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Soliton and Lump Solutions to a Fourth-order Nonlinear Wave Equation in (2+1)-Dimensions

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Abstract

This paper focuses on a (2+1)-dimensional fourth-order nonlinear wave equation with five categories of nonlinear terms, which can be reduced to spatially symmetric nonlinear models. Rational and lump solutions are derived by taking suitable limits of the soliton solutions obtained via the Hirota bilinear method. Furthermore, a kind of specific N-soliton solutions satisfying certain constraints is also obtained. Additionally, a specialized spatially symmetric model is presented to explore the corresponding lump waves. The derived lump solutions feature a critical point line, where their two spatial coordinates move at a constant velocity. Moreover, a reduced case is computed, revealing that nonlinearity and dispersion jointly control lump waves. This work enriches the solution structure of high-dimensional nonlinear equations and provides insights for describing complex dispersive phenomena.

Keywords Hirota bilinear form · Soliton solution · Lump wave · Long wave limit

1 Introduction

Nonlinear partial differential equations are used to describe natural phenomena and physical laws in many modern scientific and engineering fields such as plasma physics,

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nonlinear optics, quantum mechanics and electromagnetic fluid dynamics. By establishing and studying corresponding nonlinear evolution equations, some physical mechanisms behind natural phenomena can be revealed. One of the important fundamental issues in the subject is the exploration of exact solutions for nonlinear evolution equations. The investigation of nonlinear evolution equations has also brought many new theories and methods to the field of mathematical physics. In particular, in the process of solving nonlinear evolution equations, a series of effective analytical techniques such as the Hirota direct method [1–4], Bäcklund transformations [5–7], Darboux transformations [8], the Wronskian technique [7, 9, 10] and the Riemann-Hilbert approach [11, 12] have been developed. These methods not only enrich the theory of solving partial differential equations, but also provide reference and inspiration for the development of other mathematical branches.

Among the current solving techniques, the Hirota bilinear method, first developed by Ryogo Hirota in 1971, is an effective method for solving exact solutions of integrable nonlinear evolution equations [13]. The basic idea of this method is to transform nonlinear partial differential equations into bilinear equations through appropriate dependent variable transformations. Under new variables, exact solutions such as multiple soliton solutions can be expressed in a relatively simple form, thereby transforming the problem of solving nonlinear partial differential equations into the problem of solving bilinear equations. As a type of exponentially localized solutions, soliton solutions can characterize a wide range of nonlinear phenomena. Since the Hirota method is algebraic rather than analytic, it exhibits significant advantages in seeking multiple soliton solutions. Furthermore, within the Hirota bilinear theory, by solving the corresponding Hirota bilinear forms, various special solutions of nonlinear equations can also be generated, including complexiton solutions [14–16] and lump solutions [17, 18]. Similar to solitons, lump waves are typically expressed as rational functions of the independent variables, which decay to zero in all directions in the spatial domain [19–21]. Unlike some other solutions that may have singularities, lump solutions are non-singular. The long wave limit method, a key soliton theory technique, derives rational solutions for nonlinear evolution equations by applying a long wave limit to N-soliton solutions obtained by direct methods. By adequately choosing parameters of the two-soliton solution, the long wave limit of the solution yields a two-dimensional, non-singular lump decaying in all directions [17, 18]. The long wave limit method can be applied not only to local nonlinear equations but also to nonlocal integrable systems to obtain high-order rational solutions and rogue waves [22, 23].

To summarize, the (2+1)-dimensional nonlinear equations emerge to model multi-dimensional nonlinear phenomena in physics and engineering, generalize lower-dimensional theories, and challenge mathematical techniques for solving complex nonlinear systems. The Kadomtsev-Petviashvili (KP) equation is a well-known integrable system used to describe weakly nonlinear dispersive wave propagation in two dimensions [24]. The KP equation is given by

$$(u_t + 6uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0,\ \sigma^2 = \pm 1, \tag{1.1}$$

which is regarded as the KPI equation if $\sigma^2 = -1$ and the KPII equation if $\sigma^2 = 1$. The difference between the two lies in the sign of the u_{yy} term, which leads to distinct behaviors of the solutions and physical interpretations in different applications. The KPI equation has a large number of lump solutions while the KPII equation does not typically exhibit such solutions. By applying a direct symbolic computation, the KPI equation (1.1) possesses a class of lump solutions as follows [20]:

$$u = 2(\ln f)_{xx},\ f = \Big(a_1x + a_2y + \frac{a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6}{a_1^2 + a_5^2}t + a_4\Big)^2 + \Big(a_5x + a_6y + \frac{2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2}{a_1^2 + a_5^2}t + a_8\Big)^2 + \frac{3(a_1^2 + a_5^2)^3}{(a_1a_6 - a_2a_5)^2}, \quad (1.2)$$

where the involved parameters a_i's are arbitrary but $a_1a_6 - a_2a_5 \neq 0$. Such lump solutions can also be obtained by performing the long wave limit procedure on the soliton solutions [25]. Many other integrable equations have lump solutions such as the Davey-Stewartson II equation [17], the B-type Kadomtsev-Petviashvili (BKP) equation [26] and the Ishimori-I equation [27]. It has been further found that many non-integrable models may possess lump solutions, including the generalized KP, BKP and Sawada-Kotera equations in (2+1)-dimensions [28–32].

Motivated by the existing research, the main purpose of this study is to discuss a (2+1) dimensional fourth-order nonlinear evolution equation and search for its lump solutions by taking the long wave limit of soliton solutions. The framework of the paper is organized as follows. In Section 2, we would like to consider a fourth-order nonlinear wave equation containing five fourth-order derivative terms, which can reduce to spatially symmetric nonlinear models. A class of one-lump solutions will be presented by performing long wave limits of the corresponding two-soliton solutions. Moreover, a class of specific N-soliton solutions satisfying certain constraints will also be obtained. In Section 3, a special spatially symmetric model will be investigated to shed light on the presented results. Finally, our concluding remarks will be given in the last section.

2 Soliton and Lump Solutions

2.1 A Model Including Five Types of Nonlinear Terms

In this section, we first introduce a (2+1)-dimensional fourth-order nonlinear equation

$$\alpha_1[u_{xxxx} + 6(u_xv)_x] + \alpha_2[u_{yyyy} + 6(u_yw)_y] + \alpha_3[u_{xxxt} + 3(u_xp_t + u_tv)_x] + \alpha_4[u_{yyyt} + 3(u_yq_t + u_tw)_y] + \alpha_5(u_{xxyy} + 4uu_{xy} + 5u_xu_y + u_{yy}v + u_{xx}w + v_xw_y) + \beta_1u_{xt} + \beta_2u_{yt} + \beta_3u_{xy} + \beta_4u_{xx} + \beta_5u_{yy} = 0, \quad (2.1)$$

where $v_y = u_x, w_x = u_y, p_x = v, q_y = w$, the coefficients $\alpha_i, 1 \leq i \leq 5$ are real constants that are not all zero, and $\beta_i, 1 \leq i \leq 5$ are real constants satisfying

$\beta_1^2 + \beta_2^2 \neq 0$. It is easy to see that the coefficients a_i, $1 \leq i \leq 5$ involve five types of fourth-order derivative terms. Through the dependent variable transformations

$$u = 2(\ln f)_{xy}, v = 2(\ln f)_{xx}, w = 2(\ln f)_{yy}, p = 2(\ln f)_x, q = 2(\ln f)_y, \quad (2.2)$$

the above nonlinear equation (2.1) is converted into the following bilinear form:

$$(\alpha_1 D_x^4 + \alpha_2 D_y^4 + \alpha_3 D_x^3 D_t + \alpha_4 D_y^3 D_t + \alpha_5 D_x^2 D_y^2 + \beta_1 D_x D_t + \beta_2 D_y D_t + \beta_3 D_x D_y + \beta_4 D_x^2 + \beta_5 D_y^2) f \cdot f = 0, \quad (2.3)$$

where D_x, D_y and D_t are Hirota's bilinear derivatives [1]. Equation (2.1) contains many meaningful integrable models as its special examples, especially the spatially symmetric nonlinear models. By taking the choices

$$\alpha_1 = \beta_1 = 1, \beta_5 = -1, \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \beta_2 = \beta_3 = \beta_4 = 0,$$

and

$$\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = \beta_1 = \beta_3 = \beta_5 = 0, \alpha_3 = \beta_2 = \beta_4 = 1,$$

the above form (2.3) becomes the bilinear KPI equation

$$(D_x^4 + D_x D_t - D_y^2) f \cdot f = 0, \quad (2.4)$$

and the bilinear Hirota-Satsuma-Ito (HSI) equation in (2+1)-dimensions [33]

$$(D_x^3 D_t + D_y D_t + D_x^2) f \cdot f = 0, \quad (2.5)$$

respectively. Upon setting $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\beta_4 = \beta_5$ and $\alpha_3 = \alpha_4 = 0$ in Eq. (2.1), the Hirota bilinear form (2.3) gives a spatially symmetric generalized bilinear KP model in (2+1)-dimensions [34]:

$$[\alpha_1(D_x^4 + D_y^4) + \alpha_5 D_x^2 D_y^2 + \beta_1(D_x D_t + D_y D_t) + \beta_3 D_x D_y + \beta_4(D_x^2 + D_y^2)] f \cdot f = 0. \quad (2.6)$$

Moreover, taking $\alpha_1 = \alpha_2 = \alpha_5 = \beta_3 = 0, \alpha_3 = \alpha_4 = \alpha, \beta_1 = \beta_2 = \beta_4 = \beta_5 = 1$, then Eq. (2.3) gives the following spatially symmetric bilinear HSI model [35]:

$$(\alpha D_x^3 D_t + D_y D_t + D_x^2 + \alpha D_y^3 D_t + D_x D_t + D_y^2) f \cdot f = 0. \quad (2.7)$$

These equations all possess lump waves generated from positive quadratic wave functions through symbolic calculations with Maple.

In conclusion, the (2+1)-dimensional fourth-order nonlinear equation (2.1), containing five types of fourth-order derivative terms, refines existing models by capturing richer nonlinear dynamics, enhancing descriptions of real-world systems with concurrent nonlinear mechanisms, and offering a more precise characterization of (2+1)-dimensional phenomena. Notably, this new model can be reduced to both the KP equation and the Hirota-Satsuma-Ito equation. Though both this new model and the KP equation are (2+1)-dimensional, the new model incorporates unique higher-order and nonlinear features beyond the KP framework.

2.2 One-soliton, Two-soliton and Lump Solutions

As is well known, if a nonlinear model equation is expressed as a Hirota bilinear equation, one-soliton and two-soliton solutions can always be calculated based on the corresponding Hirota bilinear form. However, the existence of three-soliton solutions is guaranteed by passing the three-soliton solution test [36–39]. Now, assume the wave variables are

$$\eta_i = k_i(p_i x + l_i y + w_i t + \delta_i) + \eta_i^{(0)}, \; 1 \le i \le N, \tag{2.8}$$

where $k_i, p_i, l_i, 1 \le i \le N$, are constants to be determined, $\delta_i, \eta_i^{(0)}, 1 \le i \le N$, are arbitrary constant shifts, and the dispersion relations are satisfied:

$$w_i = \frac{-\alpha_1 k_i^2 p_i^4 - \alpha_2 k_i^2 l_i^4 - \alpha_5 k_i^2 l_i^2 p_i^2 - \beta_3 l_i p_i - \beta_4 p_i^2 - \beta_5 l_i^2}{\alpha_3 k_i^2 p_i^3 + \alpha_4 k_i^2 l_i^3 + \beta_1 p_i + \beta_2 l_i}, \; 1 \le i \le N. \tag{2.9}$$

Proposition 2.1 *Let η_1 be defined by (2.8), where w_1 is given by (2.9). Then the Hirota bilinear Eq. (2.3) admits a class of one-soliton solutions*

$$f = 1 + e^{\eta_1}. \tag{2.10}$$

Moreover, the first class of low-order rational solutions to Eq. (2.1) is given by

$$u = -\frac{2p_1 l_1}{\theta_1^2}, \; v = -\frac{2p_1^2}{\theta_1^2}, \; w = -\frac{2l_1^2}{\theta_1^2}, \; p = \frac{2p_1}{\theta_1}, \; q = \frac{2l_1}{\theta_1}, \tag{2.11}$$

where

$$\theta_1 = p_1 x + l_1 y - \frac{\beta_3 p_1 l_1 + \beta_4 p_1^2 + \beta_5 l_1^2}{\beta_1 p_1 + \beta_2 l_1} t + \delta_1. \tag{2.12}$$

Proof Substituting (2.10) into (2.3) yields the dispersion relation w_1 given by (2.9), which means that $f = 1 + e^{\eta_1}$ is a class of solutions of Eq. (2.3). In (2.10), setting $e^{\eta_1^{(0)}} = -1$ and taking the limit as $k_1 \to 0$, we expand f as

$$f = -k_1\theta_1 + O(k_1^2), \tag{2.13}$$

where θ_1 is defined by (2.12). Due to the logarithmic derivative transformations (2.2), f corresponds to θ_1. Using the transformations (2.2), we obtain the first class of low-order rational solutions (2.11) for Eq. (2.1). This completes the proof.

Next, we introduce a set of constants

$$A_{ij} = -\frac{P(k_i p_i - k_j p_j, k_i l_i - k_j l_j, k_i w_i - k_j w_j)}{P(k_i p_i + k_j p_j, k_i l_i + k_j l_j, k_i w_i + k_j w_j)}, 1 \le i < j \le N, \quad (2.14)$$

where the polynomial function P and constants w_i, $1 \le i \le N$, are defined by

$$P(x, y, t) = \alpha_1 x^4 + \alpha_2 y^4 + \alpha_3 x^3 t + \alpha_4 y^3 t + \alpha_5 x^2 y^2 + \beta_1 xt + \beta_2 yt + \beta_3 xy + \beta_4 x^2 + \beta_5 y^2 \quad (2.15)$$

and (2.9), respectively.

Proposition 2.2 *Let η_1, η_2 be defined by (2.8), where (2.9) holds. Then the Hirota bilinear Eq.(2.3) has the following two-soliton solutions:*

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2}, \quad (2.16)$$

where the constant A_{12} is defined by (2.14) with (2.15). Furthermore, suppose the coefficients β_i, $1 \le i \le 5$, satisfy $\beta_1^2\beta_5 - \beta_1\beta_2\beta_3 + \beta_2^2\beta_4 \ne 0$. Then the second class of rational solutions to Eq. (2.1) can be expressed as

$$u = 2(\ln f)_{xy}, v = 2(\ln f)_{xx}, w = 2(\ln f)_{yy}, p = 2(\ln f)_x, q = 2(\ln f)_y, \quad (2.17)$$

$$f = \theta_1\theta_2 + B_{12}, \quad (2.18)$$

where

$$\theta_i = p_i x + l_i y - \frac{\beta_3 p_i l_i + \beta_4 p_i^2 + \beta_5 l_i^2}{\beta_1 p_i + \beta_2 l_i} t + \delta_i, i = 1, 2, \quad (2.19)$$

$$B_{12} = \frac{b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 + b_4\alpha_4 + b_5\alpha_5}{(l_1 p_2 - l_2 p_1)^2(\beta_1^2\beta_5 - \beta_1\beta_2\beta_3 + \beta_2^2\beta_4)}, \quad (2.20)$$

with b_i, $1 \le i \le 5$, being given by

$$\begin{aligned}
b_1 &= 12 p_1^2 p_2^2 (p_2\beta_1 + l_2\beta_2)(p_1\beta_1 + l_1\beta_2),\\
b_2 &= 12 l_1^2 l_2^2 (p_2\beta_1 + l_2\beta_2)(p_1\beta_1 + l_1\beta_2),\\
b_3 &= -6 p_1^2 p_2^2 (p_1 l_2 + p_2 l_1)(\beta_1\beta_3 + \beta_2\beta_4) - 6 p_1 p_2 l_1 l_2 (p_1 l_2 + p_2 l_1)\beta_2\beta_5\\
&\quad -12 p_1^3 p_2^3 \beta_1\beta_4 - 12 p_1^2 p_2^2 l_1 l_2 \beta_2\beta_3 - 6 p_1 p_2 (p_1^2 l_2^2 + p_2^2 l_1^2)\beta_1\beta_5,\\
b_4 &= -6 l_1^2 l_2^2 (p_1 l_2 + p_2 l_1)(\beta_2\beta_3 + \beta_1\beta_5) - 6 p_1 p_2 l_1 l_2 (p_1 l_2 + p_2 l_1)\beta_1\beta_4\\
&\quad -12 l_1^3 l_2^3 \beta_2\beta_5 - 12 l_1^2 l_2^2 p_1 p_2 \beta_1\beta_3 - 6 l_1 l_2 (p_1^2 l_2^2 + p_2^2 l_1^2)\beta_2\beta_4,\\
b_5 &= 2(p_1\beta_1 + l_1\beta_2)(p_2\beta_1 + l_2\beta_2)(p_2^2 l_1^2 + 4 p_1 p_2 l_1 l_2 + p_1^2 l_2^2).
\end{aligned} \quad (2.21)$$

Proof According to Hirota's bilinear theory [1], under the dispersion relations (2.9) and the constant A_{12} defined by (2.14), there always exists a class of two-soliton solutions (2.16) to the Hirota bilinear equation (2.3). For the phase shift A_{12} defined by (2.14) with (2.9) and (2.15), choosing $e^{\eta_i^{(0)}} = -1$ and taking $k_i \to 0$ with $k_1/k_2 = O(1)$, $p_i = O(1)$ and $l_i = O(1)$ for $i = 1, 2$, we can obtain

$$A_{12} = 1 + \frac{k_1k_2(b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 + b_4\alpha_4 + b_5\alpha_5)}{(l_1p_2 - l_2p_1)^2(\beta_1^2\beta_5 - \beta_1\beta_2\beta_3 + \beta_2^2\beta_4)} + O(\mathbf{k}^3), \qquad (2.22)$$

where b_i, $1 \le i \le 5$, are given by (2.21) and $\mathbf{k} = \max(k_1, k_2)$. As a result, the class of two-soliton solutions (2.16) becomes

$$f = k_1k_2\left[\theta_1\theta_2 + \frac{b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 + b_4\alpha_4 + b_5\alpha_5}{(l_1p_2 - l_2p_1)^2(\beta_1^2\beta_5 - \beta_1\beta_2\beta_3 + \beta_2^2\beta_4)}\right] + O(\mathbf{k}^3), \qquad (2.23)$$

where $\theta_i, i = 1, 2$, are given by (2.19). Since k_1k_2 can be neglected from (2.23) as previously discussed, f is equivalent to $\theta_1\theta_2 + B_{12}$, with B_{12} defined by (2.20) and b_i, $1 \le i \le 5$, by (2.21). Therefore, through the transformations (2.2), the second class of rational solutions to Eq. (2.1) is expressed as (2.17) together with (2.18)-(2.21). This completes the proof.

In general, f generated by Proposition 2.2 is generally singular at some position. To get non-singular rational solutions of the (2+1)-dimensional nonlinear equation (2.1), we seek quadratic function solutions to Eq. (2.3), which can generate lump solutions to Eq. (2.1) through the transformations (2.2).

Theorem 2.1 *Suppose that the coefficients of the second-order linear terms satisfy* $\beta_1^2\beta_5 - \beta_1\beta_2\beta_3 + \beta_2^2\beta_4 \neq 0$. *Then the (2+1)-dimensional bilinear equation (2.3) admits a class of quadratic function solutions as follows:*

$$f = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6y + a_7t + a_8)^2 + a_9, \qquad (2.24)$$

where

$$a_3 = -\frac{(a_1^2 + a_5^2)(a_2\beta_3 + a_1\beta_4)\beta_1 + (a_2^2 + a_6^2)(a_1\beta_3 + a_2\beta_5)\beta_2}{(a_1\beta_1 + a_2\beta_2)^2 + (a_5\beta_1 + a_6\beta_2)^2} - \frac{(a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6)\beta_1\beta_5 + (a_1^2a_2 - a_2a_5^2 + 2a_1a_5a_6)\beta_2\beta_4}{(a_1\beta_1 + a_2\beta_2)^2 + (a_5\beta_1 + a_6\beta_2)^2}, \qquad (2.25)$$

$$a_7 = -\frac{(a_1^2 + a_5^2)(a_6\beta_3 + a_5\beta_4)\beta_1 + (a_2^2 + a_6^2)(a_5\beta_3 + a_6\beta_5)\beta_2}{(a_1\beta_1 + a_2\beta_2)^2 + (a_5\beta_1 + a_6\beta_2)^2} - \frac{(a_5a_6^2 - a_2^2a_5 + 2a_1a_2a_6)\beta_1\beta_5 + (a_5^2a_6 - a_1^2a_6 + 2a_1a_2a_5)\beta_2\beta_4}{(a_1\beta_1 + a_2\beta_2)^2 + (a_5\beta_1 + a_6\beta_2)^2}, \qquad (2.26)$$

$$a_9 = \frac{c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5}{(a_1a_6 - a_2a_5)^2(\beta_1^2\beta_5 - \beta_1\beta_2\beta_3 + \beta_2^2\beta_4)}, \tag{2.27}$$

with the parameters c_i, $1 \le i \le 5$, in (2.27) being given by

$$\begin{aligned}
c_1 &= -3(a_1^2 + a_5^2)^2[(a_1\beta_1 + a_2\beta_2)^2 + (a_5\beta_1 + a_6\beta_2)^2],\\
c_2 &= -3(a_2^2 + a_6^2)^2[(a_1\beta_1 + a_2\beta_2)^2 + (a_5\beta_1 + a_6\beta_2)^2],\\
c_3 &= 3(a_1^2 + a_5^2)^2(a_1a_2 + a_5a_6)(\beta_1\beta_3 + \beta_2\beta_4) + 3(a_1^2 + a_5^2)^2(a_2^2 + a_6^2)\beta_2\beta_3\\
&\quad +3(a_1^2 + a_5^2)(a_2^2 + a_6^2)(a_1a_2 + a_5a_6)\beta_2\beta_5 + 3(a_1^2 + a_5^2)^3\beta_1\beta_4\\
&\quad +3[(a_2^2 - a_6^2)a_1^2 + 4a_1a_2a_5a_6 - (a_2^2 - a_6^2)a_5^2](a_1^2 + a_5^2)\beta_1\beta_5,\\
c_4 &= 3(a_2^2 + a_6^2)^2(a_1a_2 + a_5a_6)(\beta_1\beta_5 + \beta_2\beta_3) + 3(a_2^2 + a_6^2)^2(a_1^2 + a_5^2)\beta_1\beta_3\\
&\quad +3(a_1^2 + a_5^2)(a_2^2 + a_6^2)(a_1a_2 + a_5a_6)\beta_1\beta_4 + 3(a_2^2 + a_6^2)^3\beta_2\beta_5\\
&\quad +3[(a_1^2 - a_5^2)a_2^2 + 4a_1a_2a_5a_6 - (a_1^2 - a_5^2)a_6^2](a_2^2 + a_6^2)\beta_2\beta_4,\\
c_5 &= -[(a_1\beta_1 + a_2\beta_2)^2 + (a_5\beta_1 + a_6\beta_2)^2][3(a_1a_2 + a_5a_6)^2 + (a_1a_6 - a_2a_5)^2].
\end{aligned} \tag{2.28}$$

Here, the involved six real parameters $a_1, a_2, a_4, a_5, a_6, a_8$ are arbitrary but satisfy $a_1a_6 - a_2a_5 \neq 0$.

Proof We select parameters:

$$p_1 = a_1 + \mathrm{I}a_5,\ l_1 = a_2 + \mathrm{I}a_6,\ \delta_1 = a_4 + \mathrm{I}a_8,\ p_2 = p_1^*,\ l_2 = l_1^*,\ \delta_2 = \delta_1^*, \tag{2.29}$$

where $a_1, a_2, a_4, a_5, a_6, a_8 \in \mathbb{R}$ and $\mathrm{I} = \sqrt{-1}$. Here the asterisk represents the complex conjugate. This leads to $\theta_1 = \theta_2^*$ in (2.19). Substituting (2.29) into (2.18) gives

$$f = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6y + a_7t + a_8)^2 + a_9,$$

with a_3, a_7, a_9 determined by (2.25), (2.26) and (2.27), respectively. The constants c_i, $1 \le i \le 5$, involved in (2.27) are defined by (2.28). All other parameters are arbitrarily but must satisfy $a_1a_6 - a_2a_5 \neq 0$. The proof is finished.

It is easy to observe that the two frequency parameters a_3 and a_7 generated from the wave variables (2.19) represent a type of dispersion relations in (2+1)-dimensional nonlinear dispersive waves, independent of the fourth-order coefficients. The constant term parameter a_9, generated by the phase shift (2.20) describes a complex relation with wave numbers and plays a key role in constructing lump waves within the Hirota bilinear form. The solutions in (2.24) are positive quadratic function if and only if the parameter $a_9 > 0$, and so, we require two basic positivity conditions for $a_9 > 0$ as

$$a_1a_6 - a_2a_5 \neq 0, \tag{2.30}$$

and

$$(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5)(\beta_1^2\beta_5 - \beta_1\beta_2\beta_3 + \beta_2^2\beta_4) > 0, \quad (2.31)$$

where $c_i, 1 \leq i \leq 5$, are determined by (2.28). These conditions guarantee the essential properties of lump waves. Therefore, a class of lump solutions u, v, w can be formulated via the transformations (2.2) under the two basic conditions (2.30) and (2.31). □

2.3 *N*-soliton solutions

In general, the Hirota bilinear equation (2.3) has a class of three-soliton solutions

$$\begin{aligned} f = {} & 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} \\ & + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3}, \end{aligned} \quad (2.32)$$

if and only if the corresponding three-soliton condition [37–39]

$$\begin{aligned} \sum_{\sigma_1,\sigma_2,\sigma_3=\pm1} & P(\sigma_1\bar{p}_1 + \sigma_2\bar{p}_2 + \sigma_3\bar{p}_3)P(\sigma_1\bar{p}_1 - \sigma_2\bar{p}_2)P(\sigma_2\bar{p}_2 - \sigma_3\bar{p}_3) \\ & P(\sigma_1\bar{p}_1 - \sigma_3\bar{p}_3) = 0, \end{aligned} \quad (2.33)$$

is satisfied, where $\eta_i, A_{ij}, 1 \leq i, j \leq 3$, and P are defined by (2.8), (2.14) and (2.15), respectively, and $\bar{p}_i = (k_i p_i, k_i l_i, k_i w_i), 1 \leq i \leq 3$. It is direct to check that the general model (2.3) doesn't have three-soliton solutions [3]. As we know, a general class of (1+1)-dimensional bilinear generalized KdV equations that possess N-soliton solutions is expressed as [40]

$$(aD_x^4 + bD_x^3D_t + cD_x^2 + dD_xD_t)f \cdot f = 0, \quad (2.34)$$

where a, b, c, d are arbitrary constants satisfying $b^2 + d^2 \neq 0$. By utilizing a dimensional reduction, we can obtain specific N-soliton solutions of Eq. (2.3) that satisfy certain constraints.

Theorem 2.2 *Let us impose the conditions*

$$l_i = \gamma p_i, 1 \leq i \leq N, \quad (2.35)$$

in the wave variables (2.8), where γ is a non-zero constant. If γ satisfies

$$(\alpha_1 + \alpha_2\gamma^4 + \alpha_5\gamma^2)(\beta_1 + \beta_2\gamma) - (\alpha_3 + \alpha_4\gamma^3)(\beta_3\gamma + \beta_4 + \beta_5\gamma^2) \neq 0, \quad (2.36)$$

then the Hirota bilinear Eq. (2.3) has a class of specific N-soliton solutions:

$$f = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} \mu_i \mu_j \ln A_{ij}\Big), \tag{2.37}$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_N)$, $\mu = 0, 1$ indicates that each μ_i sets 0 or 1, and

$$\begin{aligned}\eta_i &= k_i\Big[p_i(x+\gamma y) - \frac{(\alpha_1+\alpha_2\gamma^4+\alpha_5\gamma^2)k_i^2 p_i^3 + (\beta_3\gamma+\beta_4+\beta_5\gamma^2)p_i}{(\alpha_3+\alpha_4\gamma^3)k_i^2 p_i^2 + (\beta_1+\beta_2\gamma)}t\Big] \\ &\quad +\eta_i^{(0)}, \ 1 \le i \le N, \qquad (2.38) \\ A_{ij} &= \frac{[(\alpha_3+\alpha_4\gamma^3)(k_i^2p_i^2 - k_ik_jp_ip_j + k_j^2p_j^2) + 3(\beta_1+\beta_2\gamma)](k_ip_i - k_jp_j)^2}{[(\alpha_3+\alpha_4\gamma^3)(k_i^2p_i^2 + k_ik_jp_ip_j + k_j^2p_j^2) + 3(\beta_1+\beta_2\gamma)](k_ip_i + k_jp_j)^2}, \ 1 \le i < j \le N, \qquad (2.39)\end{aligned}$$

with $\eta_i^{(0)}$' s being arbitrary constant phase shifts. If γ satisfies

$$\begin{aligned}&(\alpha_1+\alpha_2\gamma^4+\alpha_5\gamma^2)(\beta_1+\beta_2\gamma) - (\alpha_3+\alpha_4\gamma^3)(\beta_3\gamma+\beta_4+\beta_5\gamma^2) = 0, \\ &(\alpha_3+\alpha_4\gamma^3)^2 + (\beta_1+\beta_2\gamma)^2 \neq 0,\end{aligned} \tag{2.40}$$

then the Hirota bilinear Eq. (2.3) has a set of resonant solutions:

$$f = 1 + \varepsilon_1 e^{\eta_1} + \cdots + \varepsilon_N e^{\eta_N}, \tag{2.41}$$

where η_i, $1 \le i \le N$, are defined by (2.38), and ε_i' s are arbitrary constants.

Proof Taking an invertible linear transform of x, y, and t,

$$x' = x + \gamma y, t' = t, \tag{2.42}$$

then the Hirota bilinear equation (2.3) becomes

$$\begin{aligned}&\big[(\alpha_1+\alpha_2\gamma^4+\alpha_5\gamma^2)D_{x'}^4 + (\alpha_3+\alpha_4\gamma^3)D_{x'}^3 D_{t'} \\ &\quad +(\beta_1+\beta_2\gamma)D_{x'}D_{t'} + (\beta_3\gamma+\beta_4+\beta_5\gamma^2)D_{x'}^2\big] f \cdot f = 0,\end{aligned} \tag{2.43}$$

which presents a class of bilinear generalized KdV equations (2.34). According to the algorithm proposed in Ref. [40] for verifying the Hirota condition, each equation in (2.43) satisfies the N-soliton conditions and possesses N-soliton solutions. Under the condition (2.40), the N-soliton solution to each bilinear equation in (2.43) has the form of (2.37), where

$$\eta_i = k_i\left[p_i x' - \frac{(\alpha_1+\alpha_2\gamma^4+\alpha_5\gamma^2)k_i^2p_i^3+(\beta_3\gamma+\beta_4+\beta_5\gamma^2)p_i}{(\alpha_3+\alpha_4\gamma^3)k_i^2p_i^2+(\beta_1+\beta_2\gamma)}t'\right] + \eta_i^{(0)},\ 1\le i\le N, \tag{2.44}$$

and A_{ij}, $1 \le i < j \le N$, are defined by (2.39). Substituting the variable transformation (2.42) into the above N-soliton solution yields (2.37) with (2.38) and (2.39). Therefore, we obtain the class of specific N-soliton solutions (2.37) to the Hirota bilinear Eq. (2.3). Let us next set

$$\Delta = (\alpha_1+\alpha_2\gamma^4+\alpha_5\gamma^2)(\beta_1+\beta_2\gamma) - (\alpha_3+\alpha_4\gamma^3)(\beta_3\gamma+\beta_4+\beta_5\gamma^2). \tag{2.45}$$

By using the conditions (2.35), it is direct to compute that

$$w_i = -\frac{(\alpha_1+\alpha_2\gamma^4+\alpha_5\gamma^2)k_i^2p_i^3+(\beta_3\gamma+\beta_4+\beta_5\gamma^2)p_i}{(\alpha_3+\alpha_4\gamma^3)k_i^2p_i^2+(\beta_1+\beta_2\gamma)},\ 1\le i\le N, \tag{2.46}$$

$$P(k_ip_i-k_jp_j,\gamma k_ip_i-\gamma k_jp_j,k_iw_i-k_jw_j) = \frac{-k_ik_jp_ip_j\Delta[(\alpha_3+\alpha_4\gamma^3)(k_i^2p_i^2-k_ik_jp_ip_j+k_j^2p_j^2)+3(\beta_1+\beta_2\gamma)](k_ip_i-k_jp_j)^2}{[(\alpha_3+\alpha_4\gamma^3)k_i^2p_i^2+(\beta_1+\beta_2\gamma)][(\alpha_3+\alpha_4\gamma^3)k_j^2p_j^2+(\beta_1+\beta_2\gamma)]},\ 1\le i,j\le N, \tag{2.47}$$

where P is defined by (2.15). Under the condition (2.40), the final expression in (2.47) is equal to zero. Therefore, we have shown that the Hirota bilinear Eq. (2.3) has a set of resonant solutions [41] given by (2.41).

We see that the conditions (2.35) reduce the (2+1)-dimensional case (2.3) to the (1+1)-dimensional case. Taking $N = 3$, a class of specific three-soliton solutions f is given by (2.32), where η_i and A_{ij}, $1 \le i, j \le 3$, are defined by (2.38) and (2.39), respectively. For Eq. (2.1), let us take

$$\alpha_1=1,\alpha_2=-1,\alpha_3=1,\alpha_4=2,\alpha_5=1,\beta_1=1,\beta_2=2,\beta_3=-1,\beta_4=2,\beta_5=1. \tag{2.48}$$

Via the transformations (2.2), a special three-soliton solution (u, v, w) determined by the expressions (2.32), (2.38) and (2.39) with the parameter values

$$k_1=1,k_2=-1,k_3=\frac{3}{2},\gamma=2,\ p_1=1,\ p_2=2,\ p_3=-1,\ \eta_1^{(0)}=\eta_2^{(0)}=\eta_3^{(0)}=0, \tag{2.49}$$

is plotted in Figs.1 and 2. As shown in Fig. 1, in each of the plots for u, v and w, we can observe that the three line-like soliton waves run parallel to each other in the (x, y) plane. This parallel behavior originates from the constraint $p_i/p_j = l_i/l_j$. This constraint ensures that the direction of propagation of each soliton wave is the

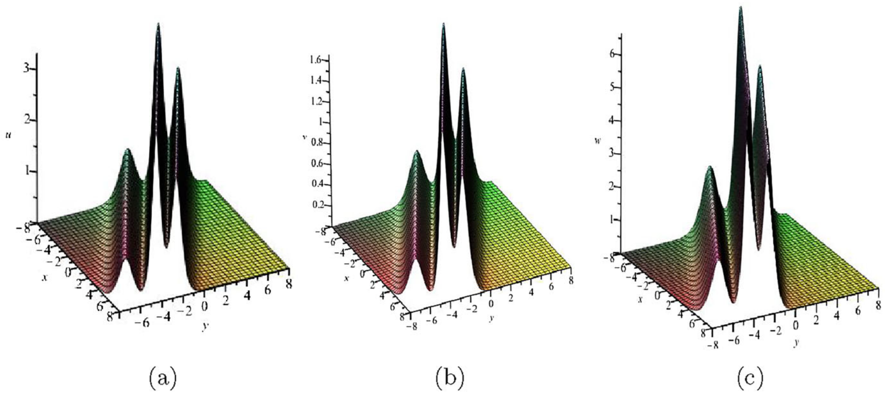

Fig. 1 Three-soliton solution (u, v, w) (2.2) generated by the function (2.32) with (2.38) and (2.39) under the selections (2.48) and (2.49) at $t = 0$

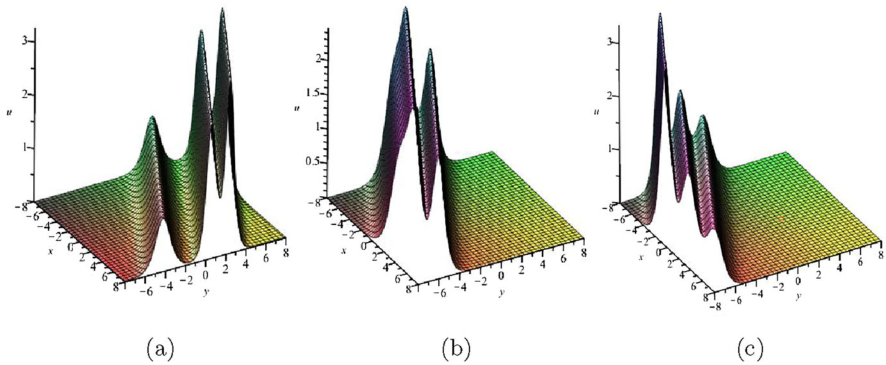

Fig. 2 The propagation of the three-soliton solution u (2.2) generated by the function (2.32) with (2.38) and (2.39) under the selections (2.48) and (2.49) at $t = -20$ in (a), $t = 15$ in (b), and $t = 30$ in (c)

same, resulting in their parallelism. In the u plot, the heights of the three solitons are different. Similarly, for v and w plots, the amplitudes of the three solitons in each plot are distinct. This means that the maximum values of u, v, and w for their respective solitons are different. Figure 2 displays the propagation process of the special three-soliton solution corresponding to the field u. It can be seen that the wave behind gradually catches up with the wave ahead and interacts with it. After a period of time, the three solitons separate again and continue to propagate in the reverse order of their initial sequence. Before and after the interaction, the original shapes and propagation velocities of these solitons remain unchanged. This is a typical characteristic of soliton dynamics in nonlinear systems, demonstrating the unique propagation and interaction behaviors of the three-soliton solution for the u field.

3 A Spatial Symmetric Nonlinear Model

In this section, a special spatial symmetric nonlinear model will be provided to seek the corresponding lump solutions and consider their characteristic dynamical properties.

Let us first take $\alpha_1 = \alpha_2 = 0, \alpha_3 = \alpha_4, \beta_1 = \beta_2, \beta_4 = \beta_5$. The fourth-order nonlinear equation (2.1) becomes the following spatial symmetric (2+1)-dimensional model equation:

$$\begin{aligned} &\alpha_3[u_{xxxt} + 3(u_x p_t + u_t v)_x] + \alpha_3[u_{yyyt} + 3(u_y q_t + u_t w)_y] \\ &+\alpha_5(u_{xxyy} + 4uu_{xy} + 5u_x u_y + u_{yy} v + u_{xx} w + v_x w_y) \\ &+\beta_1(u_{xt} + u_{yt}) + \beta_3 u_{xy} + \beta_4(u_{xx} + u_{yy}) = 0, \end{aligned} \tag{3.1}$$

with $v_y = u_x, w_x = u_y, p_x = v, q_y = w$, which possesses a Hirota bilinear form

$$\begin{aligned} &[\alpha_3(D_x^3 D_t + D_y^3 D_t) + \alpha_5 D_x^2 D_y^2 + \beta_1(D_x D_t + D_y D_t) + \beta_3 D_x D_y \\ &+\beta_4(D_x^2 + D_y^2)] f \cdot f = 0, \end{aligned} \tag{3.2}$$

under the dependent variable transformations (2.2). Based on the obtained results (2.25)-(2.27), a straightforward substitution gives a set of solutions for the parameters in (2.24), where

$$\begin{aligned} a_3 = &-\frac{\beta_3[a_2(a_1^2 + a_5^2) + a_1(a_2^2 + a_6^2)]}{\beta_1[(a_1 + a_2)^2 + (a_5 + a_6)^2]} \\ &-\frac{\beta_4[(a_1 + a_2)(a_1^2 + a_2^2) + a_1(a_5^2 + 2a_5 a_6 - a_6^2) - a_2(a_5^2 - 2a_5 a_6 - a_6^2)]}{\beta_1[(a_1 + a_2)^2 + (a_5 + a_6)^2]}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} a_7 = &-\frac{\beta_3[a_6(a_1^2 + a_5^2) + a_5(a_2^2 + a_6^2)]}{\beta_1[(a_1 + a_2)^2 + (a_5 + a_6)^2]} \\ &-\frac{\beta_4[(a_5 + a_6)(a_5^2 + a_6^2) + a_5(a_1^2 + 2a_1 a_2 - a_2^2) - a_6(a_1^2 - 2a_1 a_2 - a_2^2)]}{\beta_1[(a_1 + a_2)^2 + (a_5 + a_6)^2]}, \end{aligned} \tag{3.4}$$

$$a_9 = \frac{3\alpha_3\beta_3 c_1' + 3\alpha_3\beta_4 c_2' + \alpha_5\beta_1 c_3'}{\beta_1(2\beta_4 - \beta_3)(a_1 a_6 - a_2 a_5)^2}. \tag{3.5}$$

The above polynomials c_i', $1 \le i \le 3$, are defined as follows:

$$\begin{aligned} c_1' &= (a_1^2 + a_5^2)^2(a_1 a_2 + a_5 a_6 + a_2^2 + a_6^2) + (a_2^2 + a_6^2)^2(a_1 a_2 + a_5 a_6 + a_1^2 + a_5^2), \\ c_2' &= (a_1^2 + a_2^2 + a_5^2 + a_6^2)[(a_1^2 - a_6^2)^2 + (a_2^2 - a_5^2)^2 + 2(a_1 a_5 + a_2 a_6)^2 \\ &\quad +(a_1 a_2 + a_5 a_6)(a_1^2 + a_2^2 + a_5^2 + a_6^2)], \\ c_3' &= -[(a_1 + a_2)^2 + (a_5 + a_6)^2][3(a_1 a_2 + a_5 a_6)^2 + (a_1 a_6 - a_2 a_5)^2]. \end{aligned} \tag{3.6}$$

Based on the above solution formulas, we require the following basic condition:

$$\beta_1(2\beta_4 - \beta_3)(a_1a_6 - a_2a_5) \neq 0. \tag{3.7}$$

It is easy to see that $a_1a_6 - a_2a_5 \neq 0$ implies $(a_1 + a_2)^2 + (a_5 + a_6)^2 \neq 0$. Moreover, under the condition (3.7), the parameter a_9 is positive if and only if

$$(3\alpha_3\beta_3c_1' + 3\alpha_3\beta_4c_2' + \alpha_5\beta_1c_3')\beta_1(2\beta_4 - \beta_3) > 0. \tag{3.8}$$

If $\alpha_3 \neq 0$, we impose either

$$\begin{aligned} &a_1a_2 + a_5a_6 > 0,\ \alpha_3\beta_1\beta_3(2\beta_4 - \beta_3) \geq 0,\ \alpha_3\beta_1\beta_4(2\beta_4 - \beta_3) > 0, \\ &\alpha_5(2\beta_4 - \beta_3) \leq 0, \end{aligned} \tag{3.9}$$

or

$$\begin{aligned} &a_1a_2 + a_5a_6 \geq 0,\ \alpha_3\beta_1\beta_3(2\beta_4 - \beta_3) > 0,\ \alpha_3\beta_1\beta_4(2\beta_4 - \beta_3) \geq 0, \\ &\alpha_5(2\beta_4 - \beta_3) \leq 0, \end{aligned} \tag{3.10}$$

then the condition (3.8) can be satisfied. If $\alpha_3 = 0$ and $\alpha_5 \neq 0$, the condition

$$\alpha_5(2\beta_4 - \beta_3) < 0, \beta_1(a_1a_6 - a_2a_5) \neq 0, \tag{3.11}$$

guarantees that the solutions for u, v, w are analytic and localized in all directions in the space. Evidently, the conditions specified by (3.9), (3.10) and (3.11) incorporate the coefficients, α_3, α_5, of the nonlinear terms and the coefficients, $\beta_1, \beta_3, \beta_4$, of the dispersion terms. This implies that both the nonlinearity and the dispersion jointly influence the lump waves of the model equation (3.2). For example, a reduced case is presented as follows.

Example 3.1 We take $\alpha_3 = \beta_4 = 0, \beta_3 = \sigma^2, \sigma^2 = \pm 1$, and $\alpha_5\beta_1 \neq 0$, then the spatial symmetric (2+1)-dimensional bilinear equation (3.2) is reduced to

$$(\alpha_5D_x^2D_y^2 + \beta_1D_xD_t + \beta_1D_yD_t + \sigma^2D_xD_y)f \cdot f = 0. \tag{3.12}$$

According to (3.5), we have

$$a_9 = \frac{\alpha_5[(a_1 + a_2)^2 + (a_5 + a_6)^2][3(a_1a_2 + a_5a_6)^2 + (a_1a_6 - a_2a_5)^2]}{\sigma^2(a_1a_6 - a_2a_5)^2}. \tag{3.13}$$

If $\sigma^2 = 1$, then the condition for the existence of lump waves reads

$$a_1a_6 - a_2a_5 \neq 0, \alpha_5 > 0, \tag{3.14}$$

and if $\sigma^2 = -1$, the condition for the existence of lump waves reads

$$a_1a_6 - a_2a_5 \neq 0, \alpha_5 < 0. \tag{3.15}$$

Taking $\sigma^2 = 1, \alpha_5 = 2, \beta_1 = -1$, and considering the following special values of the free parameters:

$$a_1 = 1,\ a_2 = -1,\ a_4 = 3,\ a_5 = 2,\ a_6 = 1,\ a_8 = 6, \tag{3.16}$$

the corresponding f defined by (2.24) becomes

$$f = \left(x - y - \frac{1}{3}t + 3\right)^2 + (2x + y + t + 6)^2 + 24, \tag{3.17}$$

which provides a lump solution (u, w, v) to the reduced case (3.12) via the logarithmic transformations (2.2). The profiles of the lump solution are vividly depicted in Fig. 3. From the three-dimensional plots, it is observed that the lump solution u displays two peaks paired with two hollows, while v and w are characterized by one peak with two hollows. The curves show the spatiotemporal evolution of three fields u, v and w. All fields exhibit localized peaks near $x \approx 0$ with rapid decay, reflecting the rational localization property of lump solutions.

Secondly, we discuss critical points of the quadratic function f defined by (2.24). For this purpose, we need to consider solutions to the system

$$\frac{\partial f}{\partial x}(x(t), y(t), t) = 0,\ \frac{\partial f}{\partial y}(x(t), y(t), t) = 0. \tag{3.18}$$

Due to the condition given by $a_1a_6 - a_2a_5 \neq 0$, the system (3.18) is equivalent to

$$\xi_1 = a_1x + a_2y + a_3t + a_4 = 0,\ \xi_2 = a_5x + a_6y + a_7t + a_8 = 0. \tag{3.19}$$

Based on (3.3) and (3.4), solving the above system (3.19) for x and y, critical points of the quadratic function f can be written as:

$$x(t) = \frac{\beta_3(a_2^2 + a_6^2) + \beta_4(a_1^2 + 2a_1a_2 - a_2^2 + a_5^2 + 2a_5a_6 - a_6^2)}{\beta_1[(a_1 + a_2)^2 + (a_5 + a_6)^2]}t + \frac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \tag{3.20}$$

$$y(t) = \frac{\beta_3(a_1^2 + a_5^2) - \beta_4(a_1^2 - 2a_1a_2 - a_2^2 + a_5^2 - 2a_5a_6 - a_6^2)}{\beta_1[(a_1 + a_2)^2 + (a_5 + a_6)^2]}t - \frac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}, \tag{3.21}$$

at any time t. Clearly, the critical points define a linear path along which the two spatial coordinates advance at a constant rate.

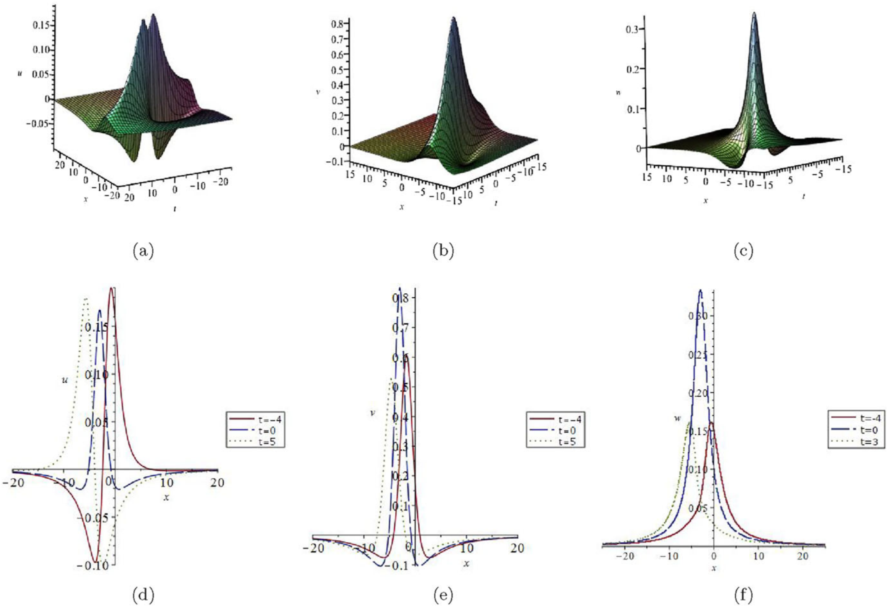

Fig. 3 Three-dimensional plots and x-curves of the lump solution (u, v, w) determined by (2.2) with the function f given by (3.17) when $y = 0$ in (a),(b),(c) and (d),(e),(f), respectively

4 Concluding Remarks

In summary, by applying an appropriate limiting procedure to the soliton solutions derived via the Hirota bilinear method, we have obtained soliton solutions and lump waves for a fourth-order nonlinear equation containing five types of nonlinear terms in (2+1)-dimensions. The derived specific N-soliton solutions can be transformed into resonant solutions when appropriate parameter configurations are selected. We have also proposed a technique to deduce lump waves by extending the involved parameters to the complex field. In addition, we have provided a special spatially symmetric nonlinear model for exploring the corresponding lump waves. The resulting lump solutions possess a line of critical points, whose two spatial coordinates move with constant velocities. Furthermore, we have calculated a reduced case, which demonstrates the importance of the second-order linear dispersion terms and fourth-order nonlinear terms for the existence of lump solutions.

This study establishes a connection for transforming specific N-soliton solutions into resonant solutions through parameter adjustment, and extends relevant parameters to the complex field, which provides an effective technical pathway for lump wave derivation. Studying these lump solutions is novel and important because the highly general fourth-order (2+1)-dimensional equation, with five fourth-order terms and reducing to key integrable models, reveals unique non-singular, spatially decaying lumps with moving critical lines, shaped by high-order derivative interactions. These solutions enrich soliton theory by extending localized wave understanding to higher-dimensional, higher-order systems and clarify dispersion relations and phase shift mechanisms through parameters like a_3, a_7, and a_9.

We note that the lump solutions obtained in this paper can also be generated from positive quadratic function solutions using symbolic calculations with Maple. However, our work adopts the long wave limit method that extends the involved parameters to the complex field, which offers distinct clarity. It reveals that parameters a_3 and a_7 directly characterize a class of dispersion relations in (2+1)-dimensional nonlinear dispersive waves. The constant term a_9, related to the phase shifts, exhibits a complex relationship with wave numbers. In particular, the fourth-order derivative terms such as $D_x^3 D_t$, $D_y^3 D_t$ and $D_x^2 D_y^2$ enhance the complexity of the structure of a_9.

The novelty and originality of our work lies in the introduction of five fourth-order bilinear terms, which produce nonlinear terms in the model equation that have not appeared in the literature. As is well recognized, nonlinearity interacts with various types of dispersion to generate both lump and soliton waves. Our study makes a novel contribution to this important area of nonlinear wave research.

Given that more nonlinear phenomena in the real world are described by interaction solutions between lump waves and other interesting waves, both homoclinic and heteroclinic waves [42, 43], we hope to explore lump solutions and interaction solutions to nonlinear evolution equations of various orders and dimensions in the future.

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Author Contributions Li Cheng. wrote the main manuscript text and prepared figures 1-3. Wen-Xiu Ma contributed to this research through supervision and methodology development. All authors reviewed the manuscript.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

Competing interests The authors declare no competing interests.

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