Framework of extended BKP equations possessing N-wave solutions in (2+1)-dimensions

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INTRODUCTION

There are many interesting methods to explore exact solutions of nonlinear wave equations, including the Hirota's bilinear method,1–4 the Lie symmetry method5 and the Riemann–Hilbert approach.6–8 As is well known, the Hirota's bilinear method historically developed is pretty powerful in constructing soliton solutions for integrable and nonintegrable nonlinear equations by perturbation.1,2 Soliton solutions, generated from combinations of multiple exponential waves on the basis of Hirota bilinear forms, could be expressed as Wronskian or Grammian determinants. Typical nonlinear equations, such as the Korteweg–de Vries (KdV) equation, the Kadomtsev–Petviashvili (KP) equation, and the Jimbo–Miwa equation, possess Wronskian or Grammian determinant solutions.1,9–14 For some higher-dimensional soliton equations, there exist Pfaffian solutions complementing Wronskian and Grammian determinant solutions,15,16 due to Pfaffians generalized determinants. Soliton solutions of the B-type Kadomtsev–Petviashvili (BKP) equation were expressed in terms of Pfaffians with odd weight variables by Hirota.1,17

Adding to the diversity of solitons, N-wave resonant solutions, which are linear combinations of exponential traveling waves, were presented by the linear superposition principle. In previous studies,18,19 a sufficient and necessary criterion for the existence of N-wave resonant solutions to Hirota bilinear equations was established, and an algorithm was also discussed about constructing Hirota bilinear equations possessing N-wave resonant solutions of linear combinations of exponential waves. A few examples with N-wave resonant solutions satisfying and not satisfying the dispersion relation were furnished in previous studies.20–25
The BKP hierarchy, the KP hierarchy of the B-type, was proposed by Date, Jimbo, Kashiwara, and Miwa.\textsuperscript{26} The first two bilinear forms can be written as follows:

\begin{align}
[(D_3 - D_1^2)D_{-1} + 3D_1] \tau \cdot \tau &= 0, \quad (1.1) \\
(D_1^6 - 5D_1^4D_3 - 5D_3^2 + 9D_1D_3) \tau \cdot \tau &= 0, \quad (1.2)
\end{align}

where the Hirota bilinear differential operators are defined by

\[ D_n^m f \cdot g = (\partial_{x_1} - \partial_{x_1'})^m (\partial_{x_2} - \partial_{x_2'})^n f(x_1, x_2)g(x_1', x_2'), \]

with \( n_1 \) and \( n_2 \) being arbitrary nonnegative integers.\textsuperscript{1} Here \( \tau \) is a function of the variables \( x_k, k = 1, 2, 3, \ldots \) and \( D_k \equiv D_{x_k}^2 \).

We first consider a \( 2N \times 2N \) skew-symmetric matrix \( A = (a_{ij})_{1 \leq i, j \leq 2N} \). The Pfaffian \( Pf(A) \) of \( A \) is defined conventionally as follows:\textsuperscript{1}

\[
Pf(A) = (a_{12} a_{13} \ldots a_{1,2N}) = \\
\begin{vmatrix}
| a_{12} & a_{13} & \ldots & a_{1,2N} \\
| a_{23} & a_{24} & \ldots & a_{2,2N} \\
| \vdots & \vdots & \ddots & \vdots \\
| a_{2N-1,2} & a_{2N-1,3} & \ldots & a_{2N-1,2N} \\
\end{vmatrix}
\]

\[ = \sum_{\sigma} \text{sgn}(\sigma)a_{i_1,i_2}a_{i_3,i_4} \ldots a_{i_{2N-1},i_{2N}}, \quad (1.3)\]

where the summation is taken over all permutations

\[ \sigma = \begin{pmatrix} 1 & 2 & \ldots & 2N \\ i_1 & i_2 & \ldots & i_{2N} \end{pmatrix} \]

with

\[ i_1 < i_2, i_3 < i_4, \ldots, i_{2N-1} < i_{2N}, i_1 < i_3 < \ldots < i_{2N-1}. \]

and \( \text{sgn}(\sigma) = \pm 1 \) means the parity of the permutations \( \sigma \). For instance, when \( N = 1, 2 \), the Pfaffians can be written as

\[
Pf(A) = (a_{1,2}) = a_{12},
\]

\[
Pf(A) = (a_{1,2,3,4}) = a_{13}a_{14} - a_{13}a_{24} + a_{14}a_{23}. \quad (1.4)
\]

It is known that Pfaffian identities are similar to the Jacobi identity for determinants. The fundamental Pfaffian identity is\textsuperscript{1,27}

\[
(a_{1,2,3,4,1,2,\ldots,2N})(1,2,\ldots,2N) = (a_{1,2,3,4,1,2,\ldots,2N})(a_{1,2,3,4,1,2,\ldots,2N}) - (a_{1,2,3,4,1,2,\ldots,2N})(a_{1,2,3,4,1,2,\ldots,2N})
\]

\[ + (a_{1,2,3,4,1,2,\ldots,2N})(a_{1,2,3,4,1,2,\ldots,2N}). \quad (1.5)\]

Hirota has shown that the BKP equations in bilinear forms reduce to the identity of Pfaffians and that the \( \tau \) function of the BKP Equations (1.1) and (1.2) is written by the Pfaffian form\textsuperscript{27}

\[
\tau = Pf(a_{ij}, 1 \leq i, j \leq 2N), \quad a_{ij} = c_{ij} + \int D_x f_i(x)f_j(x)dx, \quad i, j = 1, 2, \ldots, 2N, \quad (1.6)
\]

where the constants are skew-symmetric \( c_{ij} = -c_{ji} \) and all \( f_i(x) \) satisfy the following linear differential equations:

\[
\frac{\partial}{\partial x_n} f_i(x) = \frac{\partial^n}{\partial x^n} f_i(x), \quad n = -1, 1, 3, 5, \ldots, \quad (1.7)
\]
with each \( f_i \) having the boundary condition \( f_i \to 0 \) as \( x \to -\infty \) for \( i = 1, 2, \ldots, 2N \) and \( \frac{\partial}{\partial x_{-1}} f_i(x) \) being defined by

\[
\frac{\partial}{\partial x_{-1}} f_i(x) = \int_{x_{-1}}^{x} f_i(x) \, dx.
\]

(1.8)

In the presented paper, we would like to investigate a universal property of the Hirota differential operators and apply it to construct of Hirota bilinear equations possessing Pfaffian formulations in (2+1)-dimensions, based on the bilinear BKP hierarchy. Moreover, we will show that for those extended Hirota bilinear equations, there exist \( N \)-wave resonant solutions formulated by linear combinations of exponential functions in terms of the linear superposition principle and compute illustrative examples to shed light on the presented results. Finally, a few concluding remarks will be given.

### 2. CONSTRUCTION METHOD FOR EXTENDED BILINEAR BKP EQUATIONS

Firstly, let us characterize a universal property of the \( D \)-operator which plays a key role in constructing extended (2+1)-dimensional bilinear BKP equations.

**Lemma 2.1.** If functions \( f_i, i = 1, 2, \ldots, 2N \) of the variables \((x, y, t)\) in the Pfaffian form (1.6) satisfy

\[
\begin{aligned}
\partial_y f_i &= (a_{-1} \partial_{x_{-1}} + a_1 \partial_{x_1} + a_2 \partial_{x_2} + \ldots + a_{2m-1} \partial_{x_{2m-1}}) f_i \\
&\equiv a_{-1} \int_{x_{-1}}^{x} f_i \, dx + (a_1 \partial_{x_1} + a_2 \partial_{x_2} + \ldots + a_{2m-1} \partial_{x_{2m-1}}) f_i,
\end{aligned}
\]

(2.1a)

\[
\begin{aligned}
\partial_t f_i &= (b_{-1} \partial_{x_{-1}} + b_1 \partial_{x_1} + b_2 \partial_{x_2} + \ldots + b_{2n-1} \partial_{x_{2n-1}}) f_i \\
&\equiv b_{-1} \int_{x_{-1}}^{x} f_i \, dx + (b_1 \partial_{x_1} + b_2 \partial_{x_2} + \ldots + b_{2n-1} \partial_{x_{2n-1}}) f_i,
\end{aligned}
\]

(2.1b)

where \( m, n \) are nonnegative integers and \( a_1, a_2, \ldots, a_m \) and \( b_1, b_2, \ldots, b_n \) are arbitrary constants, then the Pfaffian \( \tau \) defined by (1.6) yields

\[
D^\theta_k D^\theta_l \cdot \tau = (a_{-1} D_{-1} + a_1 D_1 + \ldots + a_{2m-1} D_{2m-1})^\theta (b_{-1} D_{-1} + b_1 D_1 + \ldots + b_{2n-1} D_{2n-1})^\theta \tau \cdot \tau,
\]

(2.2)

where \( D_k \equiv D_{x_k} \).

**Proof.** From the Pfaffian definition, we first write the Pfaffian \( \tau \) as follows:

\[
\tau = (1, 2, \ldots, 2N) = (\ast), \quad (i, j) = a_{ij}.
\]

(2.3)

Under the conditions (2.1a) and (2.1b), the differentials of the elements \((i, j), (1 \leq i \leq j \leq 2N)\) are expressed as

\[
\begin{aligned}
\frac{\partial}{\partial y}(i, j) &= a_{-1} \left( f_i \frac{\partial f_j}{\partial x_{-1}} - f_j \frac{\partial f_i}{\partial x_{-1}} \right) + a_1 \left( \frac{\partial f_i}{\partial x} f_j - f_i \frac{\partial f_j}{\partial x} \right) + \ldots + a_{2m-1} \left[ \frac{\partial^{2m-1} f_i}{\partial x^{2m-1}} \frac{\partial f_j}{\partial x^{m}} \right]
\end{aligned}
\]

(2.4a)

\[
\begin{aligned}
\frac{\partial}{\partial t}(i, j) &= b_{-1} \left( f_i \frac{\partial f_j}{\partial x_{-1}} - f_j \frac{\partial f_i}{\partial x_{-1}} \right) + b_1 \left( \frac{\partial f_i}{\partial x} f_j - f_i \frac{\partial f_j}{\partial x} \right) + \ldots + b_{2n-1} \left[ \frac{\partial^{2n-1} f_i}{\partial x^{2n-1}} \frac{\partial f_j}{\partial x^{n}} \right]
\end{aligned}
\]

(2.4b)
Thus, we have
\[
\frac{\partial}{\partial y} (i, j) = a_{-1} \frac{\partial}{\partial x_{-1}} (i, j) + a_1 \frac{\partial}{\partial x_1} (i, j) + \ldots + a_{2m-1} \frac{\partial}{\partial x_{2m-1}} (i, j),
\]
(2.5a)

\[
\frac{\partial}{\partial t} (i, j) = b_{-1} \frac{\partial}{\partial x_{-1}} (i, j) + b_1 \frac{\partial}{\partial x_1} (i, j) + \ldots + b_{2n-1} \frac{\partial}{\partial x_{2n-1}} (i, j).
\]
(2.5b)

Applying the differential rules for Pfaffians introduced in previous studies, then we can get the following derivatives for \(r\) with respect to the variables \(y\) and \(t\):

\[
\frac{\partial r}{\partial y} = (a_{-1} \partial_{x_{-1}} + a_1 \partial_{x_1} + \ldots + a_{2m-1} \partial_{x_{2m-1}}) r,
\]
(2.6a)

\[
\frac{\partial r}{\partial t} = (b_{-1} \partial_{x_{-1}} + b_1 \partial_{x_1} + \ldots + b_{2n-1} \partial_{x_{2n-1}}) r.
\]
(2.6b)

Next, let us suppose that

\[
(a_1 + a_2 + \ldots + a_m)^g = \sum_{\lambda_1, \lambda_2, \ldots, \lambda_m = 1}^{\lambda_1 + \lambda_2 + \ldots + \lambda_m = m} \gamma_{\lambda_1, \lambda_2, \ldots, \lambda_m} \prod_{i=1}^{m} a_i^{\lambda_i},
\]
(2.7)

where \(\gamma_{\lambda_1, \lambda_2, \ldots, \lambda_m} = 1, 2, \ldots, C_{m+n-1}^n\), are expansion coefficients. By using the definition of the \(D\)-operator and the conditions (2.6b) and (2.6a), we have

\[
D^g_{x_1} D^g_{x_1} \tau \cdot \tau = (\partial_x - \partial_{x'})^g (\partial_t - \partial_{t'})^g r(x, y, t) \tau(x, y', t') |_{y'=y, t'=t}
\]

\[
= [a_{-1} (\partial_{x_{-1}} - \partial_{x_{-1}'} ) + a_1 (\partial_{x_1} - \partial_{x_1'} ) + \ldots + a_{2m-1} (\partial_{x_{2m-1}} - \partial_{x_{2m-1}'} )]^g
\]

\[
\times [b_{-1} (\partial_{x_{-1}} - \partial_{x_{-1}'} ) + b_1 (\partial_{x_1} - \partial_{x_1'} ) + \ldots + b_{2n-1} (\partial_{x_{2n-1}} - \partial_{x_{2n-1}'} )]^g r(x, y, t) \tau(x', y, t) |_{x'=x}
\]

\[
= \sum_{\lambda_1+\lambda_2+\ldots+\lambda_{2m-1} = g}^{C_{2m-1}^g} \gamma_{\lambda_1, \lambda_2, \ldots, \lambda_{2m-1}} \prod_{i=0}^{m} (a_{2i-1} (\partial_{x_{2i-1}} - \partial_{x_{2i-1}'}))^g
\]

\[
\times \sum_{\lambda_1+\lambda_2+\ldots+\lambda_{2n-1} = g}^{C_{2n-1}^g} \gamma_{\lambda_1, \lambda_2, \ldots, \lambda_{2n-1}} \prod_{j=0}^{n} (b_{2j-1} (\partial_{x_{2j-1}} - \partial_{x_{2j-1}'}))^g r(x, y, t) \tau(x', y, t) |_{x'=x}
\]

\[
= \sum_{\lambda_1+\lambda_2+\ldots+\lambda_{2m-1} = g}^{C_{2m-1}^g} \gamma_{\lambda_1, \lambda_2, \ldots, \lambda_{2m-1}} \prod_{i=0}^{m} (a_{2i-1} D_{2i-1})^g
\]

\[
\times \sum_{\lambda_1+\lambda_2+\ldots+\lambda_{2n-1} = g}^{C_{2n-1}^g} \gamma_{\lambda_1, \lambda_2, \ldots, \lambda_{2n-1}} \prod_{j=0}^{n} (b_{2j-1} D_{2j-1})^g r(x, y, t) \tau(x', y, t) |_{x'=x}
\]

\[
= (a_{-1} D_{-1} + a_1 D_1 + \ldots + a_{2m-1} D_{2m-1})^g (b_{-1} D_{-1} + b_1 D_1 + \ldots + b_{2n-1} D_{2n-1})^g r \cdot \tau.
\]
(2.8)

It means that the property holds.

In what follows, we will furnish extended (2+1)-dimensional BKP equations using the identity of Pfaffians and Lemma 2.1 described above. For clarity, we give two propositions as follows.
**Proposition 2.1.** Let a group of functions \( f_i = f_i(x, y, t), i = 1, 2, \ldots, 2N, \) satisfy the following linear differential equations:

\[
\frac{\partial f_i}{\partial y} = a_{-1} \int_{-\infty}^{x} f_i \, dx + a_1 \frac{\partial f_i}{\partial x} + a_3 \frac{\partial^3 f_i}{\partial x^3},
\]

(2.9a)

\[
\frac{\partial f_i}{\partial t} = b_{-1} \int_{-\infty}^{x} f_i \, dx + b_1 \frac{\partial f_i}{\partial x} + b_3 \frac{\partial^3 f_i}{\partial x^3}.
\]

(2.9b)

Then the Pfaffian \( \tau_N = \text{Pf}(a_{ij})_{1 \leq i \leq 2N} \) defined by (1.6) solves the following bilinear equation

\[
(c_1 D_x^2 + c_2 D_x^3 D_y + c_3 D_x^2 D_t + c_4 D_x D_y + c_5 D_x D_t + c_6 D_y D_t + c_7 D_y^2 + c_8 D_t^2) \tau \cdot \tau = 0,
\]

(2.10)

with

\[
\begin{align*}
    c_1 &= a_{-1} a_1 b_1^2 - a_{-1} a_3 b_1 b_3 + a_1^2 b_1 - a_1 a_3 b_1 b_3, \\
    c_2 &= a_3 b_1 b_3 - a_{-1} b_2^3, \\
    c_3 &= a_{-1} a_3 b_3 - a_2^2 b_1, \\
    c_4 &= -a_{-1} b_1 b_3 + 2a_1 b_2 b_3 - a_2 b_1 b_3, \\
    c_5 &= -a_{-1} a_1 b_1 + 2a_{-1} a_3 b_1 - a_1 a_3 b_1 - a_3 b_1, \\
    c_7 &= 3a_2^2 b_2^3 + a_{-1} a_1 b_1 b_3 - 6a_{-1} a_3 b_1 b_3 - a_{-1} a_3 b_1^2 - a_1^2 b_1 b_3, \\
    c_8 &= -a_{-1} b_3, \\
    c_9 &= -a_{-1} a_3,
\end{align*}
\]

where \( a_{-1}, a_1, a_3, b_{-1}, b_1, \) and \( b_3 \) are arbitrary constants but \( a_{-1} b_3 - a_3 b_{-1} \neq 0. \)

**Proof.** Using the linear differential Equations (2.9a) and (2.9b) and Lemma 2.1, then the Pfaffian \( \tau = \text{Pf}(a_{ij})_{1 \leq i \leq 2N} \) defined by (1.6) yields

\[
D_x^3 \tau \cdot \tau = D_x^2 \tau \cdot \tau, \quad D_x^4 \tau \cdot \tau = D_x^3 (a_{-1} D_{-1} + a_1 D_1 + a_3 D_3) \tau \cdot \tau,
\]

\[
D_t^2 D_x \tau \cdot \tau = D_t^2 (b_{-1} D_{-1} + b_1 D_1 + b_3 D_3) \tau \cdot \tau,
\]

\[
D_t D_x \tau \cdot \tau = D_t (a_{-1} D_{-1} + a_1 D_1 + a_3 D_3) \tau \cdot \tau,
\]

\[
D_t D_x \tau \cdot \tau = (a_{-1} D_{-1} + a_1 D_1 + a_3 D_3) (b_{-1} D_{-1} + b_1 D_1 + b_3 D_3) \tau \cdot \tau
\]

\[
\ldots \ldots
\]

Substituting the above results into the left-side of Equation (2.10), a direct computation leads to

\[
(c_1 D_x^2 + c_2 D_x^3 D_y + c_3 D_x^2 D_t + c_4 D_x D_y + c_5 D_x D_t + c_6 D_y D_t + c_7 D_y^2 + c_8 D_t^2) \tau \cdot \tau = 0
\]

where we have applied the conditions (2.11) and the Pfaffian identity of the BKP Equation (1.1). Therefore, this shows that \( \tau \) defined by (1.6) solves the \((2+1)\)-dimensional bilinear Equation (2.10). \( \square \)
If we take $a_1 = a_3 = 0$ in (2.9), then Equation (2.10) is given by

$$(c_2D_x^2D_y + c_4D_yD_x + c_6D_yD_t + c_7D_x^2 + c_9D_y^2)\tau \cdot \tau = 0,$$

(2.13)

with

$$c_2 = -a_{-1}b_{-1}^2, c_4 = -a_{-1}b_1b_3, c_6 = a_{-1}b_1, c_7 = 3a_{-1}^2b_{-1}^2, c_8 = -b_{-1}b_1,$$

(2.14)

and the other $c_i$'s are zero. A direct computation leads to

$$a_{-1} = -\frac{c_7}{3c_2}, b_{-1} = \frac{c_7c_6}{3c_2c_6}, b_1 = \frac{c_4}{c_6}, b_3 = \frac{c_2}{c_6},$$

(2.15)

where $c_2c_6c_7 \neq 0$. In particular, if $c_2 = 1$, the Hirota bilinear Equation (2.13), which is similar to the $(2+1)$-dimensional bilinear Ito equation investigated in Du and Lou, has the following sufficient conditions on the Pfaffian form solution defined by (1.6):

$$\frac{\partial f_1}{\partial y} = -\frac{c_2}{3} \int_{-\infty}^{x} f_1 \, dx, \quad \frac{\partial f_1}{\partial t} = \frac{c_7c_6}{3c_2} \int_{-\infty}^{x} f_1 \, dx - \frac{c_4}{c_6} \frac{\partial^3 f_1}{\partial x^3} - \frac{1}{c_6} \frac{\partial^3 f_1}{\partial x^3},$$

(2.16)

where $c_6c_7 \neq 0$. Similarly, the following bilinear equation

$$(D_x^3D_t + c_3D_xD_t + c_6D_yD_t + c_7D_x^2 + c_9D_t^2)\tau \cdot \tau = 0$$

(2.17)

possesses a Pfaffian form solution as follows:

$$\frac{\partial f_1}{\partial y} = \frac{c_7c_6}{3c_2} \int_{-\infty}^{x} f_1 \, dx - \frac{c_5}{c_6} \frac{\partial f_1}{\partial x} - \frac{1}{c_6} \frac{\partial^3 f_1}{\partial x^3}, \quad \frac{\partial f_1}{\partial t} = -\frac{c_7}{3} \int_{-\infty}^{x} f_1 \, dx,$$

(2.18)

where $c_6c_7 \neq 0$.

When $a_{-1} = a_1 = b_1 = 0$, the combined bilinear Equation (2.10) reduces to

$$(c_2D_x^2D_y + c_4D_yD_x + c_6D_yD_t + c_7D_x^2 + c_9D_y^2)\tau \cdot \tau = 0,$$

(2.19)

with

$$c_2 = a_3b_{-1}b_3, c_3 = -a_2^2b_{-1}, c_6 = a_3b_{-1}, c_7 = 3a_2^2b_{-1}^2, c_8 = -b_{-1}b_3,$$

and the other $c_i$'s are zero. A similar direct computation provides us with a set of sufficient conditions on the Pfaffian form solution for the bilinear Equation (2.19):

$$\frac{\partial f_1}{\partial y} = -\frac{c_3}{c_6} \frac{\partial^3 f_1}{\partial x^3}, \quad \frac{\partial f_1}{\partial t} = -\frac{c_7}{3c_3} \int_{-\infty}^{x} f_1 \, dx + \frac{c_2}{c_6} \frac{\partial^3 f_1}{\partial x^3},$$

(2.20)

where $c_3c_6c_7 \neq 0, c_7 = 3c_3^2$ and $c_2c_6 = c_3c_8$.

We also note that in this proposition, if $a_{-1} = b_3 = 1$ and $a_1 = a_3 = b_{-1} = b_1 = 0$, then Equation (2.10) becomes the bilinear BKP Equation (1.1).

**Proposition 2.2.** Let a group of functions $f_i = f_i(x,y,t), i = 1, 2, \ldots, 2N$, satisfy the following linear differential equations:

$$\frac{\partial f_1}{\partial y} = a_1 \frac{\partial f_1}{\partial x} + a_2 \frac{\partial^3 f_1}{\partial x^3} + a_5 \frac{\partial^5 f_1}{\partial x^5},$$

(2.21a)

$$\frac{\partial f_1}{\partial t} = b_1 \frac{\partial f_1}{\partial x} + b_3 \frac{\partial^3 f_1}{\partial x^3} + b_5 \frac{\partial^5 f_1}{\partial x^5}.$$  

(2.21b)
Then the Pfaffian $\tau_N = \text{Pf}(a_{ij})_{1 \leq i,j \leq 2N}$ defined by (1.6) solves the following bilinear equation

$$(\rho D_x^2 + c_1 D_x^4 + c_2 D_x^2 D_y + c_3 D_y^2 D_x + c_4 D_x D_y + c_5 D_y^2) \tau \cdot \tau = 0,$$  \hspace{1cm} (2.22)

with

$$c_1 = 5(a_1 b_5 - a_5 b_1)(a_2 b_5 - a_3 b_3), \quad c_2 = -5b_5(a_3 b_5 - a_5 b_3),$$

$$c_3 = 5a_5(a_1 b_5 - a_5 b_3), \quad c_4 = 10a_1 b_5^2 - 9a_3 b_5 b_3 - 10a_1 b_1 b_5 + 9a_5 b_3^2,$$

$$c_5 = -10a_1 a_5 b_5 + 9a_5^2 b_5 - 9a_3 a_5 b_3 + 10a_5^2 b_1, \quad c_6 = 10a_5 b_5,$$

$$c_7 = -5(a_1 b_5 - a_5 b_1)^2 + 9(a_1 b_3 - a_3 b_1)(a_2 b_5 - a_3 b_3),$$

$$c_8 = -5b_5^2, \quad c_9 = -5a_5^2, \quad \rho = (a_3 b_5 - a_5 b_3)^2,$$  \hspace{1cm} (2.23)

where $a_1, a_3, b_1, b_3,$ and $b_5$ are arbitrary constants but $a_3 b_5 - a_5 b_3 \neq 0$.

Proof. By Lemma 2.1, we have

$$D_x^2 \tau \cdot \tau = D_y^2 \tau \cdot \tau, \quad D_y^2 \tau \cdot \tau = D_x^2 \tau \cdot \tau,$$

$$D_x^2 D_y \tau \cdot \tau = D_y^2 (a_1 D_1 + a_3 D_3 + a_5 D_5) \tau \cdot \tau,$$

$$D_x^2 D_x \tau \cdot \tau = D_y^2 (b_1 D_1 + b_3 D_3 + b_5 D_5) \tau \cdot \tau,$$

$$D_y^2 \tau \cdot \tau = (a_1 D_1 + a_3 D_3 + a_5 D_5)^2 \tau \cdot \tau.$$  \hspace{1cm} ...

Similarly, substituting the above results into the left-side of Equation (2.22), we obtain

$$(\rho D_x^2 + c_1 D_x^4 + c_2 D_x^2 D_y + c_3 D_y^2 D_x + c_4 D_x D_y + c_5 D_y^2) \tau \cdot \tau = \left[ \rho D_1^2 + (c_1 + c_2 + c_3) D_1^4 + (c_4 + c_5) D_3^2 D_1 + (c_1 + c_3 + c_5) D_3^2 D_5 \right.$$

$$+ (c_3 a_3 + c_5 b_3 + c_6 a_1 b_5 + c_6 a_3 b_1 + 2c_6 a_3 a_1 + 2c_6 b_1 b_3) D_1 D_3$$

$$+ (c_3 a_5 + c_5 b_5 + c_6 a_5 b_3 + c_6 a_5 b_3 + 2c_6 a_5 a_1 + 2c_6 b_1 b_5) D_1 D_5$$

$$+ (c_3 a_1 b_5 + c_5 a_3 b_3 + 2c_6 a_1 a_3 + 2c_6 b_1 b_3) D_3 D_3$$

$$+ (c_3 a_3 + c_5 b_3 + c_6 a_1 b_5 + c_6 a_3 b_1 + 2c_6 a_3 a_1 + 2c_6 b_1 b_3) D_3 D_5$$

$$+ (c_3 a_1 b_5 + c_5 a_3 b_3 + 2c_6 a_1 a_3 + 2c_6 b_1 b_5) D_5 D_5) \tau \cdot \tau$$

$$= (a_3 b_5 - a_5 b_3)^2 (D_3^2 - 5D_1^2 D_3 - 5D_3^2 + 9D_1 D_3) \tau \cdot \tau = 0,$$  \hspace{1cm} (2.24)

where we have applied the conditions (2.23) and the Pfaffian identity of the BKP equation (1.2). Therefore, this shows that $\tau$ defined by (1.6) solves the (2+1)-dimensional bilinear equation (2.22).

Let us set $a_1 = a_5 = 0$ in (2.21), then Equation (2.22) can be rewritten as

$$(D_x^2 + c_2 D_x^4 + c_4 D_x D_y + c_5 D_x D_t + c_7 D_y^2 + c_6 D_y^2) \tau \cdot \tau = 0,$$  \hspace{1cm} (2.25)

with

$$c_2 = -\frac{5}{a_5}, \quad c_4 = -\frac{9b_3}{a_3 b_5}, \quad c_5 = \frac{9}{b_5}, \quad c_7 = -\frac{9b_1}{b_5}, \quad c_8 = -\frac{5}{a_5^2},$$

and the other $c_i$’s are zero. Through direct computations, we have the Pfaffian form solution of the Hirota bilinear Equation (2.25) defined by (1.6) with

$$\frac{\partial f_i}{\partial y} = \frac{5}{c_2} \frac{\partial^3 f_i}{\partial x^3}, \quad \frac{\partial f_i}{\partial t} = -c_7 \frac{\partial f_i}{c_5} \frac{\partial x}{\partial x} + 5c_4 \frac{\partial^3 f_i}{c_5} \frac{\partial x^3}{\partial x} + 9 \frac{\partial^5 f_i}{c_5} \frac{\partial x^5}{\partial x},$$  \hspace{1cm} (2.27)
where \( c_2 c_5 \neq 0 \) and \( c_2^2 = -5c_8 \).

Moreover, if we take \( a_1 = a_5 = b_1 = b_3 = 0 \) and \( a_3 = b_5 = 1 \), then Equation (2.22) transforms into the (2+1)-dimensional bilinear BKP Equation (1.2).

3 | RESONANT SOLUTIONS AND APPLICATIONS

As we know, Hirota bilinear equations may possess linear subspaces of solutions. The linear superposition principle presented by Ma and Fan plays a vital role in constructing linear combination solutions of exponential waves to Hirota bilinear equations,\(^{18,19} \) particularly higher-dimensional bilinear equations. In this section, we would like to present \( N \)-wave resonant solutions by means of Pfaffian solutions for the above extended bilinear BKP equations.

Let us first describe the linear superposition principle for constructing exponential wave function solutions.\(^{18,19} \)

**Theorem 3.1 (Linear superposition principle).** Let \( N \)-wave variables

\[
\eta_i = k_{1,i} x_1 + k_{2,i} x_2 + \ldots + k_{M,i} x_M, \quad 1 \leq i \leq N, \tag{3.1}
\]

where \( k_{i,j} \) are constants and a Hirota bilinear equation, denoted by

\[
P(D_{x_1}, D_{x_2}, \ldots, D_{x_M})f \cdot f = 0, \tag{3.2}
\]

where \( P \) is an even polynomial in the indicated variables satisfying

\[
P(k_{1,i} - k_{1,j}, \ldots, k_{M,i} - k_{M,j}) = 0, \quad 1 \leq i \neq j \leq N, \tag{3.3}
\]

then any linear combination of the exponential waves \( e^{\psi_i}, 1 \leq i \leq N, \) solves the bilinear Equation (3.2) and \( f \) is called an \( N \)-wave resonant solution to Equation (3.2).

**Theorem 3.1** has been proven in previous studies\(^{18,19} \). For the bilinear Equation (2.10), the corresponding polynomial reads:

\[
P(x, y, t) = c_1 x^4 + c_2 x^3 y + c_3 x^2 y + c_4 x y + c_5 x t + c_6 y t + c_7 x^2 + c_8 y^2 + c_9 z^2, \tag{3.4}
\]

where \( c_i, 1 \leq i \leq 9 \), are defined by (2.11). Associated with Equation (2.10), let us take \( c_{ij} = 1 \) and consider solutions to the system (2.9) of the form

\[
f_i = e^{\psi_i}, \quad \xi_i = k_i x + (a_{-1} k_i^{-1} + a_1 k_i + a_3 k_i^3) y + (b_{-1} k_i^{-1} + b_1 k_i + b_3 k_i^3) t, \tag{3.5}
\]

\[
f_j = e^{\psi_j}, \quad \xi_j = -k_j x - (a_{-1} k_j^{-1} + a_1 k_j + a_3 k_j^3) y - (b_{-1} k_j^{-1} + b_1 k_j + b_3 k_j^3) t, \tag{3.6}
\]

where \( k_i \) and \( k_j, i \neq j \) are arbitrary constants. Then we can obtain one-soliton solutions of Equation (2.10)

\[
\tau = (i, j) = 1 + \frac{k_i + k_j}{k_i - k_j} e^{\psi_i + \psi_j}. \tag{3.7}
\]

Substituting the above solutions into (2.10), we have

\[
\begin{align*}
(c_1 D_x^4 + c_2 D_x^3 D_y + c_3 D_x^2 D_t + c_4 D_x D_y + c_5 D_y D_t + c_6 D_y^2 + c_7 D_x^2 \\
+ c_8 D_y^2 + c_9 D_t^2) \left( 1 + \frac{k_i + k_j}{k_i - k_j} e^{\psi_i + \psi_j} \right) \cdot \left( 1 + \frac{k_i + k_j}{k_i - k_j} e^{\psi_i + \psi_j} \right) &= 0, \tag{3.8}
\end{align*}
\]

where \( c_i, 1 \leq i \leq 9 \), are defined by (2.11). By using the properties for bilinear derivatives introduced in Hirota\(^1 \)

\[
D_{i}^{m} D_{j}^{n} f \cdot 1 = \partial_{x}^{m} \partial_{x}^{n} f, \quad D_{i}^{m} D_{j}^{n} e^{\psi_i} \cdot e^{\psi_j} = 0,
\]
then the identity (3.8) yields

\[ P \left( k_i - k_j, \left( a_{-1} k_i^{-1} + a_1 k_i + a_3 k_i^3 \right) - \left( a_{-1} k_j^{-1} + a_1 k_j + a_3 k_j^3 \right), \right) \]
\[ (b_{-1} k_i^{-1} + b_1 k_i + b_3 k_i^3) - (b_{-1} k_j^{-1} + b_1 k_j + b_3 k_j^3) = 0, \]

where the polynomial \( P(x, y, t) \) is defined by (3.4). Based on the linear superposition principle in Theorem 3.1, we find that the Hirota bilinear Equation (2.10) corresponding to the polynomial (3.4) has the following N-wave resonant solution

\[ \tau = \sum_{i=1}^{N} \epsilon_i e^{k_i x + (a_{-1} k_i^{-1} + a_1 k_i + a_3 k_i^3) y + (b_{-1} k_i^{-1} + b_1 k_i + b_3 k_i^3) t}, \]

(3.10)

where the \( \epsilon_i \)'s, \( k_i \)'s, and \( a_{-1}, a_1, a_3, b_{-1}, b_1, b_3 \) are arbitrary constants but \( a_{-1} b_3 - a_3 b_{-1} \neq 0 \). Similarly, the bilinear equation (2.22) possesses the following resonant multiple wave solution expressed as

\[ \tau = \sum_{i=1}^{N} \epsilon_i e^{k_i x + (a_{-1} k_i^{-1} + a_1 k_i + a_3 k_i^3) y + (b_{-1} k_i^{-1} + b_1 k_i + b_3 k_i^3) t}, \]

(3.11)

where the \( \epsilon_i \)'s, \( k_i \)'s, and \( a_1, a_3, b_1, b_3, b_5 \) are arbitrary constants but \( a_3 b_5 - a_5 b_3 \neq 0 \).

In the following, three application examples will be provided to illustrate the Hirota bilinear equations possessing their particular Pfaffian and N-wave resonant solutions.

**Example 1.** The (2+1)-dimensional generalized Hirota–Satsuma–Ito (HSI) equation

Let us first consider the (2+1)-dimensional generalized HSI equation

\[ g_{\text{HSI}} := u_{tt} + u_{xxxx} + 6u_x u_t + 3u_{xx} v_t + \beta u_{xxt} + u_{ytt} + a u_{xxt} = 0, \]

\[ [v_x = -u, \]

(3.12)

where the constant \( \alpha \neq 0 \), but the constant \( \beta \) is arbitrary. Under the typical transformation \( u = 2(\ln \tau)_{xx} \), this equation is mapped into

\[ (D_t^2 + D_x D_x^3 + \beta D_x D_x + D_t D_y + a D_x^2) \tau \cdot \tau = 0. \]

(3.13)

Following expression (2.18), a set of sufficient conditions, which makes the Pfaffian determinant a solution to the above bilinear equation (3.13), can be expressed as follows:

\[ \frac{\partial f_i}{\partial y} = \frac{\alpha}{3} \int_{-\infty}^{\infty} f_i dx - \beta \frac{\partial f_i}{\partial x} - \frac{\partial^3 f_i}{\partial x^3}, \quad \frac{\partial f_i}{\partial t} = -\frac{\alpha}{3} \int_{-\infty}^{\infty} f_i dx. \]

(3.14)

Also, the (2+1)-dimensional equation \( g_{\text{HSI}} \) (3.12) has the following N-wave solution:

\[ u = 2(\ln \tau)_{xx}, \tau = \sum_{i=1}^{N} \epsilon_i e^{k_i x + (\frac{\alpha}{\beta} k_i^{-1} - \beta k_i^{-1} k_i^3) y - \frac{\alpha}{\beta} k_i^{-1} t}, \]

(3.15)

where the \( \epsilon_i \)'s and \( k_i \)'s are arbitrary constants.

**Example 2.** The (2+1)-dimensional fourth-order nonlinear equation

The second example is a special fourth-order nonlinear equation in (2+1)-dimensions as follows:

\[ \alpha [3(u_x u_t)_x + u_{xxxx}] + \beta [3(u_x u_t)_x + u_{xxxx}] + \frac{\alpha}{\beta} u_{ytt} + u_{xx} + u_{yy} = 0, \]

(3.16)
where the constants $\alpha$ and $\beta$ satisfy $\alpha \beta \neq 0$. Via the dependent variable transformation $u = 2(\ln \tau)_{x}$, this equation is written as

$$
\left( aD_{x}^{5}D_{t} + \beta D_{x}^{3}D_{y} + \frac{\alpha}{\beta} D_{y}D_{x} + D_{x}^{2} + D_{y}^{2} \right) \tau \cdot \tau = 0.
$$

(3.17)

Based on the sufficient conditions (2.20) in Proposition 2.1, the corresponding Pfaffian form solution reads as

$$
\frac{\partial f_{i}}{\partial y} = -\beta \frac{\partial^{3} f_{i}}{\partial x^{3}}, \quad \frac{\partial f_{i}}{\partial t} = -\frac{1}{3a} \int_{-\infty}^{x} f_{i} dx + \frac{\beta^{2}}{\alpha} \frac{\partial^{3} f_{i}}{\partial x^{3}}.
$$

(3.18)

In the meantime, we can obtain the resonant multiple wave solutions $u$ to Equation (3.16) as

$$
u = 2(\ln \tau)_{x}, \quad \tau = \sum_{i=1}^{N} \epsilon_{i} \left( e^{k_{x_{i}}x - \beta k_{y_{i}}y} \right) \left( \frac{\epsilon^{k^{-1}}}{x} \right)^{k_{i}} t,
$$

(3.19)

where the $\epsilon_{i}$’s and $k_{i}$’s are arbitrary constants.

**Example 3.** The $(2+1)$-dimensional fifth-order KdV equation

As the third example, we consider the following $(2+1)$-dimensional fifth-order KdV equation

$$
36u_{t} = -u_{xxxxx} - 15(uu_{xx})_{x} - 45u^{2}u_{x} + 5u_{xx} + 15 uu_{x} + 15u_{x} \int u_{y} dx + 5 \int u_{xy} dx,
$$

(3.20)

which was first proposed by Konopelchenko and Dubovsky. This equation has a bilinear form under the logarithmic transformation $u = 2(\ln \tau)_{x}$:

$$(36D_{y}D_{t} + D_{x}^{6} - 5D_{x}^{3}D_{y} - 5D_{y}^{2}) \tau \cdot \tau = 0,
$$

(3.21)

for which there exists a set of sufficient conditions on the Pfaffian form solution defined by (1.6) with

$$
\frac{\partial f_{i}}{\partial y} = \frac{\partial^{3} f_{i}}{\partial x^{3}}, \quad \frac{\partial f_{i}}{\partial t} = \frac{1}{4} \frac{\partial^{3} f_{i}}{\partial x^{3}}.
$$

(3.22)

In addition, the following $N$-wave resonant solution can be determined

$$
u = 2(\ln \tau)_{x}, \quad \tau = \sum_{i=1}^{N} \epsilon_{i} \left( e^{k_{x}x + k_{y}y} + \frac{1}{x} k_{i} t \right),
$$

(3.23)

where the $\epsilon_{i}$’s and $k_{i}$’s are arbitrary constants.

## 4 CONCLUDING REMARKS

In summary, by means of a universal property of Hirota differential operators and the fundamental Pfaffian identity, we built Hirota bilinear equations in $(2+1)$-dimensions that possess the Pfaffian form solutions. The resulting sufficient conditions on Pfaffian formulas show us the $N$-wave resonant solutions of linear combinations of exponential waves that the considered Hirota bilinear equations carry. Applications were made for the $(2+1)$-dimensional generalized HSI equation, the special fourth-order $(2+1)$-dimensional nonlinear equation, and the fifth-order KdV equation, thereby presenting their particular Pfaffian and multiple wave solutions.

We also remark that $N$-wave solutions formed by linear combinations of hyperbolic and trigonometric functions can be obtained for the considered Hirota bilinear equations, since the solutions of the BKP hierarchy are expressed in terms...
of Pfaffians with odd weight variables. As an example, the (2+1)-dimensional fifth-order KdV equation has the following mixed-type function solutions like complexiton solutions:\textsuperscript{16,23}

\[ u = 2\ln(\tau), \quad \tau = \sum_{i=1}^{N} \left[ \varepsilon_i \cosh \left( k_i x + k_i^3 y + \frac{1}{4} k_i^5 t \right) + \lambda_i \cos \left( k_i x - k_i^3 y + \frac{1}{4} k_i^5 t \right) \right], \]

and

\[ \tau = \sum_{i=1}^{N} \left[ \varepsilon_i \sinh \left( k_i x + k_i^3 y + \frac{1}{4} k_i^5 t \right) + \lambda_i \sin \left( k_i x - k_i^3 y + \frac{1}{4} k_i^5 t \right) \right], \]

where the \( \varepsilon_i, \lambda_i, \) and \( k_i \) are arbitrary constants.

Additionally, we remark that we only considered some specific Hirota bilinear equations possessing Pfaffian and \( N \)-wave resonant solutions. There should exist more interesting problems which need to be discussed. We expect to investigate integrable properties\textsuperscript{35,36} and high-order lump solutions\textsuperscript{37–44} to the presented Hirota bilinear equations in our future works.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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REFERENCES

35. Ma WX, Yong XL, Lü X. Soliton solutions to the B-type Kadomtsev-Petviashvili equation under general dispersion relations. *Wave Motion*. 2021;103:102719.

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