Received: 23 September 2020 | Revised: 27 June 2021 | Accepted: 6 August 2021
DOI: 10.1002/mma.7809

RESEARCH ARTICLE

WILEY

Framework of extended BKP equations possessing *N*-wave solutions in (2+1)-dimensions

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Communicated by: J. Banasiak

A universal property in terms of Hirota differential operators is presented, with an aim to construct Hirota bilinear equations possessing Pfaffian formulations in (2+1)-dimensions. It is then shown that there exist linear combination solutions of exponential waves to various extended Hirota bilinear equations. Applications are made for the (2+1)-dimensional generalized Hirota–Satsuma–Ito (HSI) equation, the special fourth-order (2+1)-dimensional nonlinear equation, and the fifth-order Korteweg–de Vries (KdV) equation, thereby presenting their particular Pfaffian and multiple wave solutions.

KEYWORDS

BKP hierarchy, Hirota's bilinear form, N-wave solutions, Pfaffian formulation

MSC CLASSIFICATION

35Q51; 35Q53; 37K40

1 | INTRODUCTION

There are many interesting methods to explore exact solutions of nonlinear wave equations, including the Hirota's bilinear method,[1–4] the Lie symmetry method[5] and the Riemann–Hilbert approach.[6–8] As is well known, the Hirota's bilinear method historically developed is pretty powerful in constructing soliton solutions for integrable and nonintegrable nonlinear equations by perturbation.[1,2] Soliton solutions, generated from combinations of multiple exponential waves on the basis of Hirota bilinear forms, could be expressed as Wronskian or Grammian determinants. Typical nonlinear equations, such as the Korteweg–de Vries (KdV) equation, the Kadomtsev–Petviashvili (KP) equation, and the Jimbo–Miwa equation, possess Wronskian or Grammian determinant solutions.[1,9-14] For some higher-dimensional soliton equations, there exist Pfaffian solutions complementing Wronskian and Grammian determinant solutions,[15,16] due to Pfaffians generalized determinants. Soliton solutions of the B-type Kadomtsev–Petviashvili (BKP) equation were expressed in terms of Pfaffians with odd weight variables by Hirota.[1,17]

Adding to the diversity of solitons, *N*-wave resonant solutions, which are linear combinations of exponential traveling waves, were presented by the linear superposition principle. In previous studies,[18,19] a sufficient and necessary criterion for the existence of *N*-wave resonant solutions to Hirota bilinear equations was established, and an algorithm was also discussed about constructing Hirota bilinear equations possessing *N*-wave resonant solutions of linear combinations of exponential waves. A few examples with *N*-wave resonant solutions satisfying and not satisfying the dispersion relation were furnished in previous studies.[20–25]

The BKP hierarchy, the KP hierarchy of the B-type, was proposed by Date, Jimbo, Kashiwara, and Miwa.[26] The first two bilinear forms can be written as follows:

$$[(D_3 - D_1^3)D_{-1} + 3D_1^2]\tau \cdot \tau = 0, \tag{1.1}$$

$$(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5)\tau \cdot \tau = 0, \tag{1.2}$$

where the Hirota bilinear differential operators are defined by

$$D_{x_1}^{n_1} D_{x_2}^{n_2} f \cdot g = (\partial_{x_1} - \partial_{x_1'})^{n_1}(\partial_{x_2} - \partial_{x_2'})^{n_2} f(x_1, x_2) g(x_1', x_2')|_{x_1'=x_1, x_2'=x_2},$$

with n_1 and n_2 being arbitrary nonnegative integers.[1] Here τ is a function of the variables x_k, $k = 1, 2, 3, \ldots$ and $D_k \equiv D_{x_k}$.

We first consider a $2N \times 2N$ skew-symmetric matrix $A = (a_{ij})_{1 \le i,j \le 2N}$. The Pfaffian Pf$(A)$ of A is defined conventionally as follows:[1]

$$\begin{aligned} \mathrm{Pf}(A) = (\alpha_1, \alpha_2, \ldots, \alpha_{2N}) &= \left| \begin{array}{cccc} a_{12} & a_{13} & \ldots & a_{1,2N} \\ & a_{23} & \ldots & a_{2,2N} \\ & & \ddots & \vdots \\ & & & a_{2N-1,2N} \end{array} \right| \\ &= \sum_{\sigma} \mathrm{sgn}(\sigma) a_{i_1 i_2} a_{i_3 i_4} \ldots a_{i_{2N-1} i_{2N}}, \end{aligned} \tag{1.3}$$

where the summation is taken over all permutations

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & 2N \\ i_1 & i_2 & \ldots & i_{2N} \end{pmatrix}$$

with

$$i_1 < i_2, i_3 < i_4, \ldots, i_{2N-1} < i_{2N}, i_1 < i_3 < \ldots < i_{2N-1},$$

and $\mathrm{sgn}(\sigma) = \pm 1$ means the parity of the permutations σ. For instance, when $N = 1, 2$, the Pfaffians can be written as

$$\begin{aligned} \mathrm{Pf}(A) &= (\alpha_1, \alpha_2) = a_{12}, \\ \mathrm{Pf}(A) &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}. \end{aligned} \tag{1.4}$$

It is known that Pfaffian identities are similar to the Jacobi identity for determinants. The fundamental Pfaffian identity is[1,27]

$$\begin{aligned} (a_1, a_2, a_3, a_4, 1, 2, \ldots, 2N)(1, 2, \ldots, 2N) = &\ (a_1, a_2, 1, 2, \ldots, 2N)(a_3, a_4, 1, 2, \ldots, 2N) \\ &- (a_1, a_3, 1, 2, \ldots, 2N)(a_2, a_4, 1, 2, \ldots, 2N) \\ &+ (a_1, a_4, 1, 2, \ldots, 2N)(a_2, a_3, 1, 2, \ldots, 2N). \end{aligned} \tag{1.5}$$

Hirota has shown that the BKP equations in bilinear forms reduce to the identity of Pfaffians and that the τ function of the BKP Equations (1.1) and (1.2) is written by the Pfaffian form[27]

$$\tau = \mathrm{Pf}(a_{ij})_{1 \le i \le j \le 2N}, \ a_{ij} = c_{ij} + \int^x D_x f_i(x) f_j(x)\,\mathrm{d}x, \ i, j = 1, 2, \ldots, 2N, \tag{1.6}$$

where the constants are skew-symmetric $c_{ij} = -c_{ji}$ and all $f_i(x)$ satisfy the following linear differential equations:

$$\frac{\partial}{\partial x_n} f_i(x) = \frac{\partial^n}{\partial x^n} f_i(x), \qquad n = -1, 1, 3, 5, \ldots, \tag{1.7}$$

with each f_i having the boundary condition $f_i \to 0$ as $x \to -\infty$ for $i = 1, 2, \dots, 2N$ and $\frac{\partial}{\partial x_{-1}} f_i(x)$ being defined by

$$\frac{\partial}{\partial x_{-1}} f_i(x) = \int^x f_i(x)\mathrm{d}x. \tag{1.8}$$

In the presented paper, we would like to investigate a universal property of Hirota differential operators and apply it to construct of Hirota bilinear equations possessing Pfaffian formulations in (2+1)-dimensions, based on the bilinear BKP hierarchy. Moreover, we will show that for those extended Hirota bilinear equations, there exist N-wave resonant solutions formulated by linear combinations of exponential functions in terms of the linear superposition principle and compute illustrative examples to shed light on the presented results. Finally, a few concluding remarks will be given.

2 | CONSTRUCTION METHOD FOR EXTENDED BILINEAR BKP EQUATIONS

Firstly, let us characterize a universal property of the D-operator which plays a key role in constructing extended (2+1)-dimensional bilinear BKP equations.

Lemma 2.1. *If functions $f_i, i = 1, 2, \dots, 2N$ of the variables (x, y, t) in the Pfaffian form (1.6) satisfy*

$$\begin{aligned}\partial_y f_i &= (a_{-1}\partial_{x_{-1}} + a_1\partial_{x_1} + a_3\partial_{x_3} \dots + a_{2m-1}\partial_{x_{2m-1}})f_i \\ &\equiv a_{-1}\int^x f_i \mathrm{d}x + (a_1\partial_x + a_3\partial_x^3 + \dots + a_{2m-1}\partial_x^{2m-1})f_i,\end{aligned} \tag{2.1a}$$

$$\begin{aligned}\partial_t f_i &= (b_{-1}\partial_{x_{-1}} + b_1\partial_{x_1} + b_3\partial_{x_3} + \dots + b_{2n-1}\partial_{x_{2n-1}})f_i \\ &\equiv b_{-1}\int^x f_i \mathrm{d}x + (b_1\partial_x + b_3\partial_x^3 + \dots + b_{2n-1}\partial_x^{2n-1})f_i,\end{aligned} \tag{2.1b}$$

where m, n are nonnegative integers and $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_n$ are arbitrary constants, then the Pfaffian τ defined by (1.6) yields

$$D_y^p D_t^q \tau\cdot\tau = (a_{-1}D_{-1} + a_1D_1 + \dots + a_{2m-1}D_{2m-1})^p(b_{-1}D_{-1} + b_1D_1 + \dots + b_{2n-1}D_{2n-1})^q\tau\cdot\tau, \tag{2.2}$$

where $D_k \equiv D_{x_k}$.

Proof. From the Pfaffian definition, we first write the Pfaffian τ as follows:

$$\tau = (1, 2, \dots, 2N) = (\bullet),\ (i, j) = a_{ij}. \tag{2.3}$$

Under the conditions (2.1a) and (2.1b), the differentials of the elements (i,j), $(1 \le i \le j \le 2N)$ are expressed as

$$\begin{aligned}\frac{\partial}{\partial y}(i,j) &= a_{-1}\left(f_i\frac{\partial f_j}{\partial x_{-1}} - \frac{\partial f_i}{\partial x_{-1}}f_j\right) + a_1\left(\frac{\partial f_i}{\partial x}f_j - f_i\frac{\partial f_j}{\partial x}\right) + \dots + a_{2m-1}\left[\frac{\partial^{2m-1}f_i}{\partial x^{2m-1}}f_j\right.\\ &\left.-f_i\frac{\partial^{2m-1}f_j}{\partial x^{2m-1}} + 2\sum_{r=1}^{m-1}(-1)^r\left(\frac{\partial^{2m-1-r}f_i}{\partial x^{2m-1-r}}\frac{\partial^r f_j}{\partial x^r} - \frac{\partial^r f_i}{\partial x^r}\frac{\partial^{2m-1-r}f_j}{\partial x^{2m-1-r}}\right)\right],\end{aligned} \tag{2.4a}$$

$$\begin{aligned}\frac{\partial}{\partial t}(i,j) &= b_{-1}\left(f_i\frac{\partial f_j}{\partial x_{-1}} - \frac{\partial f_i}{\partial x_{-1}}f_j\right) + b_1\left(\frac{\partial f_i}{\partial x}f_j - f_i\frac{\partial f_j}{\partial x}\right) + \dots + b_{2n-1}\left[\frac{\partial^{2n-1}f_i}{\partial x^{2n-1}}f_j\right.\\ &\left.-f_i\frac{\partial^{2n-1}f_j}{\partial x^{2n-1}} + 2\sum_{r=1}^{n-1}(-1)^r\left(\frac{\partial^{2n-1-r}f_i}{\partial x^{2n-1-r}}\frac{\partial^r f_j}{\partial x^r} - \frac{\partial^r f_i}{\partial x^r}\frac{\partial^{2n-1-r}f_j}{\partial x^{2n-1-r}}\right)\right].\end{aligned} \tag{2.4b}$$

Thus, we have

$$\frac{\partial}{\partial y}(i,j)=a_{-1}\frac{\partial}{\partial x_{-1}}(i,j)+a_1\frac{\partial}{\partial x_1}(i,j)+\ \dots\ +a_{2m-1}\frac{\partial}{\partial x_{2m-1}}(i,j), \tag{2.5a}$$

$$\frac{\partial}{\partial t}(i,j)=b_{-1}\frac{\partial}{\partial x_{-1}}(i,j)+b_1\frac{\partial}{\partial x_1}(i,j)+\ \dots\ +b_{2n-1}\frac{\partial}{\partial x_{2n-1}}(i,j). \tag{2.5b}$$

Applying the differential rules for Pfaffians introduced in previous studies,[1,27,28] then we can get the following derivatives for τ with respect to the variables y and t:

$$\frac{\partial\tau}{\partial y}=(a_{-1}\partial_{x_{-1}}+a_1\partial_{x_1}+\ \dots\ +a_{2m-1}\partial_{x_{2m-1}})\tau, \tag{2.6a}$$

$$\frac{\partial\tau}{\partial t}=(b_{-1}\partial_{x_{-1}}+b_1\partial_{x_1}+\ \dots\ +b_{2n-1}\partial_{x_{2n-1}})\tau. \tag{2.6b}$$

Next, let us suppose that

$$(a_1+a_2+\ \dots\ +a_m)^n=\sum_{\substack{\lambda_1+\lambda_2+\dots+\lambda_m=n\\ r=1,2,\dots,\mathrm{C}^{\mathrm{n}}_{\mathrm{m+n-1}}\\ \lambda_i=0,1,2,\dots}}\gamma_{nr}\prod_{i=1}^{m}a_i^{\lambda_i}, \tag{2.7}$$

where $\gamma_{nr}, r\ =\ 1,2,\ \dots\ ,\mathrm{C}^{\mathrm{n}}_{\mathrm{m+n-1}}$, are expansion coefficients. By using the definition of the D-operator and the conditions (2.6b) and (2.6a), we have

$$\begin{aligned}
D_y^pD_t^q\tau\cdot\tau&=(\partial_y-\partial_{y'})^p(\partial_t-\partial_{t'})^q\tau(x,y,t)\tau(x,y',t')|_{y'=y,t'=t}\\
&=\left[a_{-1}(\partial_{x_{-1}}-\partial_{x'_{-1}})+a_1(\partial_{x_1}-\partial_{x'_1})+\ \dots\ +a_{2m-1}(\partial_{x_{2m-1}}-\partial_{x'_{2m-1}})\right]^p\\
&\quad\times\left[b_{-1}(\partial_{x_{-1}}-\partial_{x'_{-1}})+b_1(\partial_{x_1}-\partial_{x'_1})+\ \dots\ +b_{2n-1}(\partial_{x_{2n-1}}-\partial_{x'_{2n-1}})\right]^q\tau(x,y,t)\tau(x',y,t)|_{x'=x}\\
&=\sum_{\substack{r=1\\ \lambda_{-1}+\lambda_1+\dots+\lambda_{2m-1}=p\\ \lambda_{2i-1}=0,1,2,\dots}}^{\mathrm{C}^p_{m+p}}\gamma_{pr}\prod_{i=0}^{m}[a_{2i-1}(\partial_{x_{2i-1}}-\partial_{x'_{2i-1}})]^{\lambda_{2i-1}}\\
&\quad\times\sum_{\substack{s=1\\ \omega_{-1}+\omega_1+\dots+\omega_{2n-1}=q\\ \omega_{2j-1}=0,1,2,\dots}}^{\mathrm{C}^q_{n+q}}\gamma_{qs}\prod_{j=0}^{n}[b_{2j-1}(\partial_{x_{2j-1}}-\partial_{x'_{2j-1}})]^{\omega_{2j-1}}\tau(x,y,t)\tau(x',y,t)|_{x'=x}\\
&=\sum_{\substack{r=1\\ \lambda_{-1}+\lambda_1+\dots+\lambda_{2m-1}=p\\ \lambda_{2i-1}=0,1,2,\dots}}^{\mathrm{C}^p_{m+p}}\gamma_{pr}\prod_{i=0}^{m}(a_{2i-1}D_{2i-1})^{\lambda_{2i-1}}\\
&\quad\times\sum_{\substack{s=1\\ \omega_{-1}+\omega_1+\dots+\omega_{2n-1}=q\\ \omega_{2j-1}=0,1,2,\dots}}^{\mathrm{C}^q_{n+q}}\gamma_{qs}\prod_{j=0}^{n}(b_{2j-1}D_{2j-1})^{\omega_{2j-1}}\tau(x,y,t)\tau(x',y,t)|_{x'=x}\\
&=(a_{-1}D_{-1}+a_1D_1+\ \dots\ +a_{2m-1}D_{2m-1})^p(b_{-1}D_{-1}+b_1D_1+\ \dots\ +b_{2n-1}D_{2n-1})^q\tau\cdot\tau.
\end{aligned} \tag{2.8}$$

□

It means that the property holds.

In what follows, we will furnish extended (2+1)-dimensional BKP equations using the identity of Pfaffians and Lemma 2.1 described above. For clarity, we give two propositions as follows.

Proposition 2.1. *Let a group of functions $f_i = f_i(x, y, t), i = 1, 2, \dots 2N$, satisfy the following linear differential equations:*

$$\frac{\partial f_i}{\partial y} = a_{-1} \int_{-\infty}^{x} f_i \,\mathrm{d}x + a_1 \frac{\partial f_i}{\partial x} + a_3 \frac{\partial^3 f_i}{\partial x^3}, \tag{2.9a}$$

$$\frac{\partial f_i}{\partial t} = b_{-1} \int_{-\infty}^{x} f_i \,\mathrm{d}x + b_1 \frac{\partial f_i}{\partial x} + b_3 \frac{\partial^3 f_i}{\partial x^3}. \tag{2.9b}$$

Then the Pfaffian $\tau_N = \mathrm{Pf}(a_{ij})_{1\le i \le j \le 2N}$ defined by (1.6) solves the following bilinear equation

$$(c_1 D_x^4 + c_2 D_x^3 D_y + c_3 D_x^3 D_t + c_4 D_x D_y + c_5 D_x D_t + c_6 D_y D_t + c_7 D_x^2 + c_8 D_y^2 + c_9 D_t^2)\tau \cdot \tau = 0, \tag{2.10}$$

with

$$\begin{aligned}
&c_1 = a_{-1}a_1 b_3^2 - a_{-1}a_3 b_1 b_3 + a_3^2 b_{-1} b_1 - a_1 a_3 b_{-1} b_3,\ c_2 = a_3 b_{-1} b_3 - a_{-1} b_3^2,\\
&c_3 = a_{-1} a_3 b_3 - a_3^2 b_{-1},\ c_4 = -a_{-1} b_1 b_3 + 2a_1 b_{-1} b_3 - a_3 b_{-1} b_1,\\
&c_5 = -a_{-1} a_1 b_3 + 2a_{-1} a_3 b_1 - a_1 a_3 b_{-1},\ c_6 = a_{-1} b_3 + a_3 b_{-1},\\
&c_7 = 3a_{-1}^2 b_3^2 + a_{-1} a_1 b_1 b_3 - 6a_{-1} a_3 b_{-1} b_3 - a_{-1} a_3 b_1^2 - a_1^2 b_{-1} b_3\\
&\qquad + a_1 a_3 b_{-1} b_1 + 3a_3^2 b_{-1}^2,\ c_8 = -b_{-1} b_3,\ c_9 = -a_{-1} a_3,
\end{aligned} \tag{2.11}$$

where a_{-1}, a_1, a_3, b_{-1}, b_1 and b_3 are arbitrary constants but $a_{-1}b_3 - a_3 b_{-1} \neq 0$.

Proof. Using the linear differential Equations (2.9a) and (2.9b) and Lemma 2.1, then the Pfaffian $\tau = \mathrm{Pf}(a_{ij})_{1\le i \le j \le 2N}$ defined by (1.6) yields

$$D_x^4 \tau \cdot \tau = D_1^4 \tau \cdot \tau,\ D_x^3 D_y \tau \cdot \tau = D_1^3 (a_{-1} D_{-1} + a_1 D_1 + a_3 D_3)\tau \cdot \tau,$$

$$D_x^3 D_t \tau \cdot \tau = D_1^3 (b_{-1} D_{-1} + b_1 D_1 + b_3 D_3)\tau \cdot \tau,$$

$$D_x D_y \tau \cdot \tau = D_1 (a_{-1} D_{-1} + a_1 D_1 + a_3 D_3)\tau \cdot \tau,$$

$$D_y D_t \tau \cdot \tau = (a_{-1} D_{-1} + a_1 D_1 + a_3 D_3)(b_{-1} D_{-1} + b_1 D_1 + b_3 D_3)\tau \cdot \tau$$

$$\cdots\ \cdots$$

Substituting the above results into the left-side of Equation (2.10), a direct computation leads to

$$\begin{aligned}
&(c_1 D_x^4 + c_2 D_x^3 D_y + c_3 D_x^3 D_t + c_4 D_x D_y + c_5 D_x D_t\\
&+ c_6 D_y D_t + c_7 D_x^2 + c_8 D_y^2 + c_9 D_t^2)\tau \cdot \tau\\
= &\big[(c_1 + c_2 a_1 + c_3 b_1) D_1^4 + (c_2 a_{-1} + c_3 b_{-1}) D_1^3 D_{-1} + (c_2 a_3 + c_3 b_3) D_1^3 D_3\\
&+ (c_4 a_{-1} + c_5 b_{-1} + c_6 a_{-1} b_1 + c_6 a_1 b_{-1} + 2c_8 a_{-1} a_1 + 2c_9 b_{-1} b_1) D_1 D_{-1}\\
&+ (c_6 a_{-1} b_3 + c_6 a_3 b_{-1} + 2c_8 a_{-1} a_3 + 2c_9 b_{-1} b_3) D_{-1} D_3\\
&+ (c_4 a_3 + c_5 b_3 + c_6 a_1 b_3 + c_6 a_3 b_1 + 2c_8 a_1 a_3 + 2c_9 b_1 b_3) D_1 D_3\\
&+ (c_6 a_{-1} b_{-1} + c_8 a_{-1}^2 + c_9 b_{-1}^2) D_{-1}^2 + (c_6 a_3 b_3 + c_8 a_3^2 + c_9 b_3^2) D_3^2\\
&+ (c_4 a_1 + c_5 b_1 + c_6 a_1 b_1 + c_7 + c_8 a_1^2 + c_9 b_1^2) D_1^2\big]\, \tau \cdot \tau\\
= &(a_{-1} b_3 - a_3 b_{-1})^2 [(D_3 - D_1^3) D_{-1} + 3D_1^2]\tau \cdot \tau = 0,
\end{aligned} \tag{2.12}$$

where we have applied the conditions (2.11) and the Pfaffian identity of the BKP Equation (1.1). Therefore, this shows that τ defined by (1.6) solves the (2+1)-dimensional bilinear Equation (2.10). □

If we take $a_1 = a_3 = 0$ in (2.9), then Equation (2.10) is given by

$$(c_2 D_x^3 D_y + c_4 D_x D_y + c_6 D_y D_t + c_7 D_x^2 + c_8 D_y^2)\tau \cdot \tau = 0, \tag{2.13}$$

with

$$c_2 = -a_{-1} b_3^2, c_4 = -a_{-1} b_1 b_3, c_6 = a_{-1} b_3, c_7 = 3a_{-1}^2 b_3^2, c_8 = -b_{-1} b_3, \tag{2.14}$$

and the other c_i's are zero. A direct computation leads to

$$a_{-1} = -\frac{c_7}{3c_2}, b_{-1} = \frac{c_7 c_8}{3c_2 c_6}, b_1 = -\frac{c_4}{c_6}, b_3 = -\frac{c_2}{c_6}, \tag{2.15}$$

where $c_2 c_6 c_7 \neq 0$. In particular, if $c_2 = 1$, the Hirota bilinear Equation (2.13), which is similar to the $(2+1)$-dimensional bilinear Ito equation investigated in Du and Lou,[29] has the following sufficient conditions on the Pfaffian form solution defined by (1.6):

$$\frac{\partial f_i}{\partial y} = -\frac{c_7}{3}\int_{-\infty}^{x} f_i \mathrm{d}x, \quad \frac{\partial f_i}{\partial t} = \frac{c_7 c_8}{3c_6}\int_{-\infty}^{x} f_i \mathrm{d}x - \frac{c_4}{c_6}\frac{\partial f_i}{\partial x} - \frac{1}{c_6}\frac{\partial^3 f_i}{\partial x^3}, \tag{2.16}$$

where $c_6 c_7 \neq 0$. Similarly, the following bilinear equation

$$(D_x^3 D_t + c_5 D_x D_t + c_6 D_y D_t + c_7 D_x^2 + c_9 D_t^2)\tau \cdot \tau = 0 \tag{2.17}$$

possesses a Pfaffian form solution as follows:

$$\frac{\partial f_i}{\partial y} = \frac{c_7 c_9}{3c_6}\int_{-\infty}^{x} f_i \mathrm{d}x - \frac{c_5}{c_6}\frac{\partial f_i}{\partial x} - \frac{1}{c_6}\frac{\partial^3 f_i}{\partial x^3}, \quad \frac{\partial f_i}{\partial t} = -\frac{c_7}{3}\int_{-\infty}^{x} f_i \mathrm{d}x, \tag{2.18}$$

where $c_6 c_7 \neq 0$.

When $a_{-1} = a_1 = b_1 = 0$, the combined bilinear Equation (2.10) reduces to

$$(c_2 D_x^3 D_y + c_3 D_x^3 D_t + c_6 D_y D_t + c_7 D_x^2 + c_8 D_y^2)\tau \cdot \tau = 0, \tag{2.19}$$

with

$$c_2 = a_3 b_{-1} b_3, c_3 = -a_3^2 b_{-1}, c_6 = a_3 b_{-1}, c_7 = 3a_3^2 b_{-1}^2, c_8 = -b_{-1} b_3,$$

and the other c_i's are zero. A similar direct computation provides us with a set of sufficient conditions on the Pfaffian form solution for the bilinear Equation (2.19):

$$\frac{\partial f_i}{\partial y} = -\frac{c_3}{c_6}\frac{\partial^3 f_i}{\partial x^3}, \quad \frac{\partial f_i}{\partial t} = -\frac{c_7}{3c_3}\int_{-\infty}^{x} f_i \mathrm{d}x + \frac{c_2}{c_6}\frac{\partial^3 f_i}{\partial x^3}, \tag{2.20}$$

where $c_3 c_6 c_7 \neq 0$, $c_7 = 3c_6^2$ and $c_2 c_6 = c_3 c_8$.

We also note that in this proposition, if $a_{-1} = b_3 = 1$ and $a_1 = a_3 = b_{-1} = b_1 = 0$, then Equation (2.10) becomes the bilinear BKP Equation (1.1).

Proposition 2.2. *Let a group of functions $f_i = f_i(x, y, t), i = 1, 2, \ldots 2N$, satisfy the following linear differential equations:*

$$\frac{\partial f_i}{\partial y} = a_1 \frac{\partial f_i}{\partial x} + a_3 \frac{\partial^3 f_i}{\partial x^3} + a_5 \frac{\partial^5 f_i}{\partial x^5}, \tag{2.21a}$$

$$\frac{\partial f_i}{\partial t} = b_1 \frac{\partial f_i}{\partial x} + b_3 \frac{\partial^3 f_i}{\partial x^3} + b_5 \frac{\partial^5 f_i}{\partial x^5}. \tag{2.21b}$$

Then the Pfaffian $\tau_N = \mathrm{Pf}(a_{ij})_{1\le i\le j\le 2N}$ defined by (1.6) solves the following bilinear equation

$$(\rho D_x^6 + c_1 D_x^4 + c_2 D_x^3 D_y + c_3 D_x^3 D_t + c_4 D_x D_y + c_5 D_x D_t$$
$$+c_6 D_y D_t + c_7 D_x^2 + c_8 D_y^2 + c_9 D_t^2)\tau\cdot\tau = 0, \tag{2.22}$$

with

$$c_1 = 5(a_1b_5 - a_5b_1)(a_3b_5 - a_5b_3),\ c_2 = -5b_5(a_3b_5 - a_5b_3),$$
$$c_3 = 5a_5(a_3b_5 - a_5b_3),\ c_4 = 10a_1b_5^2 - 9a_3b_3b_5 - 10a_5b_1b_5 + 9a_5b_3^2,$$
$$c_5 = -10a_1a_5b_5 + 9a_3^2b_5 - 9a_3a_5b_3 + 10a_5^2b_1,\ c_6 = 10a_5b_5,$$
$$c_7 = -5(a_1b_5 - a_5b_1)^2 + 9(a_1b_3 - a_3b_1)(a_3b_5 - a_5b_3),$$
$$c_8 = -5b_5^2,\ c_9 = -5a_5^2,\ \rho = (a_3b_5 - a_5b_3)^2, \tag{2.23}$$

where a_1, a_3, a_5, b_1, b_3, and b_5 are arbitrary constants but $a_3b_5 - a_5b_3 \neq 0$.

Proof. By Lemma 2.1, we have

$$D_x^6\tau\cdot\tau = D_1^6\tau\cdot\tau,\ D_x^4\tau\cdot\tau = D_1^4\tau\cdot\tau,$$
$$D_x^3D_y\tau\cdot\tau = D_1^3(a_1D_1 + a_3D_3 + a_5D_5)\tau\cdot\tau,$$
$$D_x^3D_t\tau\cdot\tau = D_1^3(b_1D_1 + b_3D_3 + b_5D_5)\tau\cdot\tau,$$
$$D_y^2\tau\cdot\tau = (a_1D_1 + a_3D_3 + a_5D_5)^2\tau\cdot\tau,$$
$$\cdots\ \cdots$$

Similarly, substituting the above results into the left-side of Equation (2.22), we obtain

$$\begin{aligned}
&(\rho D_x^6 + c_1 D_x^4 + c_2 D_x^3 D_y + c_3 D_x^3 D_t + c_4 D_x D_y + c_5 D_x D_t\\
&+ c_6 D_y D_t + c_7 D_x^2 + c_8 D_y^2 + c_9 D_t^2)\tau\cdot\tau\\
=\ &\big[\rho D_1^6 + (c_1 + c_2a_1 + c_3b_1)D_1^4 + (c_2a_3 + c_3b_3)D_1^3D_3 + (c_2a_5 + c_3b_5)D_1^3D_5\\
&+ (c_4a_3 + c_5b_3 + c_6a_1b_3 + c_6a_3b_1 + 2c_8a_1a_3 + 2c_9b_1b_3)D_1D_3\\
&+ (c_4a_5 + c_5b_5 + c_6a_1b_5 + c_6a_5b_1 + 2c_8a_1a_5 + 2c_9b_1b_5)D_1D_5\\
&+ (c_6a_3b_5 + c_6a_5b_3 + 2c_8a_3a_5 + 2c_9b_3b_5)D_3D_5 + (c_6a_3b_3 + c_8a_3^2 + c_9b_3^2)D_3^2\\
&+(c_4a_1 + c_5b_1 + c_6a_1b_1 + c_8a_1^2 + c_9b_1^2 + c_7)D_1^2 + (c_6a_5b_5 + c_8a_5^2 + c_9b_5^2)D_5^2\big]\,\tau\cdot\tau\\
=\ &(a_3b_5 - a_5b_3)^2(D_1^6 - 5D_1^3D_3 - 5D_3^2 + 9D_1D_5)\tau\cdot\tau = 0,
\end{aligned} \tag{2.24}$$

where we have applied the conditions (2.23) and the Pfaffian identity of the BKP equation (1.2). Therefore, this shows that τ defined by (1.6) solves the (2+1)-dimensional bilinear equation (2.22). □

Let us set $a_1 = a_5 = 0$ in (2.21), then Equation (2.22) can be rewritten as

$$(D_x^6 + c_2 D_x^3 D_y + c_4 D_x D_y + c_5 D_x D_t + c_7 D_x^2 + c_8 D_y^2)\tau\cdot\tau = 0, \tag{2.25}$$

with

$$c_2 = -\frac{5}{a_3}, c_4 = -\frac{9b_3}{a_3b_5}, c_5 = \frac{9}{b_5}, c_7 = -\frac{9b_1}{b_5}, c_8 = -\frac{5}{a_3^2}, \tag{2.26}$$

and the other c_i's are zero. Through direct computations, we have the Pfaffian form solution of the Hirota bilinear Equation (2.25) defined by (1.6) with

$$\frac{\partial f_i}{\partial y} = -\frac{5}{c_2}\frac{\partial^3 f_i}{\partial x^3},\ \frac{\partial f_i}{\partial t} = -\frac{c_7}{c_5}\frac{\partial f_i}{\partial x} + \frac{5c_4}{c_2c_5}\frac{\partial^3 f_i}{\partial x^3} + \frac{9}{c_5}\frac{\partial^5 f_i}{\partial x^5}, \tag{2.27}$$

where $c_2c_5 \neq 0$ and $c_2^2 = -5c_8$.

Moreover, if we take $a_1 = a_5 = b_1 = b_3 = 0$ and $a_3 = b_5 = 1$, then Equation (2.22) transforms into the (2+1)-dimensional bilinear BKP Equation (1.2).

3 | RESONANT SOLUTIONS AND APPLICATIONS

As we know, Hirota bilinear equations may possess linear subspaces of solutions. The linear superposition principle presented by Ma and Fan plays a vital role in constructing linear combination solutions of exponential waves to Hirota bilinear equations,[18,19] particularly higher-dimensional bilinear equations. In this section, we would like to present N-wave resonant solutions by means of Pfaffian solutions for the above extended bilinear BKP equations.

Let us first describe the linear superposition principle for constructing exponential wave function solutions.[18,19]

Theorem 3.1 (Linear superposition principle). *Let N-wave variables*

$$\eta_i = k_{1,i}x_1 + k_{2,i}x_2 + \dots + k_{M,i}x_M,\ 1 \le i \le N, \tag{3.1}$$

where $k_{j,\,i}$ are constants and a Hirota bilinear equation, denoted by

$$P(D_{x_1}, D_{x_2}, \dots, D_{x_M})f \cdot f = 0, \tag{3.2}$$

where P is an even polynomial in the indicated variables satisfying

$$P(k_{1,i} - k_{1,j}, \dots, k_{M,i} - k_{M,j}) = 0,\ 1 \le i \neq j \le N, \tag{3.3}$$

then any linear combination of the exponential waves $e^{\eta_i}, 1 \le i \le N$, solves the bilinear Equation (3.2) and f is called an N-wave resonant solution to Equation (3.2).

Theorem 3.1 has been proven in previous studies[18,19] For the bilinear Equation (2.10), the corresponding polynomial reads:

$$P(x, y, t) = c_1x^4 + c_2x^3y + c_3x^3t + c_4xy + c_5xt + c_6yt + c_7x^2 + c_8y^2 + c_9z^2, \tag{3.4}$$

where $c_i, 1 \le i \le 9$, are defined by (2.11). Associated with Equation (2.10), let us take $c_{ij} = 1$ and consider solutions to the system (2.9) of the form

$$f_i = e^{\xi_i},\ \xi_i = k_ix + (a_{-1}k_i^{-1} + a_1k_i + a_3k_i^3)y + (b_{-1}k_i^{-1} + b_1k_i + b_3k_i^3)t, \tag{3.5}$$

$$f_j = e^{\xi_j},\ \xi_j = -k_jx - (a_{-1}k_j^{-1} + a_1k_j + a_3k_j^3)y - (b_{-1}k_j^{-1} + b_1k_j + b_3k_j^3)t, \tag{3.6}$$

where k_i and $k_j, i \neq j$ are arbitrary constants. Then we can obtain one-soliton solutions of Equation (2.10)

$$\tau = (i, j) = 1 + \frac{k_i + k_j}{k_i - k_j}e^{\xi_i + \xi_j}. \tag{3.7}$$

Substituting the above solutions into (2.10), we have

$$\begin{aligned}(c_1D_x^4 + c_2D_x^3D_y + c_3D_x^3D_t + c_4D_xD_y + c_5D_xD_t + c_6D_yD_t + c_7D_x^2\\ + c_8D_y^2 + c_9D_t^2)\left(1 + \frac{k_i + k_j}{k_i - k_j}e^{\xi_i + \xi_j}\right) \cdot \left(1 + \frac{k_i + k_j}{k_i - k_j}e^{\xi_i + \xi_j}\right) = 0,\end{aligned} \tag{3.8}$$

where $c_i, 1 \le i \le 9$, are defined by (2.11). By using the properties for bilinear derivatives introduced in Hirota[1]

$$D_t^mD_x^nf \cdot 1 = \partial_t^m\partial_x^nf,\ D_t^mD_x^ne^{\xi_i} \cdot e^{\xi_i} = 0,$$

then the identity (3.8) yields

$$P\Big(k_i - k_j, (a_{-1}k_i^{-1} + a_1k_i + a_3k_i^3) - (a_{-1}k_j^{-1} + a_1k_j + a_3k_j^3), \\ (b_{-1}k_i^{-1} + b_1k_i + b_3k_i^3) - (b_{-1}k_j^{-1} + b_1k_j + b_3k_j^3)\Big) = 0, \tag{3.9}$$

where the polynomia $P(x, y, t)$ is defined by (3.4). Based on the linear superposition principle in Theorem 3.1, we find that the Hirota bilinear Equation (2.10) corresponding to the polynomial (3.4) has the following N-wave resonant solution

$$\tau = \sum_{i=1}^{N} \varepsilon_i e^{k_ix+(a_{-1}k_i^{-1}+a_1k_i+a_3k_i^3)y+(b_{-1}k_i^{-1}+b_1k_i+b_3k_i^3)t}, \tag{3.10}$$

where the ε_i's, k_i's, and $a_{-1}, a_1, a_3, b_{-1}, b_1, b_3$ are arbitrary constants but $a_{-1}b_3 - a_3b_{-1} \neq 0$. Similarly, the bilinear equation (2.22) possesses the following resonant multiple wave solution expressed as

$$\tau = \sum_{i=1}^{N} \varepsilon_i e^{k_ix+(a_1k_i+a_3k_i^3+a_5k_i^5)y+(b_1k_i+b_3k_3+b_5k_i^5)t}, \tag{3.11}$$

where the ε_i's, k_i's, and $a_1, a_3, a_5, b_1, b_3, b_5$ are arbitrary constants but $a_3b_5 - a_5b_3 \neq 0$.

In the following, three application examples will be provided to illustrate the Hirota bilinear equations possessing their particular Pfaffian and N-wave resonant solutions.

Example 1. The (2+1)-dimensional generalized Hirota–Satsuma–Ito (HSI) equation

Let us first consider the (2+1)-dimensional generalized HSI equation[30]

$$\begin{aligned} \mathrm{gHSI} := u_{tt} + u_{xxxt} + 6u_xu_t + 3uu_{xt} - 3u_{xx}v_t + \beta u_{xt} + u_{yt} + \alpha u_{xx} = 0, \\ [v_x = -u, \end{aligned} \tag{3.12}$$

where the constant $\alpha \neq 0$, but the constant β is arbitrary. Under the typical transformation $u = 2(\ln\tau)_{xx}$, this equation is mapped into

$$(D_t^2 + D_tD_x^3 + \beta D_tD_x + D_tD_y + \alpha D_x^2)\tau \cdot \tau = 0. \tag{3.13}$$

Following expression (2.18), a set of sufficient conditions, which makes the Pfaffian determinant a solution to the above bilinear equation (3.13), can be expressed as follows:

$$\frac{\partial f_i}{\partial y} = \frac{\alpha}{3}\int_{-\infty}^{x} f_i\,\mathrm{d}x - \beta\frac{\partial f_i}{\partial x} - \frac{\partial^3 f_i}{\partial x^3}, \quad \frac{\partial f_i}{\partial t} = -\frac{\alpha}{3}\int_{-\infty}^{x} f_i\,\mathrm{d}x. \tag{3.14}$$

Also, the (2+1)-dimensional equation gHSI (3.12) has the following N-wave solution:

$$u = 2(\ln\tau)_{xx}, \ \tau = \sum_{i=1}^{N} \varepsilon_i e^{k_ix+\left(\frac{\alpha}{3}k_i^{-1}-\beta k_i-k_i^3\right)y-\frac{\alpha}{3}k_i^{-1}t}, \tag{3.15}$$

where the ε_i's and k_i's are arbitrary constants.

Example 2. The (2+1)-dimensional fourth-order nonlinear equation

The second example is a special fourth-order nonlinear equation in (2+1)-dimensions[31] as follows:

$$\alpha[3(u_xu_t)_x + u_{xxxt}] + \beta[3(u_xu_y)_x + u_{xxxy}] + \frac{\alpha}{\beta}u_{yt} + u_{xx} + u_{yy} = 0, \tag{3.16}$$

where the constants α and β satisfy $\alpha\beta \neq 0$. Via the dependent variable transformation $u = 2(\ln \tau)_x$, this equation is written as

$$\left(\alpha D_x^3 D_t + \beta D_x^3 D_y + \frac{\alpha}{\beta} D_y D_t + D_x^2 + D_y^2\right) \tau \cdot \tau = 0. \tag{3.17}$$

Based on the sufficient conditions (2.20) in Proposition 2.1, the corresponding Pfaffian form solution reads as

$$\frac{\partial f_i}{\partial y} = -\beta \frac{\partial^3 f_i}{\partial x^3}, \ \frac{\partial f_i}{\partial t} = -\frac{1}{3\alpha} \int_{-\infty}^{x} f_i \, \mathrm{d}x + \frac{\beta^2}{\alpha} \frac{\partial^3 f_i}{\partial x^3}. \tag{3.18}$$

In the meantime, we can obtain the resonant multiple wave solutions u to Equation (3.16) as

$$u = 2(\ln \tau)_x, \ \tau = \sum_{i=1}^{N} \varepsilon_i e^{k_i x - \beta k_i^3 y - \left(\frac{1}{3\alpha} k_i^{-1} - \frac{\beta^2}{\alpha} k_i^3\right) t}, \tag{3.19}$$

where the ε_i's and k_i's are arbitrary constants.

Example 3. The (2+1)-dimensional fifth-order KdV equation

As the third example, we consider the following (2+1)-dimensional fifth-order KdV equation[32]

$$36u_t = -u_{xxxxx} - 15(uu_{xx})_x - 45u^2 u_x + 5u_{xxy} + 15uu_y + 15u_x \int u_y \mathrm{d}x + 5 \int u_{yy} \mathrm{d}x, \tag{3.20}$$

which was first proposed by Konopelchenko and Dubovsky.[33,34] This equation has a bilinear form under the logarithmic transformation $u = 2(\ln \tau)_{xx}$:

$$(36D_x D_t + D_x^6 - 5D_x^3 D_y - 5D_y^2)\tau \cdot \tau = 0, \tag{3.21}$$

for which there exists a set of sufficient conditions on the Pfaffian form solution defined by (1.6) with

$$\frac{\partial f_i}{\partial y} = \frac{\partial^3 f_i}{\partial x^3}, \ \frac{\partial f_i}{\partial t} = \frac{1}{4} \frac{\partial^5 f_i}{\partial x^5}. \tag{3.22}$$

In addition, the following N-wave resonant solution can be determined

$$u = 2(\ln \tau)_{xx}, \tau = \sum_{i=1}^{N} \varepsilon_i e^{k_i x + k_i^3 y + \frac{1}{4} k_i^5 t}, \tag{3.23}$$

where the ε_i's and k_i's are arbitrary constants.

4 | CONCLUDING REMARKS

In summary, by means of a universal property of Hirota differential operators and the fundamental Pfaffian identity, we built Hirota bilinear equations in (2+1)-dimensions that possess the Pfaffian form solutions. The resulting sufficient conditions on Pfaffian formulas show us the N-wave resonant solutions of linear combinations of exponential waves that the consided Hirota bilinear equations carry. Applications were made for the (2+1)-dimensional generalized HSI equation, the special fourth-order (2+1)-dimensional nonlinear equation, and the fifth-order KdV equation, thereby presenting their particular Pfaffian and multiple wave solutions.

We also remark that N-wave solutions formed by linear combinations of hyperbolic and trigonometric functions can be obtained for the considered Hirota bilinear equations, since the solutions of the BKP hierarchy are expressed in terms of Pfaffians with odd weight variables. As an example, the (2+1)-dimensional fifth-order KdV equation has the following mixed-type function solutions like complexiton solutions:[16,23]

$$u = 2(\ln \tau)_{xx},\ \tau = \sum_{i=1}^{N}\left[\varepsilon_i \cosh\left(k_i x + k_i^3 y + \frac{1}{4}k_i^5 t\right) + \lambda_i \cos\left(k_i x - k_i^3 y + \frac{1}{4}k_i^5 t\right)\right],$$

and

$$\tau = \sum_{i=1}^{N}\left[\varepsilon_i \sinh\left(k_i x + k_i^3 y + \frac{1}{4}k_i^5 t\right) + \lambda_i \sin\left(k_i x - k_i^3 y + \frac{1}{4}k_i^5 t\right)\right],$$

where the ε_i' s, λ_i' s, and k_i' s are arbitrary constants.

Additionally, we remark that we only considered some specific Hirota bilinear equations possessing Pfaffian and N-wave resonant solutions. There should exist more interesting problems which need to be discussed. We expect to investigate integrable properties[35,36] and high-order lump solutions[37–44] to the presented Hirota bilinear equations in our future works.

ACKNOWLEDGEMENT

The authors express their sincere thanks to the anonymous referees for their valuable comments. There are no funders to report for this submission.

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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How to cite this article: Cheng L, Ma W-X. Framework of extended BKP equations possessing N-wave solutions in (2+1)-dimensions. *Math Meth Appl Sci*. 2022;45:749-760. https://doi.org/10.1002/mma.7809