-4\lambda\gamma\bar{v},\ \bar{B}_1=-4\lambda+2\gamma\bar{v}_x+2\gamma^2\bar{v}^2,\\&\bar{A}_2=-4\lambda\gamma\partial_x^{-1}\bar{v}_y,\ \bar{B}_2=2\gamma^2\partial_x^{-1}(\bar{v}^2)_y+2\gamma\bar{v}_y.\end{aligned}\tag{A4}$$

To prove proposition 2.1, we only need to prove that the above two expressions hold simultaneously.

Let us firstly verify that the coefficients of ψ in (A1) and (A3) are equal. According to (2.18) and (2.19), we can obtain the following relational expressions:

$$\sigma_x=-1+2\gamma\sigma v+\lambda_1\sigma^2,\tag{A5}$$

$$\sigma_{xy}=2\gamma\sigma_y v+2\gamma\sigma v_y+2\lambda_1\sigma\sigma_y,\tag{A6}$$

$$\partial_x^{-1}\left(\frac{\sigma_x}{\sigma}\right)_y=\frac{\sigma_y}{\sigma},\tag{A7}$$

$$\left(\frac{\sigma_x}{\sigma}\right)_y=\lambda_1\sigma_y+2\gamma v_y+\frac{\sigma_y}{\sigma^2}.\tag{A8}$$

By using (A5), we have

$$\begin{aligned}&(\bar{A}_1-A_1)+\lambda\sigma(\bar{B}_1-B_1)-\sigma A_{1,x}\\&\quad=4\lambda\sigma_x\sigma^{-1}(1-2\gamma\sigma v-\lambda_1\sigma^2+\sigma_x)=0,\end{aligned}\tag{A9}$$

which is equivalent to

$$A_1+\sigma A_{1,x}+\lambda\sigma B_1=\bar{A}_1+\lambda\sigma\bar{B}_1,\tag{A10}$$

with A_1, B_1 and $\bar{A}_1$, $\bar{B}_1$ being given by (A2) and (A4), respectively. To verify

$$A_2+\sigma A_{2,x}+\lambda\sigma B_2=\bar{A}_2+\lambda\sigma\bar{B}_2,\tag{A11}$$

we next only need to verify

$$(\bar{A}_2-A_2)+\lambda\sigma(\bar{B}_2-B_2)-\sigma A_{2,x}=0,\tag{A12}$$

with A_2, B_2 and $\bar{A}_2$, $\bar{B}_2$ being given by (A2) and (A4), respectively. By applying (2.14), (2.19), (A7) and (A8), a direct computation leads to

$$\begin{aligned}&(\bar{A}_2-A_2)+\lambda\sigma(\bar{B}_2-B_2)-\sigma A_{2,x}\\&\quad=\lambda\left[\tfrac{2\sigma_y}{\sigma}+4\sigma\partial_x^{-1}\left(\tfrac{\sigma_x^2}{2\sigma^2}-\tfrac{\gamma\sigma_x}{\sigma}v\right)_y-2\lambda_1\sigma\sigma_y\right].\end{aligned}\tag{A13}$$

So we need to show

$$\partial_x^{-1}\left(\frac{\sigma_x^2}{2\sigma^2}-\frac{\gamma\sigma_x}{\sigma}v\right)_y=\frac{\lambda_1\sigma_y}{2}-\frac{\sigma_y}{2\sigma^2},\tag{A14}$$

that is to say

$$\left(\frac{\sigma_x^2}{2\sigma^2}-\frac{\gamma\sigma_x}{\sigma}v\right)_y=\frac{\lambda_1\sigma_{xy}}{2}-\left(\frac{\sigma_y}{2\sigma^2}\right)_x.\tag{A15}$$

On the one hand, again by using (A8) and (A5), the left hand side of (A15) becomes

$$\begin{aligned}\left(\frac{\sigma_x^2}{2\sigma^2}-\frac{\gamma\sigma_x}{\sigma}v\right)_y=&\left(\lambda_1\sigma_y+2\gamma v_y+\frac{\sigma_y}{\sigma^2}\right)\left(-\gamma v+\frac{\sigma_x}{\sigma}\right)\\&-\frac{\gamma\sigma_x}{\sigma}v_y\\=&-\gamma\lambda_1\sigma_y v-2\gamma^2vv_y-\gamma\frac{\sigma_y}{\sigma^2}v+\lambda_1\frac{\sigma_x\sigma_y}{\sigma}\\&+\gamma\frac{\sigma_x}{\sigma}v_y+\frac{\sigma_x\sigma_y}{\sigma^3}\\=&\gamma\lambda_1\sigma_y v-\gamma\frac{\sigma_y}{\sigma^2}v-\lambda_1\frac{\sigma_y}{\sigma}\\&+\lambda_1^2\sigma\sigma_y-\frac{\gamma}{\sigma}v_y+\gamma\lambda_1\sigma v_y+\frac{\sigma_x\sigma_y}{\sigma^3}.\end{aligned}\tag{A16}$$

On the other hand, by applying (A5) and (A6), the right hand side of (A15) becomes

$$\frac{\lambda_1\sigma_{xy}}{2} - \left(\frac{\sigma_y}{2\sigma^2}\right)_x = (\gamma\sigma_y v + \gamma\sigma v_y + \lambda_1\sigma\sigma_y)\left(\lambda_1 - \frac{1}{\sigma^2}\right) + \frac{\sigma_x\sigma_y}{\sigma^3} = \gamma\lambda_1\sigma_y v + \gamma\lambda_1\sigma v_y + \lambda_1^2\sigma\sigma_y - \gamma\frac{\sigma_y}{\sigma^2}v - \frac{\gamma}{\sigma}v_y - \lambda_1\frac{\sigma_y}{\sigma} + \frac{\sigma_x\sigma_y}{\sigma^3}. \tag{A17}$$

This shows that (A11) is valid. Hence, from (A10) and (A11), we can see that the coefficients of ψ in (A1) and (A3) are equal.

Let us secondly verify that the coefficients of ψ_x in (A1) and (A3) are also equal. By employing (2.18) and (2.20), we get

$$\sigma_t = \frac{a}{c}[\sigma^2 A_{1,x}(\lambda_1) + \sigma_x B_1(\lambda_1) - \sigma B_{1,x}(\lambda_1)] + \frac{b}{c}[\sigma^2 A_{2,x}(\lambda_1) + \sigma_x B_2 - \sigma B_{2,x} - 4\lambda_1\sigma_y]. \tag{A18}$$

We now verify the following equality:

$$\frac{a}{c}(\sigma A_1 + B_1 + \sigma B_{1,x} - 2\gamma\sigma v B_1) + \frac{b}{c}(\sigma A_2 + B_2 + \sigma B_{2,x} - 2\gamma\sigma v B_2) + \sigma_t = \frac{a}{c}[\sigma\bar{A}_1 + \bar{B}_1(1 + \sigma_x - 2\gamma\sigma v)] + \frac{b}{c}[\sigma\bar{A}_2 + \bar{B}_2(1 + \sigma_x - 2\gamma\sigma v) - 4\lambda\sigma_y]. \tag{A19}$$

Via (A18), (A5), (2.14) and (2.19), a direct calculation yields

$$\begin{aligned}
&\sigma\bar{A}_1 + (1 + \sigma_x - 2\gamma\sigma v)\bar{B}_1 \\
&- (\sigma A_1 + B_1 + \sigma B_{1,x} - 2\gamma\sigma v B_1) \\
&- [\sigma^2 A_{1,x}(\lambda_1) + \sigma_x B_1(\lambda_1) - \sigma B_{1,x}(\lambda_1)] \\
&= \sigma(\bar{A}_1 - A_1) + \lambda_1\sigma^2(\bar{B}_1 - B_1) - \sigma^2 A_{1,x}(\lambda_1) \\
&+ \sigma_x(B_1 - B_1(\lambda_1)) \\
&= 4\lambda\sigma_x + \lambda_1\sigma^2(\bar{B}_1 - B_1) - \sigma^2 A_{1,x}(\lambda_1) \\
&+ \sigma_x(-4\lambda + 4\lambda_1) \\
&= \sigma[\bar{A}_1(\lambda_1) - A_1(\lambda_1)] + \lambda_1\sigma^2[\bar{B}_1(\lambda_1) - B_1(\lambda_1)] \\
&- \sigma^2 A_{1,x}(\lambda_1).
\end{aligned} \tag{A20}$$

By virtue of the formula (A9) associated with the spectral parameter λ_1 , the final expression in (A20) is equal to zero. Directly we can also compute that

$$\begin{aligned}
&\sigma\bar{A}_2 + (1 + \sigma_x - 2\gamma\sigma v)\bar{B}_2 - 4\lambda\sigma_y \\
&- (\sigma A_2 + B_2 + \sigma B_{2,x} - 2\gamma\sigma v B_2) \\
&- [\sigma^2 A_{2,x}(\lambda_1) + \sigma_x B_2 - \sigma B_{2,x} - 4\lambda_1\sigma_y] \\
&= \sigma(\bar{A}_2 - A_2) + \lambda_1\sigma^2(\bar{B}_2 - B_2) - 4\lambda\sigma_y - \sigma^2 A_{2,x}(\lambda_1) \\
&+ 4\lambda_1\sigma_y \\
&= 4\lambda\sigma_y + \lambda_1\sigma^2(\bar{B}_2 - B_2) - 4\lambda\sigma_y - \sigma^2 A_{2,x}(\lambda_1) + 4\lambda_1\sigma_y \\
&= \sigma[\bar{A}_2(\lambda_1) - A_2(\lambda_1)] + \lambda_1\sigma^2(\bar{B}_2 - B_2) \\
&- \sigma^2 A_{2,x}(\lambda_1) = 0,
\end{aligned} \tag{A21}$$

where the formulas (A5), (A7) and (A11) associated with the spectral parameter λ_1 have been applied. Therefore, based on (A20) and (A21), we show that equality (A19) holds. This completes the proof.

References

[1] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge University Press)

[2] Ma W X 2022 Soliton solutions by means of Hirota bilinear forms *Partial Differ Equ. Appl. Math.* **5** 100220

[3] Lü X and Chen S J 2023 *N*-soliton solutions and associated integrability for a novel (2+1)-dimensional generalized KdV equation *Chaos Solitons Fractals* **169** 113291

[4] Ma W X 2021 *N*-soliton solution and the Hirota condition of a (2+1)-dimensional combined equation *Math. Comput. Simul.* **190** 270–9

[5] Ma W X 2023 Matrix integrable fifth-order mKdV equations and their soliton solutions *Chin. Phys.* B **32** 020201

[6] Ma W X 2022 Reduced nonlocal integrable mKdV equations of type (−λ, λ) and their exact soliton solutions *Commun. Theor. Phys.* **74** 065002

[7] Gao D, Lü X and Peng M S 2023 Study on the (2+1)-dimensional extension of Hietarinta equation: soliton solutions and Bäcklund transformation *Phys. Scr.* **98** 095225

[8] Lü X, Ma W X and Khalique C M 2015 A direct bilinear Bäcklund transformation of a (2+1)-dimensional Korteweg–de Vries-like model *Appl. Math. Lett.* **50** 37–42

[9] Lü X and He X J 2023 Bäcklund transformation to solve the generalized (3+1)-dimensional KP-YTSF equation and kinky periodic-wave, Wronskian and Grammian solutions *J. Appl. Anal. Comput.* **13** 758–81

[10] Fan E G and Chow K W 2011 Darboux covariant Lax pairs and infinite conservation laws of the (2+1)-dimensional breaking soliton equation *J. Math. Phys.* **52** 023504

[11] Wazwaz A M 2020 Painlevé analysis for Boiti–Leon–Manna–Pempinelli equation of higher dimensions with time dependent coefficients: multiple soliton solutions *Phys. Lett.* A **384** 126310

[12] Wang Y H, Wang H and Temuer C 2014 Lax pair, conservation laws, and multi-shock wave solutions of the DJKM equation with Bell polynomials and symbolic computation *Nonlinear Dyn.* **78** 1101–7

[13] Ma W X 2024 A combined Liouville integrable hierarchy associated with a fourth-order matrix spectral problem *Commun. Theor. Phys.* **76** 075001

[14] Li Y S 1999 *Soliton and Integrable System* (Shanghai Scientific and Technological Education Publishing House) (in Chinese)

[15] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press)

[16] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)

[17] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Springer)

[18] Geng X G and Wu L H 2010 Darboux transformation and explicit solutions for Drinfel'd–Sokolov–Wilson equation *Commun. Theor. Phys.* **53** 1090–6

[19] Geng X G 1990 MKP equation and Darboux transformation *Chin. Ann. Math., 11A* **3** 265–8

[20] Ma W X 2024 Binary Darboux transformation of vector nonlocal reverse-time integrable NLS equations *Chaos Solitons Fractals* **180** 114539

[21] Lü X, Zhang L L and Ma W X 2024 Oceanic shallow-water description with (2+1)-dimensional generalized variable-coefficient Hirota–Satsuma-Ito equation: Painlevé analysis, soliton solutions, and lump solutions *Phys. Fluids* **36** 064110

[22] Chen D Y 2006 *Introduction to Solitons* (Science Press)
[23] Saha D, Chatterjee P and Raut S 2023 Multi-shock and soliton solutions of the Burgers equation employing Darboux transformation with the help of the Lax pair *Pramana-J. Phys.* **97** 54
[24] Chatterjee P, Saha D, Wazwaz A M and Raut S 2023 Explicit solutions of the Schamel-KdV equation employing Darboux transformation *Pramana-J Phys* **97** 172
[25] Pal N K, Chatterjee P and Saha A 2024 Solitons, multi-solitons and multi-periodic solutions of the generalized Lax equation by Darboux transformation and its quasiperiodic motions *Int. J. Mod. Phys.* B **38** 2440001
[26] Korteweg D J and de Vries G 1895 On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves *Philosophical Magazine 5th Series* **39** 422–43
[27] Crighton D G 1995 Applications of KdV *Acta Appl. Math.* **39** 39–67
[28] Fuchssteiner B and Fokas A S 1981 Symplectic structures, their Bäcklund transformations and hereditary symmetries *Physica* D **4** 47–66
[29] Calogero F and Degasperis A 1976 Nonlinear evolution equations solvable by the inverse spectral transform.-I *Nuovo Cimento* B **32** 201–42
[30] Bogoyavlenskii O I 1990 Overturning solitons in new two dimensional integrable equations *Math. USSR Izv.* **34** 245–59
[31] Schiff J 1992 *Painlevé Transendents, Their Asymtopics and Physical Applications* (Plenum Press) 393
[32] Yu S J, Toda K, Sasa N and Fukuyama T 1998 *N* soliton solutions to the Bogoyavlenskii–Schiff equation and a quest for the soliton solution in (3+1) dimensions *J. Phys. A: Math. Gen.* **31** 3337–47
[33] Wazwaz A M 2010 Integrable (2+1)-dimensional and (3+1)-dimensional breaking soliton equations *Phys. Scr.* **81** 035005
[34] Xu G Q 2015 Integrability of a (2+1)-dimensional generalized breaking soliton equation *Appl. Math. Lett.* **50** 16–22
[35] Gesztesy F, Schweiger W and Simon B 1991 Commutation methods applied to the mKdV-equation *Trans. Amer. Math. Soc.* **324** 465–525
[36] Toda K, Yu S J and Fukuyama T 1999 The Bogoyavlenskii–Schiff hierarchy and integrable equations in (2+1) dimensions *Rep. Math. Phys.* **44** 247–54
[37] Wang Y H and Wang H 2017 Nonlocal symmetry, CRE solvability and soliton-cnoidal solutions of the (2+1)-dimensional modified KdV-Calogero–Bogoyavlenkskii–Schiff equation *Nonlinear Dyn.* **89** 235–41
[38] Wazwaz A M 2017 Abundant solutions of various physical features for the (2+1)-dimensional modified KdV-Calogero–Bogoyavlenskii–Schiff equation *Nonlinear Dyn.* **89** 1727–32
[39] Li Y, Chaolu T and Bai Y X 2019 Rational solutions for the (2+1)-dimensional modified KdV-CBS equation *Adv. Math. Phys.* **2019** 6342042
[40] Cheng L, Ma W X, Zhang Y and Ge J Y 2022 Integrability and lump solutions to an extended (2+1)-dimensional KdV equation *Eur. Phys. J. Plus* **137** 902
[41] Lonngren K E 1998 Ion acoustic soliton experiments in a plasma *Opt. Quant. Electron* **30** 615–30
[42] Noor S, Alyousef H A, Shafee A, Shah R and El-Tantawy S A 2024 A novel analytical technique for analyzing the (3+1)-dimensional fractional Calogero–Bogoyavlenskii–Schiff equation: investigating solitary/shock waves and many others physical phenomena *Phys. Scr.* **99** 065257
[43] Lou S Y 2020 Soliton molecules and asymmetric solitons in three fifth order systems via velocity resonance *J. Phys. Commun.* **4** 041002
[44] Lou S Y 2020 A novel (2+1)-dimensional integrable KdV equation with peculiar solutions structures *Chin. Phys.* B **29** 080502
[45] Ren B, Lin J and Liu P 2020 Soliton molecules and the CRE method in the extended mKdV equation *Commun. Theor. Phys.* **72** 055005
[46] Wazwaz A M, Alhejaili W and El-Tantawy S A 2023 Analytical study on two new (3+1)-dimensional Painlevé integrable equations: Kink, lump, and multiple soliton solutions in fluid mediums *Phys. Fluids* **35** 093119
[47] Wazwaz A M, Alhejaili W and El-Tantawy S A 2024 On the Painlevé integrability and nonlinear structures to a (3+1)-dimensional Boussinesq-type equation in fluid mediums: Lumps and multiple soliton/shock solutions *Phys. Fluids* **36** 033116
[48] Ma W X 2023 Lump waves in a spatial symmetric nonlinear dispersive wave model in (2+1)-dimensions *Mathematics* **11** 4664
[49] Chen S J, Lü X and Yin Y H 2023 Dynamic behaviors of the lump solutions and mixed solutions to a (2+1)-dimensional nonlinear model *Commun. Theor. Phys.* **75** 055005
[50] Wazwaz A M, Alhejaili W and El-Tantawy S A 2023 Study on extensions of (modified) Korteweg–de Vries equations: Painlevé integrability and multiple soliton solutions in fluid mediums *Phys. Fluids* **35** 093110
[51] Cheng L and Zhang Y 2017 A KdV-Type Wronskian formulation to generalized KP, BKP and Jimbo–Miwa equations *Commun. Theor. Phys.* **68** 1–5
[52] Cheng L, Zhang Y and Ma W X 2023 Wronskian *N*-soliton solutions to a generalized KdV equation in (2+1)-dimensions *Nonlinear Dyn.* **111** 1701–14