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RESEARCH

Lump Structures and Their Dynamics in a Generalized Calogero–Bogoyavlenskii–Schiff-Like Wave Model

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Abstract

This work investigates dispersion-driven lump wave structures within a generalized (2+1)-dimensional Calogero–Bogoyavlenskii–Schiff-like framework. By employing a generalized bilinear form of the governing equation, we construct positive quadratic function solutions via symbolic computation, which in turn generate lump wave structures. The analysis shows that the stationary points of the quadratic function align along a straight trajectory in the spatial plane and propagate with constant velocity, where the lump amplitude vanishes. The emergence of these lump waves results from the interplay of eight nonlinear terms and four dispersion terms in the model.

Keywords Generalized bilinear form · Lump wave · Symbolic computation · Nonlinearity · Dispersion

1 Introduction

Closed-form exact solutions play a central role in mathematical physics and engineering, as they not only yield fundamental insights but also provide systematic frameworks for tackling complex nonlinear phenomena. However, deriving such solutions remains a highly nontrivial task, motivating extensive efforts to either obtain explicit expressions or to characterize the precise conditions under which they exist.

Within soliton theory and the study of integrable models, a wide variety of wave structures, including solitons, rogue waves, and lump waves, have been constructed through symbolic computation and analytical techniques. These localized and dispersive waveforms emerge

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from the delicate balance between nonlinearity and dispersion, and their explicit construction continues to be a central theme in exploring wave dynamics across physical and engineering applications.

In applied sciences, dipole and quadrupole solitons in photonic Moiré lattices have been analyzed based on a perturbed nonlinear Schrödinger equation [1]. The integration of deep-learning prediction with ultrafast optical setups holds great potential for advancing the application of ultrafast optical lasers in optical communication and information storage [2]. Studies have shown that incorporating common dye molecules absorbing in the near-ultraviolet and blue spectral regions enhances optical transparency at adjacent longer wavelengths [3]. Furthermore, a passive mode-locked fiber laser has been reported, within the framework of a nonlinear Schrödinger equation model, to achieve simultaneous single-wavelength tuning and multi-wavelength spacing tuning based on the split-step Fourier method [4].

Two fundamental techniques in soliton theory and the study of integrable systems are the Hirota bilinear method [5] and the inverse scattering transform (IST) [6]. The Hirota method offers a direct and systematic framework for constructing exact solutions, such as solitons and lump waves, particularly in nonlinear dispersive equations of higher dimensions, including $(2+1)$- and $(3+1)$-dimensional models [7–11]. By contrast, the IST serves as a nonlinear analogue of the Fourier transform, specifically adapted to integrable equations. It provides a powerful tool for addressing initial-value problems through associated Lax pairs [12], as well as for analyzing the long-time asymptotic behavior of dispersive waves, even in regimes without solitons [13].

Let x, y represent spatial variables and t denote time. For a given polynomial $P(x, y, t)$, a Hirota bilinear differential equation in (2+1)-dimensions can be formulated as

$$P(D_x, D_y, D_t) f \cdot f = 0, \tag{1}$$

where D_x, D_y and D_t are Hirota's bilinear operators [5], defined by

$$D_x^m D_y^n D_t^k f \cdot f = \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)^m \Big(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\Big)^n \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\Big)^k f(x, y, t) f(x', y', t')\Big|_{x'=x, y'=y, t'=t},$$

with $m, n, k \geq 0$. Through Bell polynomial theory, nonlinear PDEs for a scalar field u can often be derived from bilinear forms via logarithmic derivative transformations (see, e.g., [14]):

$$u = \beta(\ln f)_{xx},\ \beta(\ln f)_{yy},\ \beta(\ln f)_{xy},\ \beta(\ln f)_x,\ \beta(\ln f)_y, \tag{2}$$

where $\beta \neq 0$. Hirota's method enables the construction of N-soliton solutions in the exponential superposition form (see, e.g., [7, 15]):

$$f = \sum_{\lambda=0,1} \exp\Big(\sum_{i=1}^{N} \lambda_i \eta_i + \sum_{i<j} \lambda_i \lambda_j c_{ij}\Big), \tag{3}$$

where the sum runs over all binary combinations $\lambda_1, \lambda_2, \cdots, \lambda_N \in \{0, 1\}$. The phase shifts c_{ij} and wave phases η_i are specified by

$$\exp(c_{ij}) = -\frac{P(\omega_j - \omega_i, k_i - k_j, l_i - l_j)}{P(\omega_j + \omega_i, k_i + k_j, l_i + l_j)},\ \text{where } 1 \leq i < j \leq N, \tag{4}$$

and

$$\eta_i = k_i x + l_i y - \omega_i t + \eta_{i,0},\ \text{where } 1 \leq i \leq N. \tag{5}$$

The key constraint ensuring solvability is the dispersion relation:

$$P(-\omega_i, k_i, l_i) = 0, \text{ where } 1 \leq i \leq N. \tag{6}$$

Thus, the central problem reduces to verifying that a function of the form (3) indeed solves (1) under the conditions (6). A systematic algorithm for this verification, with explicit examples in (1+1)- and (2+1)-dimensions, is detailed in [15, 16].

Beyond solitons, other explicit coherent structures arise in bilinear formulations. Lump waves and rogue waves, for instance, are rationally localized in space while vanishing at infinity in all directions [17, 18]. The Kadomtsev-Petviashvili I (KPI) equation is well known to support diverse families of lump solutions [8], some of which emerge as long-wave limits of multi-soliton configurations [19]. These rationally localized structures are not limited to integrable models: lump-type solutions also appear in nonintegrable KP, BKP, KP-Boussinesq and generalized KdV-type extensions [20, 21], generalized Bogoyavlensky-Konopelchenko equations [22, 23], and even in higher-dimensional linear wave systems via linear superposition [24, 25].

A widely used approach for deriving lump solutions is the sum-of-squares ansatz, where a positive quadratic function is substituted into the bilinear equation [8, 17]. Through logarithmic derivative transformations, such quadratic forms give rise to lump solutions in diverse nonlinear wave models. In this study, we apply this method to a (2+1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff-like (gCBS-like) equation that includes eight nonlinear terms and four distinct dispersive contributions. These nonlinear and dispersive effects jointly serve as the balancing mechanisms sustaining the lump structures. By employing symbolic computation in a computer algebra system, we obtain explicit lump solutions and investigate the stationary points of the underlying quadratic function, which shed light on the associated wave dynamics. We conclude with a discussion of the results and suggest possible directions for further research.

2 A Generalized CBS-like Model

It is well known that Hirota bilinear derivatives can be extended to describe differential terms of all orders. Our analysis begins with a generalized class of bilinear differential operators, as put forward in [26]:

$$\begin{aligned} & D_{p,x}^m D_{p,y}^n D_{p,t}^k f \cdot f \\ &= \left(\frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} + \alpha_p \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} + \alpha_p \frac{\partial}{\partial t'}\right)^k f(x, y, t) f(x', y', t')\Big|_{x'=x, y'=y, t'=t}, \end{aligned} \tag{7}$$

where

$$\alpha_p^k = (-1)^{r(k)}, \; k \equiv r(k) \bmod p, \; 0 \leq r(k) < p. \tag{8}$$

For instance, when $p = 3$, the cyclic pattern is

$$\alpha_3 = -1, \; \alpha_3^2 = \alpha_3^3 = 1, \; \alpha_3^4 = -1, \; \alpha_3^5 = \alpha_3^6 = 1, \; \cdots . \tag{9}$$

For $p = 5$, one obtains

$$\alpha_5 = -1, \; \alpha_5^2 = 1, \; \alpha_5^3 = -1, \; \alpha_5^4 = \alpha_5^5 = 1, \; \alpha_5^6 = -1, \; \alpha_5^7 = 1, \; \alpha_5^8 = -1, \; \alpha_5^9 = \alpha_5^{10} = 1, \; \cdots . \tag{10}$$

Similarly, for $p = 7$, the sequence reads

$$\begin{cases} \alpha_7 = -1,\ \alpha_7^2 = 1,\ \alpha_7^3 = -1,\ \alpha_7^4 = 1,\ \alpha_7^5 = -1,\ \alpha_7^6 = \alpha_7^7 = 1, \\ \alpha_7^8 = -1,\ \alpha_7^9 = 1,\ \alpha_7^{10} = -1,\ \alpha_7^{11} = 1,\ \alpha_7^{12} = -1,\ \alpha_7^{13} = \alpha_7^{14} = 1,\ \cdots. \end{cases} \tag{11}$$

Setting $p = 3$, we propose a generalized Calogero-Bogoyavlenskii-Schiff-like (gCBS-like) bilinear equation:

$$\begin{aligned} &F_{\text{gCBS-like}}(f) \\ &:= \left(D_{3,x}^3 D_{3,y} + \sigma_1 D_{3,t} D_{3,x} + \sigma_2 D_{3,t} D_{3,y} + \sigma_3 D_{3,x} D_{3,y} + \sigma_4 D_{3,y}^2\right) f \cdot f \\ &= 2[3 f_{xx} f_{xy} + \sigma_1 (f_{tx} f - f_t f_x) + \sigma_2 (f_{ty} f - f_t f_y) + \sigma_3 (f_{xy} f - f_x f_y) \\ &\quad + \sigma_4 (f_{yy} f - f_y^2)] = 0, \end{aligned} \tag{12}$$

where $D_{3,x}$, $D_{3,y}$ and $D_{3,t}$ denote the generalized bilinear derivatives, and σ_i for $1 \le i \le 4$ are arbitrary constants. This formulation generalizes the CBS equation from the case $p = 2$ (see, e.g., [27]). By redefining the dependent variable as

$$u = 2(\ln f)_x,\ v = 2(\ln f)_y, \tag{13}$$

we obtain the associated nonlinear gCBS-like model equation:

$$\begin{aligned} P_{\text{gCBS-like}}(u, v) := &\frac{9}{8} u^2 u_x v + \frac{3}{8} u^3 u_y + \frac{3}{4} u u_{xx} v + \frac{3}{4} u_x^2 v + \frac{3}{4} u^2 u_{xy} + \frac{9}{4} u u_x u_y \\ &+ \frac{3}{2} u_{xx} u_y + \frac{3}{2} u_x u_{xy} + \sigma_1 u_{tx} + \sigma_2 u_{ty} + \sigma_3 u_{xy} + \sigma_4 u_{yy} = 0, \end{aligned} \tag{14}$$

subject to $v_x = u_y$. This new model contains eight nonlinear terms and four dispersion terms.

Special cases illustrate its structure. For $\sigma_1 = \sigma_4 = 1$ with all other coefficients vanishing, the generalized equation (14) reduces to a CBS-like form:

$$\frac{9}{8} u^2 u_x v + \frac{3}{8} u^3 u_y + \frac{3}{4} u u_{xx} v + \frac{3}{4} u_x^2 v + \frac{3}{4} u^2 u_{xy} + \frac{9}{4} u u_x u_y + \frac{3}{2} u_{xx} u_y + \frac{3}{2} u_x u_{xy} + u_{tx} + u_{yy} = 0.$$

For $\sigma_2 = \sigma_4 = 1$ with all other coefficients vanishing, we obtain another CBS-like equation:

$$\frac{9}{8} u^2 u_x v + \frac{3}{8} u^3 u_y + \frac{3}{4} u u_{xx} v + \frac{3}{4} u_x^2 v + \frac{3}{4} u^2 u_{xy} + \frac{9}{4} u u_x u_y + \frac{3}{2} u_{xx} u_y + \frac{3}{2} u_x u_{xy} + u_{ty} + u_{yy} = 0,$$

again with $v_x = u_y$.

Finally, the connection between the bilinear and nonlinear forms is exact:

$$P_{\text{gCBS-like}}(u, v) = \left[\frac{F_{\text{gCBS-like}}(f)}{f^2}\right]_x. \tag{15}$$

Hence, u and v defined by (13) solve the nonlinear model (14) whenever f satisfies the bilinear equation (12).

This new model raises natural questions concerning its integrability properties, particularly whether it admits lump solutions, a hallmark of integrable systems. In the next section, we investigate this issue with emphasis on lump structures shaped by the dispersion terms.

3 Lump Waves Governed by Dispersion

We now proceed to construct lump wave solutions of the gCBS-like model equation (14) by performing symbolic computations on its associated generalized bilinear form (12). In particular, we demonstrate that the inclusion of all four dispersion terms is essential for the emergence of lump-type solutions. Furthermore, we analyze the stationary points of the resulting quadratic function to gain insight into their spatial characteristics.

3.1 Construction of Lump Solutions Via the Sum-of-Squares Ansatz

The sum-of-squares ansatz provides a systematic approach for constructing lump solutions of higher-dimensional nonlinear evolution equations [8]. The method begins by expressing the dependent variable as a logarithmic derivative of a positive quadratic function. Typically, this quadratic form takes the sum-of-squares structure:

$$f = \xi_1^2 + \xi_2^2 + a_9,\ \xi_1 = a_1x + a_2y + a_3t + a_4,\ \xi_2 = a_5x + a_6y + a_7t + a_8, \tag{16}$$

ensuring the rational solution in all spatial directions. The parameters a_1, a_2 and a_5, a_6 represent two pairs of wave numbers, a_3 and a_7 denote the frequencies of the two underlying travelling waves, and a_4, a_8, and a_9 correspond to translation-invariant properties. Substituting this ansatz into the generalized bilinear equation (12) reduces the problem to determining the nine parameters a_i through an algebraic system. This framework provides a foundation for generating general lump wave structures of lower order in (2+1)-dimensional settings [17], with symbolic computation used to determine the coefficients.

Solving the resulting system symbolically yields explicit expressions for the frequency and constant parameters:

$$\begin{aligned} a_3 = &-\frac{1}{(a_1\sigma_1 + a_2\sigma_2)^2 + (a_5\sigma_1 + a_6\sigma_2)^2}\big[a_2(a_1^2 + a_5^2)\sigma_1\sigma_3 \\ &+(a_1a_2^2 - a_1a_6^2 + 2a_2a_5a_6)\sigma_1\sigma_4 \\ &+a_1(a_2^2 + a_6^2)\sigma_2\sigma_3 + a_2(a_2^2 + a_6^2)\sigma_2\sigma_4\big], \end{aligned} \tag{17}$$

$$\begin{aligned} a_7 = &-\frac{1}{(a_1\sigma_1 + a_2\sigma_2)^2 + (a_5\sigma_1 + a_6\sigma_2)^2}\big[a_6(a_1^2 + a_5^2)\sigma_1\sigma_3 \\ &+(2a_1a_2a_6 - a_2^2a_5 + a_5a_6^2)\sigma_1\sigma_4 \\ &+a_5(a_2^2 + a_6^2)\sigma_2\sigma_3 + a_6(a_2^2 + a_6^2)\sigma_2\sigma_4\big], \end{aligned} \tag{18}$$

and

$$\begin{aligned} a_9 = &-\frac{3(a_1a_2 + a_5a_6)(a_1^2 + a_5^2)^2\sigma_1 + 6(a_1^2 + a_5^2)(a_1a_2 + a_5a_6)^2\sigma_2}{(a_1a_6 - a_2a_5)^2(\sigma_1\sigma_4 - \sigma_2\sigma_3)} \\ &-\frac{3(a_1a_2 + a_5a_6)(a_1^2 + a_5^2)(a_2^2 + a_6^2)\sigma_2^2}{(a_1a_6 - a_2a_5)^2\sigma_1(\sigma_1\sigma_4 - \sigma_2\sigma_3)}. \end{aligned} \tag{19}$$

All remaining parameters can be chosen freely. Here, a_3 and a_7 encode the dispersion relations of the (2+1)-dimensional nonlinear dispersive system, indicating that variations in dispersion, whether positive or negative, can cause solitons to propagate with different group velocities (see also [28]). Meanwhile, a_9 depends on the wave numbers and plays a crucial role in shaping the lump structure (see also [29–31]). Higher-order dispersion relations have been previously studied in the KP hierarchy [32], and related dynamical behaviors have been explored in various generalized KP-type models (see, e.g., [33, 34]). All expressions, (17), (18) and (19), are simplified using symbolic computation.

For well-defined solutions, the following conditions on dispersion coefficients and wave numbers must hold:

$$\sigma_1(\sigma_1\sigma_4 - \sigma_2\sigma_3) \neq 0, \tag{20}$$

and

$$a_1a_6 - a_2a_5 \neq 0. \tag{21}$$

These conditions imply

$$(a_1\sigma_1 + a_2\sigma_2)^2 + (a_5\sigma_1 + a_6\sigma_2)^2 \neq 0, \tag{22}$$

ensuring the well-posedness of a_3 and a_7. Furthermore, the determinant condition (21) guarantees

$$a_1^2 + a_5^2 \neq 0,\ a_2^2 + a_6^2 \neq 0, \tag{23}$$

which in turn ensures the spatial localization of the solutions u and v defined via the logarithmic derivative transformations (13). That is, u and v decay to zero as $x^2 + y^2 \to \infty$, confirming their localization.

A sufficient condition for the positivity of f is:

$$\frac{\sigma_1}{\sigma_1\sigma_4 - \sigma_2\sigma_3} < 0,\ \frac{\sigma_2}{\sigma_1\sigma_4 - \sigma_2\sigma_3} < 0,\ a_1a_2 + a_5a_6 > 0, \tag{24}$$

or

$$\frac{\sigma_1}{\sigma_1\sigma_4 - \sigma_2\sigma_3} > 0,\ \frac{\sigma_2}{\sigma_1\sigma_4 - \sigma_2\sigma_3} < 0,\ a_1a_2 + a_5a_6 < 0, \tag{25}$$

which ensures that $a_9 > 0$ according to (19). Consequently, f defined in (16) is positive and the corresponding solutions u and v are analytic throughout the domain of x, y and t.

In summary, under the two conditions (21) and (20), the resulting solutions u and v constitute a well-defined, rationally localized lump wave solution of the gCBS-like model.

3.2 Evolution of Stationary Points

We now determine the stationary points of the quadratic function f defined in (16). These points satisfy the system

$$f_x(x(t), y(t), t) = 0,\ f_y(x(t), y(t), t) = 0.$$

Since f is quadratic in x and y, this reduces to the linear system:

$$a_1\xi_1 + a_5\xi_2 = 0,\ a_2\xi_1 + a_6\xi_2 = 0,$$

where ξ_1 and ξ_2 are defined as in (16). Assuming the non-degeneracy condition (21), we obtain

$$\xi_1 = a_1x + a_2y + a_3t + a_4 = 0,\ \xi_2 = a_5x + a_6y + a_7t + a_8 = 0, \tag{26}$$

where, by the second partial derivative test, the quadratic function f attains a local minimum at any fixed time t. Solving this system for x and y as functions of t gives the trajectory of stationary points:

$$x(t) = \frac{(a_2^2 + a_6^2)(\sigma_1\sigma_4 - \sigma_2\sigma_3)}{(a_1\sigma_1 + a_2\sigma_2)^2 + (a_5\sigma_1 + a_6\sigma_2)^2}t + \frac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \tag{27}$$

$$y(t) = \frac{(a_1^2 + a_5^2)\sigma_1\sigma_3 + 2(a_1a_2 + a_5a_6)\sigma_1\sigma_4 + (a_2^2 + a_6^2)\sigma_2\sigma_4}{(a_1\sigma_1 + a_2\sigma_2)^2 + (a_5\sigma_1 + a_6\sigma_2)^2}t - \frac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}. \tag{28}$$

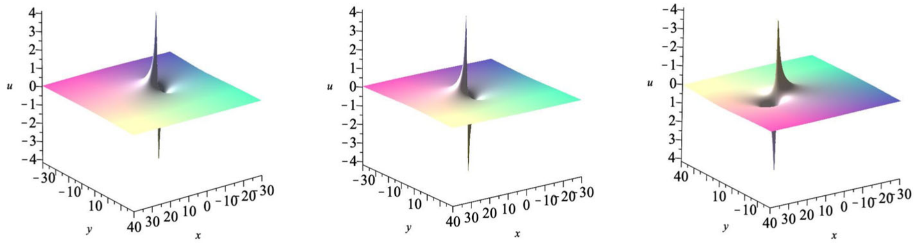

Fig. 1 3d-plots of u with $t = 0$ (left), $t = 3$ (middle) and $t = 5$ (right)

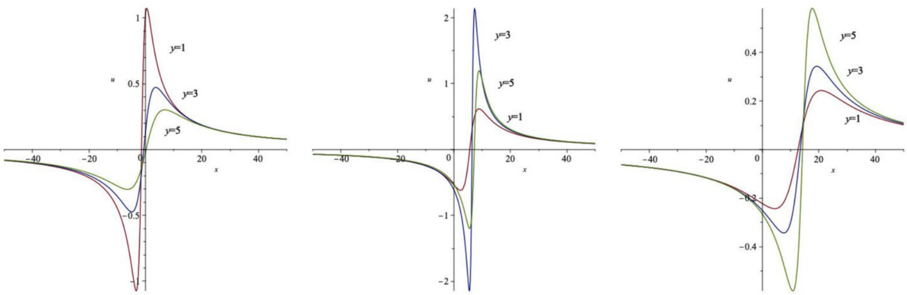

Fig. 2 x curves of u with $t = 0$ (left), $t = 3$ (middle) and $t = 6$ (right)

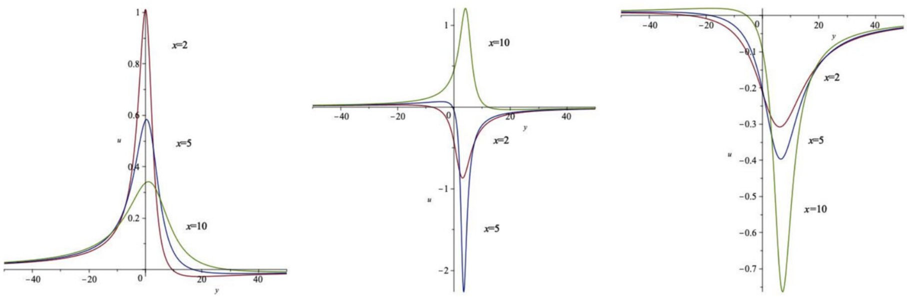

Fig. 3 y curves of u with $t = 0$ (left), $t = 3$ (middle) and $t = 6$ (right)

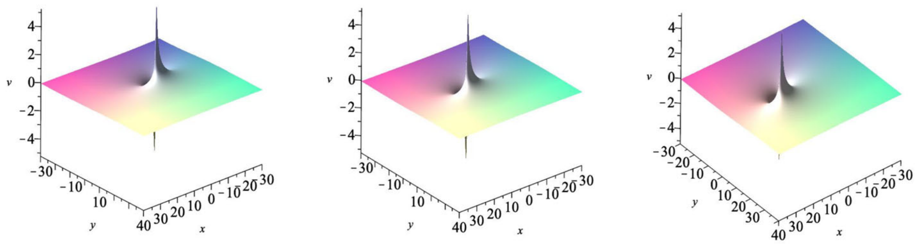

Fig. 4 3d-plots of v with $t = 0$ (left), $t = 3$ (middle) and $t = 5$ (right)

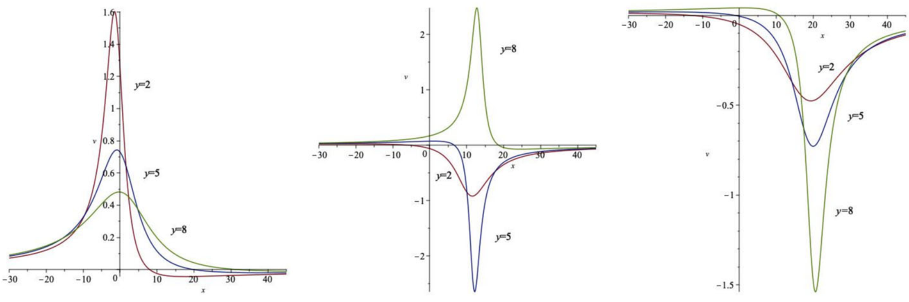

Fig. 5 x curves of v with $t = 0$ (left), $t = 5$ (middle) and $t = 8$ (right)

These expressions describe the motion of the stationary points over time. They lie along a straight line, which we refer to as the characteristic trajectory, along which both spatial coordinates advance at constant velocities. On this trajectory, the lump waves u and v vanish, while remaining rationally localized in the surrounding space. These expressions describe the trajectory of stationary points at any fixed time t.

Figures 1, 2, 3, 4, 5 and 6 present 3d and 2d visualizations of the lump waves $u = 2(\ln f)_x$ and $v = 2(\ln f)_y$, calculated using the parameter sets given below:

$$\sigma_1 = -2,\ \sigma_2 = 1,\ \sigma_3 = 3,\ \sigma_4 = 5,$$

and

$$a_1 = -1,\ a_2 = -2,\ a_4 = -3,\ a_5 = -2,\ a_6 = 2,\ a_8 = -3.$$

4 Concluding Remarks

We have analyzed a (2+1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff-like (gCBS-like) model and derived its lump wave solutions through symbolic computation using computer algebra systems. The resulting lump wave is localized and vanishes along a characteristic trajectory determined by the stationary points of the associated quadratic function.

Lump waves arise in a broad spectrum of physical and mathematical contexts, reflecting their versatility and the inherent complexity of modeling nonlinear dispersive phenomena. Previous studies have explored lump solutions in linear wave models [24, 25] as well as in nonlinear, nonintegrable models in both (2+1)-dimensions [35–41] and (3+1)-dimensions

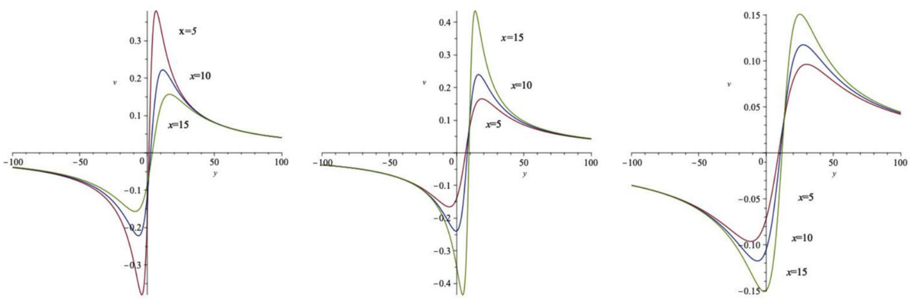

Fig. 6 y curves of v with $t = 0$ (left), $t = 8$ (middle) and $t = 12$ (right)

[33, 42–44]. The construction of lump waves frequently employs Hirota bilinear forms and their generalizations, which provide efficient tools for systematically analyzing such localized structures [17].

Moreover, lump waves display rich interactions with other coherent structures in (2+1)-dimensional integrable models, including homoclinic and heteroclinic waves [45–47]. Meanwhile, N-soliton solutions and integrability properties have been extensively investigated in both local and nonlocal integrable systems using Riemann-Hilbert methods and bi-Hamiltonian structures [48–54]. The existence, dynamics, and interactions of lump waves in (2+1)-dimensional generalizations of integrable systems, whether standard or generalized bilinear, scalar or multi-component, remain open and compelling directions for further research (see, e.g., [55–62]).

In summary, the study of lump waves offers valuable insight into nonlinear dispersive dynamics and may inform applications in physical and engineering systems where localized, coherent, and energy-concentrated structures play a critical role.

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Author Contributions L. made the computations and reviewed the manuscript. WX wrote the main manuscript text, made the computations, and reviewed the manuscript

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Ethical Approval Not applicable.

Competing interests The authors declare no competing interests.

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