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Interaction solutions of the first BKP equation

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In this paper, we study the interaction solutions of the first BKP equation by using the Hirota direct method and taking long-wave limit. In order to obtain the interaction solutions, the multi-soliton solutions are firstly derived using the Hirota direct method, then the interaction solutions are successfully constructed by properly choosing appropriate parameters and taking long-wave limit on the soliton solutions. These parameters have great influences on the propagation directions, shapes as well as energy. Moreover, the dynamic properties of these obtained solutions are illustrated vividly by some graphs. The results in this work could be used to solve nonlinear problems in nonlinear optics and engineering field.

Keywords: BKP equation; Hirota direct method; multi-soliton solutions; breather; lump wave; interaction solution.

1. Introduction

In the past decades, the study of exact solutions and integrability of nonlinear partial differential equations has been a hot topic in the field of integrable systems and soliton theory.[1–21] The integrable nonlinear evolution equations possess

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many important properties, for example, the existence of solitons, the Hamiltonian structures, an infinite number of conservation laws, the Bäcklund transformations and the Lax pairs.[22] After Gardner *et al.*[23] established the inverse scattering transform method, many effective methods for finding exact solutions have been proposed, among which are the Bäcklund transformation,[24–26] the Darboux transformation,[27,28] the Hirota bilinear method,[29,30] the variable separation approach,[31] similarity transformation,[32,33] the dressing method[34] and the inverse scattering approach.[35] Particularly, the Hirota direct method has been widely used to study nonlinear evolution equations due to its simplicity and directness. Recently, the multi-soliton and interaction solutions of nonlinear evolution equations have attracted a lot of attentions. The aim of this paper is to study the interaction solutions of the first BKP equation, which reads

$$u_{yt} - u_{xxxy} - 3(u_x u_y)_x - 3u_{xx} = 0\,. \tag{1}$$

The first BKP equation (1) has been studied by a lot of researchers.[36–39] Mixed lump-kink solutions, periodic solitary-wave solutions, Wronskian and linear superposition solutions and other solutions. In Ref. 31, the mixed lump solutions are investigated. In Ref. 32, some new exact periodic solitary-wave solutions for the $(3+1)$-dimensional generalized BKP equation were derived. In Ref. 33, the Wronskian and linear superposition solutions of the BKP equations were given. In Ref. 34, the BKP equations were studied by the multiple exp-function algorithm and some new solutions were constructed. Yue *et al.*[40] constructed some localized exact solutions of the $(3+1)$-dimensional BKP-type equation. In the present paper, we will propose some new localized solutions and the corresponding interaction solutions of the first BKP equation (1).

This paper is organized as follows. In Sec. 2, multi-soliton solutions of the first BKP equation are constructed using the Hirota bilinear method. Then, choosing appropriate parameters and taking long-wave limit, some interesting and useful interaction solutions are derived in Sec. 3. Finally, some discussions are given in Sec. 4.

2. Multi-Soliton Solutions

In fact, the multi-soliton solutions of Eq. (1) can be obtained by using the Hirota direct method. Based on Painleve analysis, Eq. (1) enjoys the following Hirota bilinear form as

$$(D_y D_t - D_x^3 D_y - 3D_x^2)\tau \cdot \tau = 0\,, \tag{2}$$

through the logarithmic transformation

$$u = 2(\ln \tau)_x, \tag{3}$$

where the Hirota bilinear derivatives D_x, D_y and D_t are defined by

$$D_x^m D_y^n D_t^k a \cdot b = (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n (\partial_t - \partial_{t'})^k a(x,y,t) b(x', y't')|_{x=x', y=y', t=t'}\,.$$

(4)

According to the standard perturbation method, the solution τ can be given in the following form:

$$\tau = 1 + \sum_{i=1}^{N} e^{\eta_i} + \sum_{i<j}^{N} A_{ij} e^{\eta_i+\eta_j} + \sum_{i<j<k}^{N} A_{ij}A_{ik}A_{jk} e^{\eta_i+\eta_j+\eta_k}$$
$$+ \cdots + \left(\prod_{i<j} A_{ij}\right) e^{\sum_{i=1}^{N}\eta_i}\,, \tag{5}$$

with $\eta_j = a_j x + b_j y + ({a_j}^3 + 3b_j^{-1}{a_j}^2)t + \eta_j^{(0)}$, $(j = 1, 2, \ldots, N)$ and

$$A_{ij} = \frac{a_i a_j b_i b_j (b_i - b_j)(a_i - a_j) - (a_i b_j - a_j b_i)^2}{a_i a_j b_i b_j (b_j + b_i)(a_j + a_i) - (a_i b_j - a_j b_i)^2}, \quad 1 \le i < j \le N\,, \tag{6}$$

where $a_j, b_j, \eta_j^{(0)}(j = 1, 2, \ldots, N)$ are arbitrary constants.

For example, after the direct computations in Maple, we obtained the first three solutions of:

When $N = 1$, we get

$$\tau = 1 + e^{\eta_1}\,, \tag{7}$$

where $\eta_1 = a_1 x + b_1 y + ({a_1}^3 + 3b_1^{-1}{a_1}^2)t + \eta_1^{(0)}$ and $a_1, b_1, \eta_1^{(0)}$ are arbitrary constants.

When $N = 2$, the function τ is

$$\tau = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2}\,, \tag{8}$$

where $\eta_j = a_j x + b_j y + ({a_j}^3 + 3b_j^{-1}{a_j}^2)t + \eta_j^{(0)}, (j = 1, 2)$

$$A_{12} = \frac{a_1 a_2 b_1 b_2 (b_1 - b_2)(a_1 - a_2) - (a_1 b_2 - a_2 b_1)^2}{a_1 a_2 b_1 b_2 (b_2 + b_1)(a_2 + a_1) - (a_1 b_2 - a_2 b_1)^2}\,,$$

and a_j, b_j and $\eta_j^{(0)}(j = 1, 2)$ are arbitrary constants.

When $N = 3$, we obtained τ as follows:

$$\tau = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12} e^{\eta_1+\eta_2} + A_{13} e^{\eta_1+\eta_3}$$
$$+ A_{23} e^{\eta_2+\eta_3} + A_{12}A_{13}A_{23} e^{\eta_1+\eta_2+\eta_3}\,, \tag{9}$$

where $\eta_j = a_j x + b_j y + ({a_j}^3 + 3b_j^{-1}{a_j}^2)t + \eta_j^{(0)}, (j = 1, 2, 3)$,

$$A_{ij} = \frac{a_i a_j b_i b_j (b_i - b_j)(a_i - a_j) - (a_i b_j - a_j b_i)^2}{a_i a_j b_i b_j (b_j + b_i)(a_j + a_i) - (a_i b_j - a_j b_i)^2}\,,$$

and a_j, b_j and $\eta_j^{(0)}(j = 1, 2, 3)$ are arbitrary constants. Then the multi-soliton solution of Eq. (1) can be got by plugging Eq. (4) with Eq. (5) into the logarithmic transformation $u = 2(\ln \tau)_x$.

3. Interaction Solutions

Based on the methods obtained in the previous section, we will derive the analytical expressions for the breather solutions, the lump solutions and the interaction solutions between the breathers and other solutions of the first BKP equation (1) by selecting appropriate parameters for the corresponding soliton solution (3). Unless otherwise specified, we take $\eta_j^{(0)} = 0$ $(j = 1, 2, 3, \ldots)$ through the whole paper.

3.1. *Breather and lump solutions*

In order to get the analytical expressions for the breather and lump solutions, we select the following appropriate parameters for the two-soliton solution in Eq. (3)

Case 1. When setting $a_1 = a_2^* = \alpha_1 + i\beta_1, b_1 = b_2 = \delta_1$, the τ function takes the following form:

$$\tau = 1 + 2\exp(\zeta_1)\cos(\beta_1 x + \omega_1 t) + A_{12}\exp(2\zeta_1)\,, \tag{10}$$

with

$$A_{12} = \frac{{\beta_1}^2}{{\beta_1}^2 + {\alpha_1}^3\delta_1 + \alpha_1{\beta_1}^2\delta_1}\,, \tag{11}$$

$$\zeta_1 = \alpha_1 x + \delta_1 y + \left[\frac{3}{\delta_1}({\alpha_1}^2 - {\beta_1}^2) + \alpha_1\left({\alpha_1}^2 - 3\,{\beta_1}^2\right)\right] t, \tag{12}$$

$$\omega_1 = \frac{\beta_1}{\delta_1}\left(3\,{\alpha_1}^2\delta_1 - {\beta_1}^2\delta_1 + 6\alpha_1\right). \tag{13}$$

Case 2. Let $a_1 = a_2 = \alpha_1, b_1 = b_2^* = \delta_1 + i\gamma_1$, then τ has the form

$$\tau = 1 + 2\exp(\hat{\zeta}_1)\cos(\gamma_1 y + \hat{\omega}_1 t) + A_{12}\exp(2\hat{\zeta}_1)\,, \tag{14}$$

with

$$A_{12} = \frac{{\gamma_1}^2}{{\gamma_1}^2 + \alpha_1{\delta_1}^3 + \alpha_1\delta_1{\gamma_1}^2}\,, \tag{15}$$

$$\hat{\zeta}_1 = \alpha_1 x + \delta_1 y + {\alpha_1}^3 t + \frac{3\,{\alpha_1}^2\delta_1}{{\delta_1}^2 + {\gamma_1}^2} t\,, \tag{16}$$

$$\hat{\omega}_1 = \frac{-3\,{\alpha_1}^2\gamma_1}{{\delta_1}^2 + {\gamma_1}^2}\,. \tag{17}$$

Case 3. Choosing $a_1 = a_2^* = \alpha_1 + i\beta_1$, $b_1 = b_2^* = \delta_1 + i\gamma_1$, we derive a new form of function τ. For simplicity, it is not given here.

Figure 1 shows the breather solitons of the first BKP equation (1) at $t = 0$ with specific parameters. Figure 1(a) displays the x-periodic breather formulated by τ function in Eq. (10) with Eqs. (11)–(13), where the parameters are $\alpha = 0$, $\beta_1 = 1$, $\delta_1 = 2$. Figure 1(b) displays the (x, y)-periodic breather in Case 3 above, where the parameters are $\alpha = -2$, $\beta_1 = \frac{1}{2}$, $\delta_1 = 1$, $\gamma_1 = 2$. Figure 1(c) displays

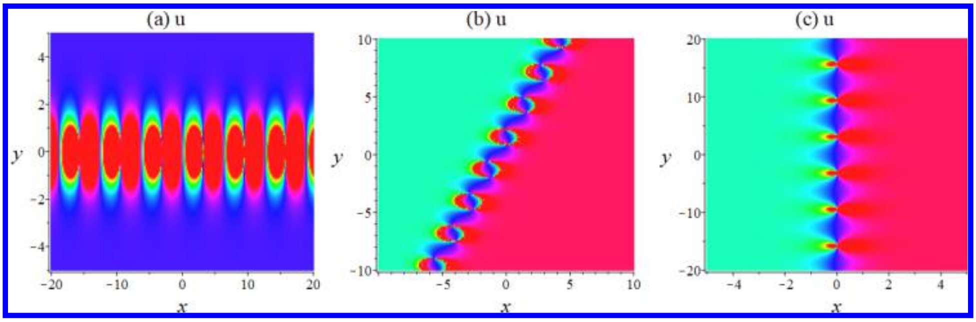

Fig. 1. (Color online) The breather solitons of the first BKP equation at $t = 0$: (a) displays the x-periodic breather formulated by τ function in Eq. (10) with $\alpha = 0$, $\beta_1 = 1$, $\delta_1 = 2$, (b) displays the (x, y)-periodic breather in Case 3 above with $\alpha = -2$, $\beta_1 = \frac{1}{2}$, $\delta_1 = 1$, $\gamma_1 = 2$ and (c) displays the y-periodic breather formulated by τ function in Eq. (14) with $\alpha_1 = 2, \delta_1 = 0, \gamma_1 = 1$.

the y-periodic breather formulated by τ function in Eq. (14) with Eqs. (15)–(17), where the parameters are $\alpha_1 = 2$, $\delta_1 = 0$, $\gamma_1 = 1$.

Case 4. Setting $b_1 = p_1 a_1$, $b_2 = p_2 a_2$, $\eta_{01} = -\eta_{02} = i\pi$ and taking $a_1 \to 0$, $a_2 \to 0$ with $\frac{a_1}{a_2} = O(1)$, then the τ function in Eq. (8) can be rewritten as

$$\tau = a_1 a_2 (\omega_0 + \omega_1 \omega_2 / (p_1 p_2)) \,, \tag{18}$$

where

$$\omega_0 = \frac{2p_1 p_2 (p_2 + p_1)}{(-p_2 + p_1)^2}, \quad \omega_j = p_j x + {p_j}^2 y + 3t, \quad j = 1, 2 \,. \tag{19}$$

Substituting Eq. (18) into the bilinear transformation in Eq. (3) leads to the lump solution of Eq. (1) as follows:

$$u = 2(\ln(\omega_0 + \omega_1 \omega_2 / (p_1 p_2)))_x \,, \tag{20}$$

where ω_0, ω_1 and ω_2 are defined in Eq. (19). Figures 2(a) and 2(b) show the density plots of the first-order lump solitons at $t = 0$ with specific parameters $p_1 = -1 - 2i$,

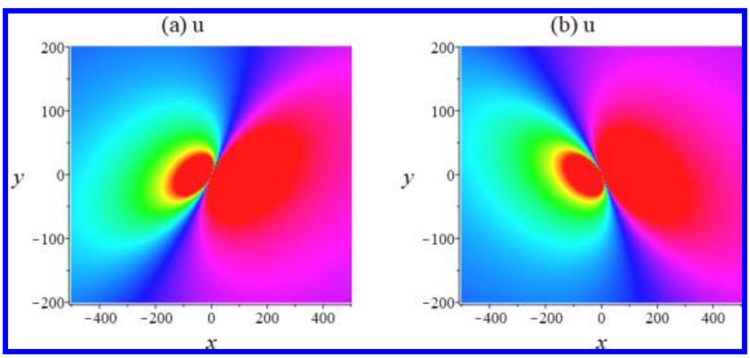

Fig. 2. (Color online) The structures of the lump wave Eq. (20) at $t = 0$ with specific parameters (a) $p_1 = -1 - 2i$, $p_2 = -1 + 2i$ and (b) $p_1 = 1 - 2i$, $p_2 = 1 + 2i$.

$p_2 = -1+2i$ and $p_1 = 1-2i$, $p_2 = 1+2i$, respectively. It is observed that the lump solitons have one peak and two valleys. It is obvious that the two-soliton solution of the first BKP equation (1) can be turned into the breather or lump solutions with the appropriate parameters. Additionally, the lump solution is the limit case of the breather solution.

3.2. *Interaction solutions between one line soliton and other solutions*

In this section, we consider the interaction solutions between one line soliton and other solutions using the three-soliton solution with function τ in Eq. (9) by selecting proper parameters. Similarly, taking

$$a_1 = \alpha_1, \quad a_2 = \alpha_1, \quad b_1 = \delta_1 + i\gamma_1, \quad b_2 = \delta_1 - i\gamma_1,$$

then the function τ in Eq. (9) becomes

$$\tau = 1 + 2e^{\hat{\zeta}_1}\cos(\gamma_1 y + \hat{\omega}_1 t) + A_{12}e^{2\hat{\zeta}_1} + e^{\hat{\zeta}_2} + e^{\hat{\zeta}_1+\hat{\zeta}_2}(A_{13}e^{i\xi_1} + A_{13}^* e^{-i\xi_1}) + A_{12}A_{13}A_{13}^* e^{2\hat{\zeta}_1+\hat{\zeta}_2} \tag{21}$$

with

$$\hat{\zeta}_1 = \alpha_1 x + \delta_1 y + {\alpha_1}^3 t + \frac{3{\alpha_1}^2\delta_1}{{\delta_1}^2+{\gamma_1}^2}t, \quad \hat{\zeta}_2 = a_3 x + b_3 y + {a_3}^3 t + 3\,{a_3}^2 b_3^{-1} t,$$

$$\hat{\omega}_1 = \frac{-3{\alpha_1}^2\gamma_1}{{\delta_1}^2+{\gamma_1}^2}, \quad \xi_1 = \gamma_1 y - \frac{3{\alpha_1}^2\gamma_1 t}{{\delta_1}^2+{\gamma_1}^2}, \quad A_{12} = \frac{{\gamma_1}^2}{{\gamma_1}^2+\alpha_1{\delta_1}^3+\alpha_1\delta_1{\gamma_1}^2},$$

$$A_{13} = \frac{\alpha_1 a_3(\delta_1+i\gamma_1)b_3(\delta_1+i\gamma_1-b_3)(\alpha_1-a_3)-(\alpha_1 b_3-a_3(\delta_1+i\gamma_1))^3}{\alpha_1 a_3(\delta_1+i\gamma_1)b_3(b_3+\delta_1+i\gamma_1)(a_3+\alpha_1)-(\alpha_1 b_3-a_3(\delta_1+i\gamma_1))^3}.$$

Substituting the function τ in Eq. (21) into the transformation (3), a new kind of exact solution to the first BKP equation (1) is obtained. To the best of our knowledge, this solution has not been reported before. Figures 3(a) and 3(b) display

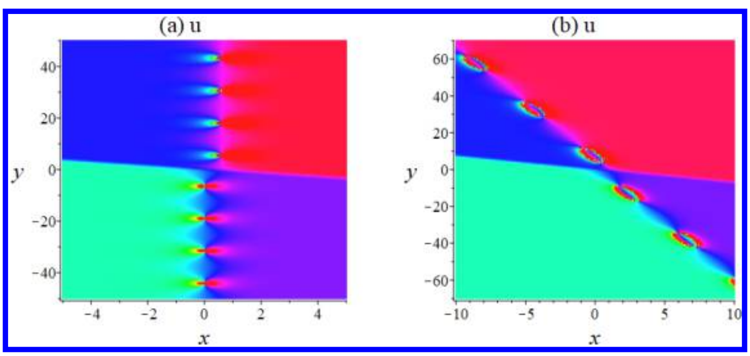

Fig. 3. (Color online) The interaction phenomena between a breather solution and a line soliton solution at $t = 0$ with specific parameters $\alpha_1 = 2$, $a_3 = 3$, $b_3 = 4$: (a) $\delta_1 = 0$, $\gamma_1 = -1/2$ and (b) $\delta_1 = 1/3$, $\gamma_1 = -1/4$.

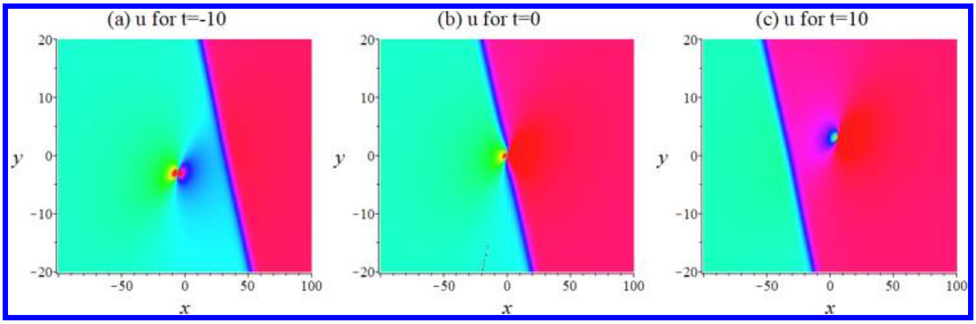

Fig. 4. (Color online) The interactions between a lump wave and a line soliton wave with specific parameters $p_1 = -1 - 3i$, $p_2 = -1 + 3i$, $a_3 = 0.5$ at different time: (a) $t = -10$, (b) $t = 0$ and (c) $t = 10$.

the interaction process between a line soliton and a breather soliton at $t = 0$ with parameters $\alpha_1 = 2, a_3 = 3, b_3 = 4$, and $\delta_1 = 0, \gamma_1 = -1/2$ and $\delta_1 = 1/3, \gamma_1 = -1/4$, respectively. So that choosing different parameters, the three-soliton solution of Eq. (1) can be turned into different interaction solutions: one breather soliton along with a line soliton.

Assuming $b_1 = p_1 a_1$, $b_2 = p_2 a_2$, $\eta_1^{(0)} = -\eta_2^{(0)} = i\pi$ and $\eta_3^{(0)} = 0$, we can rewrite function τ in Eq. (9) in another form. For simplicity, we omit it here. Figures 4(a) and 4(c) demonstrate the interaction processes between a line soliton and a lump soliton at different time with parameters $p_1 = -1 - 3i$, $p_2 = -1 + 3i$, $a_3 = 0.5$. It is also noticed that line soliton and lump soliton interact at point $(x, y) = (0, 0)$ at time $t = 0$, and the lump wave travels from the left side to the right side of the line soliton. Moreover, if selecting the proper parameters, the three-soliton solution of Eq. (1) can be turned into a interaction solutions between a line soliton solution and a lump solution. Actually, the lump wave is the limit case of the breather wave as shown in Fig. 3.

3.3. *Interaction solutions between breather and other solutions*

Finally, on the basis of the four-soliton solution for τ function in Eq. (5) with $N = 4$, a lot of interesting and complicated solutions can be derived. For simplicity, we will not present their analytical expressions herein. Actually, the four-soliton solution can be turned into two special cases with different parameters: two breather solitons (periodic solitons) or two line solitons along with a breather soliton. Figures 5(a)–5(c) show the interaction process between two breather solitons at $t = -10, 0, 10$ with parameters

$$\begin{aligned} &a_1 = 2/3, \quad a_2 = 2/3, \quad b_1 = 1/2i, \quad b_2 = -1/2i\,, \\ &a_3 = 1/2, \quad a_4 = 1/2, \quad b_3 = 1 + 1/4i, \quad b_4 = 1 - 1/4i\,. \end{aligned} \tag{22}$$

It is noticed from Fig. 5 that one breather is y-periodic along the y-direction and the other breather is periodic in the angular bisection of the x- and y-axes, we call it

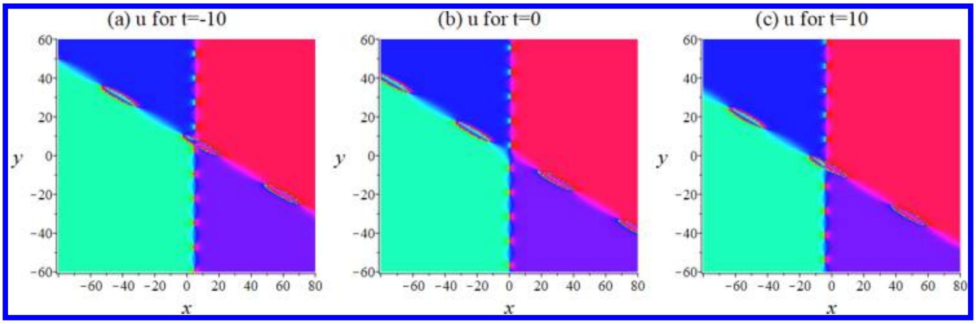

Fig. 5. (Color online) The interactions between two breather solutions with specific parameters: $a_1 = 2/3$, $a_2 = 2/3$, $b_1 = 1/2i$, $b_2 = -1/2i$, $a_3 = 1/2$, $a_4 = 1/2$, $b_3 = 1 + 1/4i$, $b_4 = 1 - 1/4i$ at different time: (a) $t = -10$, (b) $t = 0$ and (c) $t = 10$.

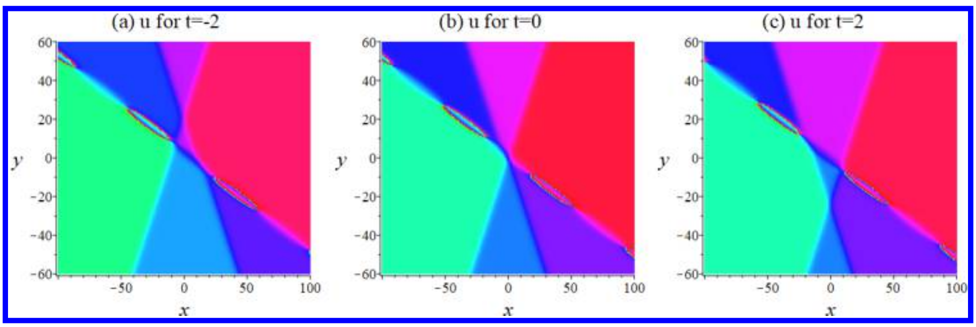

Fig. 6. (Color online) The interactions between a breather soliton and two line solitons with specific parameters: $a_1 = 2/3$, $a2 = 2/3$, $b_1 = 1/3$, $b_2 = -1/3$, $a_3 = 1/2$, $a_4 = 1/2$, $b_3 = 1 + 1/6i$, $b_4 = 1 - 1/6i$ at different time (a) $t = -2$, (b) $t = 0$ and (c) $t = 2$.

(x, y)-periodic breather, and they interact at point $(x, y) = (0, 0)$. Figures 6(a)–6(c) show the interaction processes between two line solitons and a breather solution at time $t = -2$, 0, 2 with parameters

$$\begin{aligned} &a_1 = 2/3, \quad a2 = 2/3, \quad b_1 = 1/3, \quad b_2 = -1/3\,, \\ &a_3 = 1/2, \quad a_4 = 1/2, \quad b_3 = 1 + 1/6i, \quad b_4 = 1 - 1/6i\,. \end{aligned} \tag{23}$$

It is observed from Fig. 6 that the two line solitons and a breather soliton interact at point $(x, y) = (0, 0)$ at time $t = 0$, and with the evolution of time one line soliton moves from the positive x-axis to the negative direction of the x-axis, and the other line soliton moves in an opposed way.

Figures 7(a) and 7(b) show a different case of the interaction processes between two line solitons and a breather solution at time $t = -10$, 0, 10 with parameters

$$\begin{aligned} &a_1 = 2/3, \quad a_2 = 2/3, \quad b_1 = 1/3, \quad b_2 = -1/3\,, \\ &a_3 = 1/2, \quad a_4 = 1/2, \quad b_3 = 1/6i, \quad b_4 = -1/6i\,. \end{aligned} \tag{24}$$

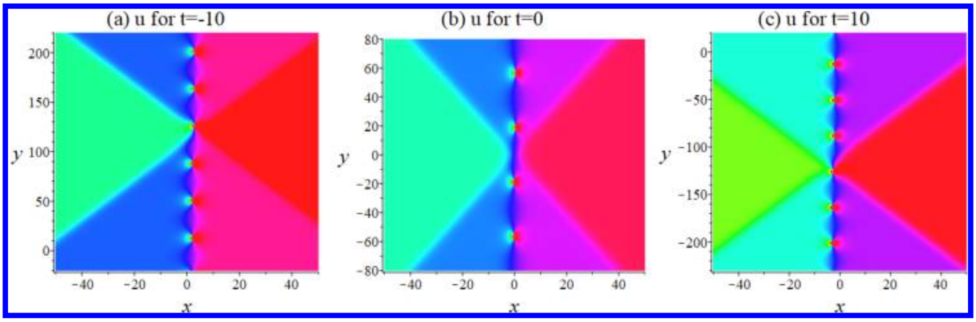

Fig. 7. (Color online) The interactions between a breather and two line solitons with specific parameters $a_1 = 2/3$, $a_2 = 2/3$, $b_1 = 1/3$, $b_2 = -1/3$, $a_3 = 1/2$, $a_4 = 1/2$, $b_3 = 1/6i$, $b_4 = -1/6i$ at different time: (a) $t = -10$, (b) $t = 0$ and (c) $t = 10$.

In Fig. 7, it is seen that the breather soliton is y-periodic and is located between two line solitons.

4. Conclusions

In this research, we studied the interaction solutions of the first BKP equation using the Hirota direct method and taking long-wave limit. First, based on the Hirota bilinear method, multi-soliton solutions of the first BKP equation are derived. Then through choose appropriate parameters, some interesting interaction solutions among some solutions, including line solitons, breathers, lumps, are constructed, respectively. The corresponding evolution profiles over time are given to illustrate their dynamic qualities. Taking long wave limit of some soliton solutions, lumps and line solitons are constructed. Moreover, the interaction between them are also found with special parameter constraints. In the same way, interaction solutions between breather and a line soliton or two line soliton or another breather are also constructed, respectively. They have different dynamic properties. It is noted that the parameters have great impact on these solutions, such as the propagation directions and shapes.

Furthermore, all the results in this paper can provide an effective way to solve some nonlinear problems in nonlinear optics, plasmas, Bose–Einstein condensates, and so on.

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