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Abundant exact solutions and interaction phenomena of the (2 + 1)-dimensional YTSF equation

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Abstract
In this paper, we study abundant exact solutions including the lump and interaction solutions to the (2+1)-dimensional Yu–Toda–Sasa–Fukuyama equation. With symbolic computation, lump solutions and the interaction solutions are generated directly based on the Hirota bilinear formulation. Analyticity and well-definedness is guaranteed through some conditions posed on the parameters. With special choices of the involved parameters, the interaction phenomena are simulated and discussed. We find the lump moves from one hump to the other hump of the two-soliton, while the lump separates from the hump of the one-soliton.

Keywords Hirota bilinear form · Lump solutions · Interaction solutions · YTSF equation

1 Introduction

The nonlinear evolution equations (NLEEs) as mathematical models have been used in many fields of science and engineering [1–10]. Soliton solutions to NLEEs, exponentially localized in certain direction, have been discovered and studied with such

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methods as the Hirota bilinear method [11–15], inverse scattering [11], Bäcklund transformation [16], Painlevé test [17], etc. In contrast to soliton solutions, lump solutions are also a kind of solutions to NLEEs, which are rationally localized in all directions in space [18,19]. An effective method has been proposed to obtain the lump solutions to the Kadomtsev–Petviashvili (KP) equation [18], where a general class of lump solutions have been derived with Maple. Many researchers obtained the lump solutions to NLEEs, for example, KPI equation [20], BKP equation [21], a generalized Kadomtsev–Petviashvili–Boussinesq equation [22], (2 + 1)-dimensional KdV equation [23,24], and BLMP equation [25].

Recently, the interaction solutions of lump-kink and lump-soliton types have become the hot topics of the research. It is reported that the interactions turn out to be elastic or completely non-elastic [26]. The elastic phenomenon is that the lump restores its shape, amplitude and velocity after colliding with another solution [27], and the completely non-elastic phenomenon is that the lump is swallowed up by another solution [28].

It is valuable in mathematics and physics to study lump solutions and the interaction phenomena for the NLEEs. The KdV equation in (1+1)-dimensional is written as

$$q_t + \Phi(q)q_x = 0, \tag{1}$$

where $\Phi(q)$ $(= \partial_x^2 + 4q + 2q_x\partial_x^{-1})$ is the strong symmetry [29], and $\partial_x^{-1} = \int dx$. The potential form of Eq. (1) is

$$q_t + q_{xxx} + 6qq_x = 0. \tag{2}$$

The KdV equation can be extended to the (2 + 1)-dimensional KP equation

$$(-4q_t + \Phi(q)q_x)_x + 3q_{yy} = 0, \tag{3}$$

or the Bogoyavlenskii–Schiff (BS) equation

$$q_t + \Phi(q)q_z = 0. \tag{4}$$

Further, both the KP equation and the BS equation can be extended to the (3 + 1)-dimensional Yu–Toda–Sasa–Fukuyama equation (YTSFE) as

$$(\Phi(q)q_z - 4q_t)_x + 3q_{yy} = 0. \tag{5}$$

Correspondingly, the potential YTSFE with $q = w_x$ can be introduced as

$$-4\,w_{xt} + w_{xxxz} + 4\,w_x w_{xz} + 2w_{xx}w_z + 3w_{yy} = 0, \tag{6}$$

where $w = w(x, y, z, t)$ is an analytic function denoting the amplitude of the relevant wave. The elastic quasi-plane wave in a lattice or interfacial wave in a two-layer liquid have been well-described by the potential YTSFE [30–32]. Equation (6) has appeared as a shallow water equation for the reacting mixtures [30], and has also been used for investigating the dynamics of solitons and nonlinear waves in fluid dynamics, weakly dispersive media, plasma physics [31]. For the potential YTSFE, the rational solutions, lump solutions and the N-lump solutions have been derived for some special cases [32]. Based on the extended homogeneous balance method, multi-soliton solutions and multi-singular soliton solutions have been obtained [33]. New kink multi-soliton solutions have been found by utilizing the three-wave method and homoclinic test approach [34]. Two new approaches, including HTA and EHTA, are introduced to get exact periodic solitary-wave and doubly periodic wave solutions [35].

Through the following transformation

$$x - z \to x, \quad y \to y, \quad t \to t, \tag{7}$$

the potential YTSFE is cast into

$$-4w_{xt} - w_{xxxx} - 6w_x w_{xx} + 3w_{yy} = 0. \tag{8}$$

By introducing the potential function $u(x, y, t) = \frac{\partial w(x,y,t)}{\partial x}$, the derivation of Eq. (8) with respect to x becomes

$$-4u_{xt} - u_{xxxx} - 6u_x^2 - 6uu_{xx} + 3u_{yy} = 0, \tag{9}$$

which is the (2 + 1)-dimensional YTSFE and will be investigated in this paper.

The extension scheme among KdV equation, BS equation, KP equation, (2 + 1)-dimensional YTSFE and (3 + 1)-dimensional YTSFE can be summarized as:

$$\begin{tikzcd}
\textbf{KdV equation} \arrow[r] \arrow[d] & \textbf{BS equation} \arrow[d] & \\
\textbf{KP equation} \arrow[r] & \textbf{(3+1) YTSF equation} \arrow[r] & \textbf{(2+1) YTSF equation}
\end{tikzcd}$$

It is shown that the KdV equation can be extended to the KP equation and the BS equation. Further, the KP equation and the BS equation can be extended to the (3 + 1)-dimensional YTSFE. Finally, the (3 + 1)-dimensional YTSFE can be transformed into the (2 + 1)-dimensional YTSFE. To our knowledge, lump solutions and interaction solutions to the (2 + 1)-dimensional YTSFE by using the bilinear method have not been reported yet.

In this paper, we will study the lump solutions to Eq. (9) in a direct manner and consider two kinds of interaction phenomena. The structure of this paper is as follows: We will firstly investigate the lump solutions to Eq. (9) in Sect. 2; The lump-soliton type interaction solutions will be derived in Sect. 3; While the lump-kink type (interaction between lump and a stripe) will be derived in Sect. 4. The conclusion will be arranged in Sect. 5.

2 Lump solutions to the (2 + 1)-dimensional YTSFE

Substituting the following dependent variable transformation

$$u(x, y, t) = 2(\ln f(x, y, t))_{xx}, \tag{10}$$

into Eq. (9), or

$$w(x, y, t) = 2(\ln f(x, y, t))_{x}, \tag{11}$$

into Eq. (8), we have

$$\begin{aligned} \mathrm{B}_{\mathrm{YTSF}} &:= (D_x^4 - 3D_y^2 + 4D_x D_t) f \cdot f \\ &= f f_{xxxx} - 4 f_x f_{xxx} + 3 f_{xx} f_{xx} - 3 f f_{yy} + 3 f_y f_y + 4 f f_{xt} - 4 f_x f_t = 0, \end{aligned} \tag{12}$$

where the operator D is defined by [13]

$$\begin{aligned} & D_x^\alpha D_y^\beta D_t^\gamma (f \cdot g) \\ & = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^\alpha \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^\beta \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^\gamma f(x, y, t) g(x', y', t')\Big|_{x'=x, y'=y, t'=t}. \end{aligned}$$

To find the lump solutions to the (2+1)-dimensional YTSFE (9), we search for quadratic function solutions to Eq. (12), and begin with

$$f = g^2 + h^2 + a_9, \tag{13}$$

with

$$\begin{aligned} g &= a_1 x + a_2 y + a_3 t + a_4, \\ h &= a_5 x + a_6 y + a_7 t + a_8, \end{aligned}$$

where a_i $(1 \le i \le 9)$ are all real parameters to be determined. With symbolic computation, we directly substitute Eq. (13) into Eq. (12), and collect the coefficients of the like power of variables x, y and t, then set them equal to zero. Solving the sets of these coefficient equations, we can get the following relations of a_i $(1 \le i \le 9)$,

$$\begin{aligned} \Bigg\{ & a_1 = a_1, a_2 = a_2, a_3 = \frac{3}{4} \frac{a_1 a_2^2 - a_1 a_6^2 + 2 a_2 a_5 a_6}{a_1^2 + a_5^2}, a_4 = a_4, a_5 = a_5, a_6 = a_6, \\ & a_7 = \frac{3}{4} \frac{2 a_1 a_2 a_6 - a_2^2 a_5 + a_5 a_6^2}{a_1^2 + a_5^2}, a_8 = a_8, a_9 = \frac{a_1^6 + 3 a_1^4 a_5^2 + 3 a_1^2 a_5^4 + a_5^6}{(a_1 a_6 - a_2 a_5)^2} \Bigg\}, \end{aligned} \tag{14}$$

which need to satisfy the following condition

$$a_1a_6 - a_2a_5 \neq 0. \tag{15}$$

At the same time, to guarantee the positiveness of f, we should let

$$a_9 > 0. \tag{16}$$

By substituting Eq. (14) into Eq. (13), the exact quadratic function solution to Eq.(12) is obtained

$$\begin{aligned} f =& \left(a_1x + a_2y + \frac{3}{4}\frac{a_1a_2^2 - a_1a_6^2 + 2\,a_2a_5a_6}{a_1^2 + a_5^2}t + a_4\right)^2 \\ &+ \left(a_5x + a_6y + \frac{3}{4}\frac{2\,a_1a_2a_6 - a_2^2a_5 + a_5a_6^2}{a_1^2 + a_5^2}t + a_8\right)^2 \\ &+ \frac{a_1^6 + 3\,a_1^4a_5^2 + 3\,a_1^2a_5^4 + a_5^6}{(a_1a_6 - a_2a_5)^2}, \end{aligned} \tag{17}$$

and the functions g and h are given as follows:

$$g = a_1x + a_2y + \frac{3}{4}\frac{a_1a_2^2 - a_1a_6^2 + 2\,a_2a_5a_6}{a_1^2 + a_5^2}t + a_4, \tag{18}$$

$$h = a_5x + a_6y + \frac{3}{4}\frac{2\,a_1a_2a_6 - a_2^2a_5 + a_5a_6^2}{a_1^2 + a_5^2}t + a_8. \tag{19}$$

Regarding u as a function of x and y, we can find the extremum point of u as

$$\left(x = \frac{a_2a_7t - a_3a_6t + a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5},\ y = -\frac{a_1a_7t - a_3a_5t + a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}\right), \tag{20}$$

and the maximum of u is $\frac{4(a_1^2+a_5^2)}{a_9}$. Therefore, six free parameters (a_1, a_2, a_4, a_5, a_6 and a_8) in Eq.(14) affect the extremum point, that is, the position of the lump solution, and four parameters a_1, a_2, a_5 and a_6 affect the maximum. We take a selection of the parameters

$$a_1 = 1, a_2 = \frac{7}{6}, a_4 = 0, a_5 = 1, a_6 = -\frac{3}{5}, a_8 = 0,$$

to plot the lump solution when $t = 1$ in Fig. 1, where the extremum point locates at $(\frac{1549}{2400}, -\frac{17}{40}) \approx (0.65, -0.43)$, and the maximum of u is $\frac{2809}{900} \approx 3.12$.

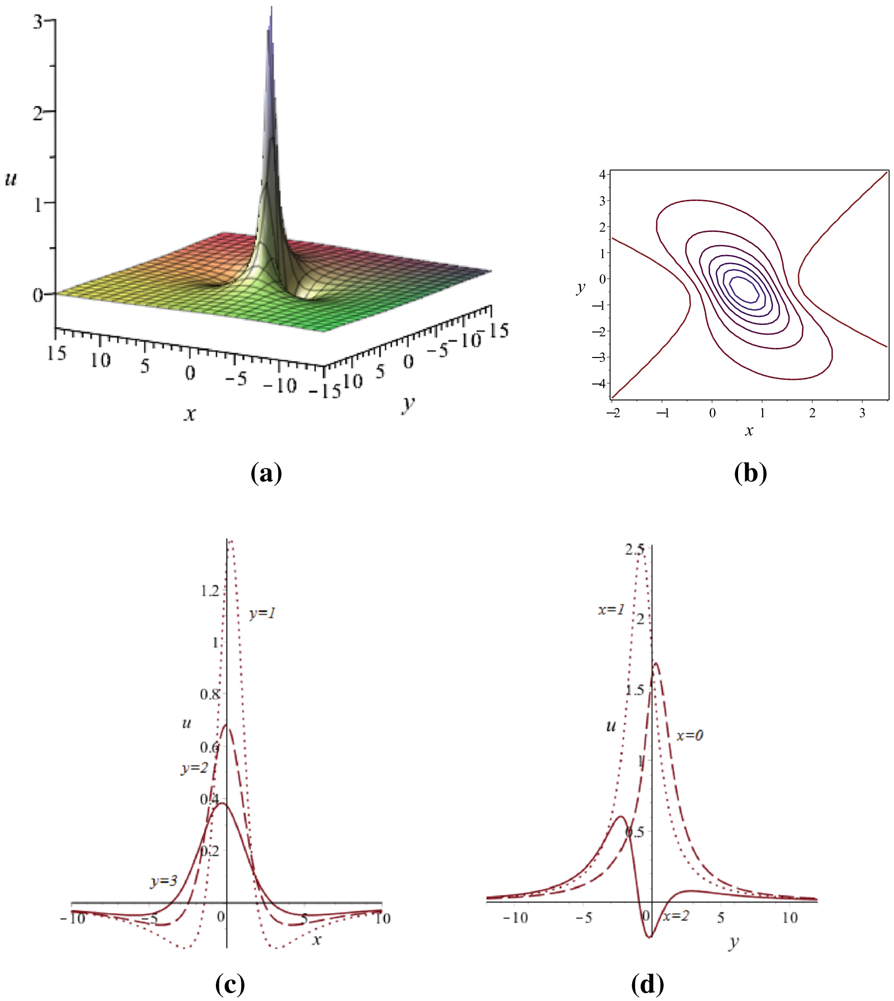


Fig. 1 Lump dynamic characteristics of u with $t = 1$: **a** 3-dimensional plot; **b** density plot; **c** x-curves and **d** y-curves

3 Lump-soliton solutions to the (2 + 1)-dimensional YTSFE

We then study the interaction between lump and soliton of the (2+1)-dimension YTSFE. We suppose f is the combination of two quadratic functions and one hyperbolic cosine function as

$$f = \xi_1^2 + \xi_2^2 + \cosh \xi_3 + a_{13}, \tag{21}$$

where ξ_i $(1 \leq i \leq 3)$ are defined by

$$\xi_1 = a_1 x + a_2 y + a_3 t + a_4,$$

$$\xi_2 = a_5 x + a_6 y + a_7 t + a_8,$$
$$\xi_3 = a_9 x + a_{10} y + a_{11} t + a_{12},$$

while a_i $(1 \le i \le 13)$ are all real parameters to be determined.

With symbolic computation, three sets of relations among the parameters a_i $(1 \le i \le 13)$ are given in Appendix A.

Based on the expression of f, we have the exact form of w as

$$\begin{aligned} w(x, y, t) = 2(\ln f(x, y, t))_x &= 2\frac{2a_1\xi_1 + 2a_5\xi_2 + a_9 \sinh \xi_3}{\xi_1^2 + \xi_2^2 + \cosh \xi_3 + a_{13}} \\ &= 2\frac{\frac{2a_1\xi_1}{\cosh \xi_3} + \frac{2a_5\xi_2}{\cosh \xi_3} + \frac{a_9 \sinh \xi_3}{\cosh \xi_3}}{\frac{\xi_1^2}{\cosh \xi_3} + \frac{\xi_2^2}{\cosh \xi_3} + 1 + \frac{a_{13}}{\cosh \xi_3}}. \end{aligned} \tag{22}$$

When $a_{11} > 0$, we have

$$\begin{aligned} &\lim_{t\to\infty} \frac{2a_1\xi_1}{\cosh \xi_3} = 0, \quad \lim_{t\to\infty} \frac{2a_5\xi_2}{\cosh \xi_3} = 0, \quad \lim_{t\to\infty} \frac{a_{13}}{\cosh \xi_3} = 0, \\ &\lim_{t\to\infty} \frac{\xi_1^2}{\cosh \xi_3} = 0, \quad \lim_{t\to\infty} \frac{\xi_2^2}{\cosh \xi_3} = 0, \end{aligned} \tag{23}$$

and

$$\lim_{t\to+\infty} \frac{\sinh \xi_3}{\cosh \xi_3} = 1, \quad \lim_{t\to-\infty} \frac{\sinh \xi_3}{\cosh \xi_3} = -1, \tag{24}$$

so we can deduce the limit of w

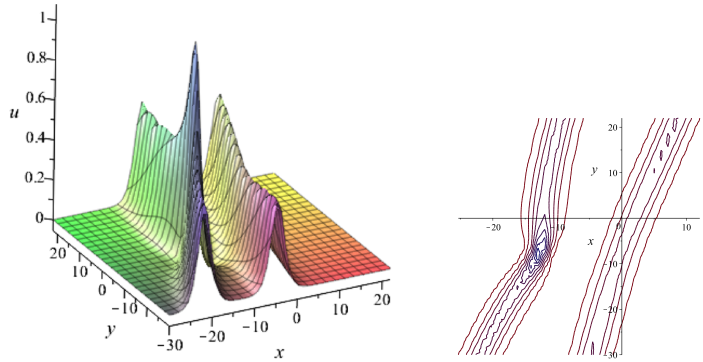
$$\lim_{t\to+\infty} w = 2a_9, \quad \lim_{t\to-\infty} w = -2a_9. \tag{25}$$

When $a_{11} < 0$, the results are opposite.

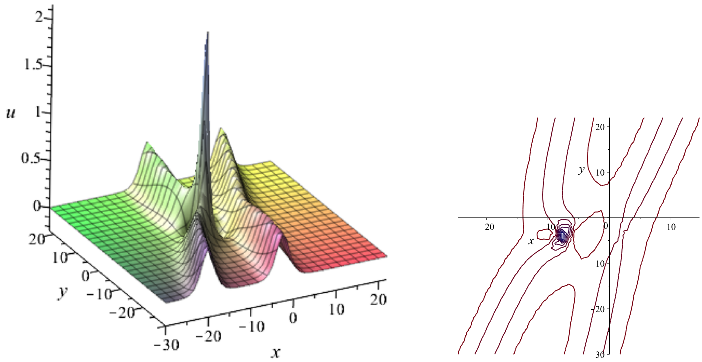
As an example, we choose the following parameters in the first case of interaction solutions with $\varepsilon = -1$ in Appendix A

$$a_1 = 2, a_2 = 0, a_4 = 0, a_5 = \frac{1}{2}, a_8 = 0, a_9 = 1, a_{12} = 2,$$

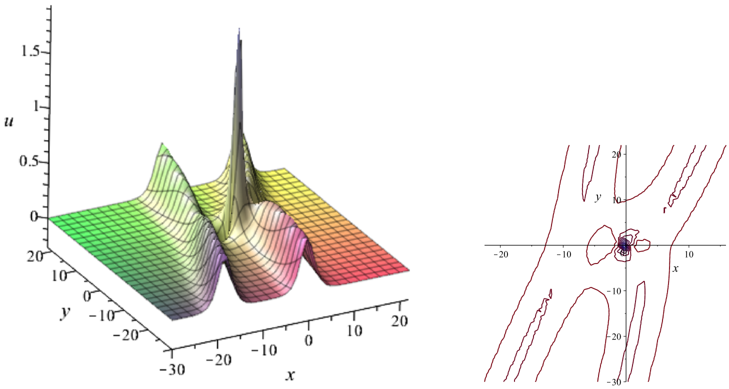
to plot Fig. 2, which are the 3-dimensional plots and contour plots. From the contour plots with different time, we can find that the lump appears and moves from one hump of the soliton to the other hump, and finally is swallowed. The maximum of lump also changes in the process, and the interaction is nonelastic.



(a) $t = -20$

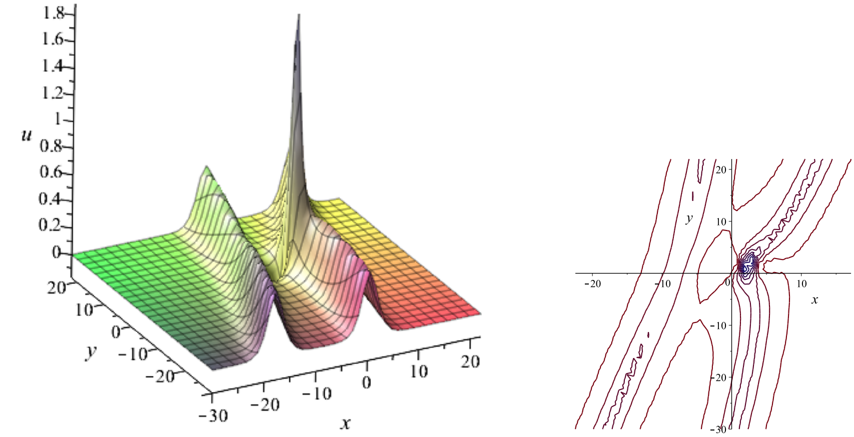


(b) $t = -10$

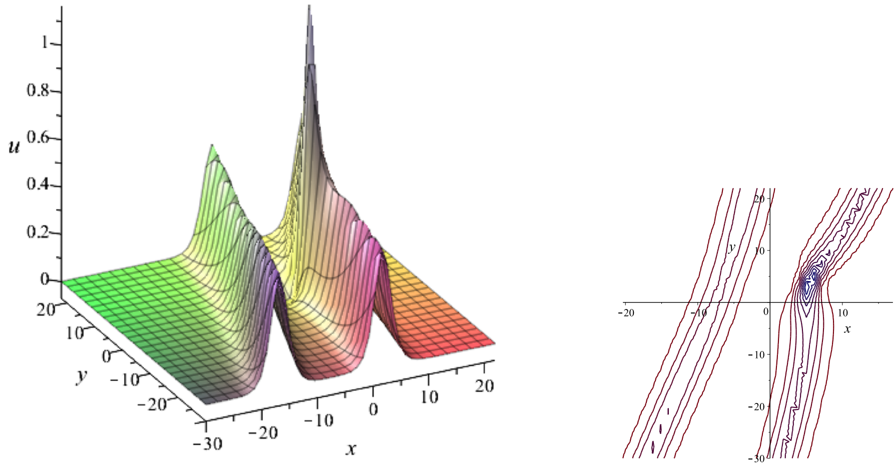


(c) $t = 0$

Fig. 2 Interaction between lump and soliton at different time **a** $t = -20$, **b** $t = -10$, **c** $t = 0$, **d** $t = 4$, **e** $t = 10$



(d) $t = 4$



(e) $t = 10$

Fig. 2 continued

4 Lump-kink solutions to the (2 + 1)-dimensional YTSFE

We then study the interaction between lump and a stripe of the (2+1)-dimensional YTSFE by defining f as

$$f = g^2 + h^2 + ke^l + a_9, \tag{26}$$

with

$$\begin{aligned} g &= a_1x + a_2y + a_3t + a_4, \\ h &= a_5x + a_6y + a_7t + a_8, \end{aligned}$$

$$l = k_1 x + k_2 y + k_3 t,$$

where a_i $(1 \leq i \leq 9)$, k_i $(1 \leq i \leq 3)$ and k are all real parameters to be determined.

As we can see, the function $f(x, y, t)$ combines two quadratic functions and an exponential function. Three sets of relations among the parameters a_i $(1 \leq i \leq 9)$ and k_i $(1 \leq i \leq 3)$ are given in Appendix B.

Based on the expression of f, we calculate the function w as

$$\begin{aligned} w(x, y, t) = 2(\ln f(x, y, t))_x &= 2\frac{2a_1 g + 2a_5 h + kk_1 e^l}{g^2 + h^2 + ke^l + a_9} \\ &= 2\frac{\frac{2a_1 g}{e^l} + \frac{2a_5 h}{e^l} + kk_1}{\frac{g^2}{e^l} + \frac{h^2}{e^l} + k + \frac{a_9}{e^l}}, \end{aligned} \tag{27}$$

in which we suppose $k_3 > 0$ and have

$$\lim_{t\to+\infty} \frac{2a_1 g}{e^l} = 0, \quad \lim_{t\to+\infty} \frac{2a_5 h}{e^l} = 0, \quad \lim_{t\to+\infty} \frac{a_9}{e^l} = 0, \quad \lim_{t\to+\infty} \frac{g^2}{e^l} = 0, \quad \lim_{t\to+\infty} \frac{h^2}{e^l} = 0,$$

so we can deduce the limit of w as

$$\lim_{t\to+\infty} w = 2k_1. \tag{28}$$

If $k_3 < 0$, then we have

$$\lim_{t\to+\infty} w = 0. \tag{29}$$

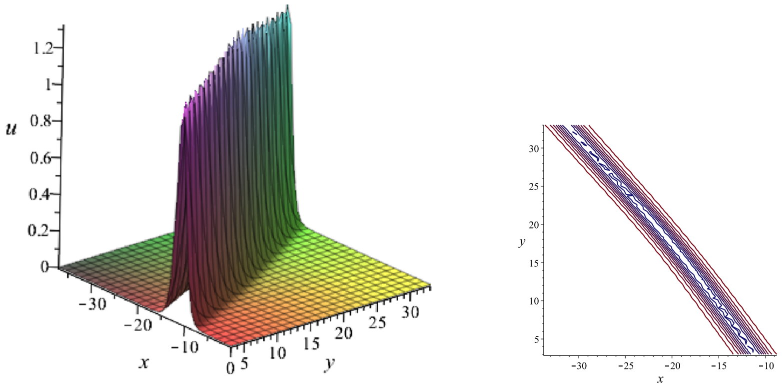
For example, we choose the following parameters in the first case of the interaction solutions

$$a_1 = 1, a_4 = 0, a_5 = 1, a_8 = 0, k_1 = -\frac{3}{2}, k_2 = -1, k = \frac{1}{2}, \tag{30}$$

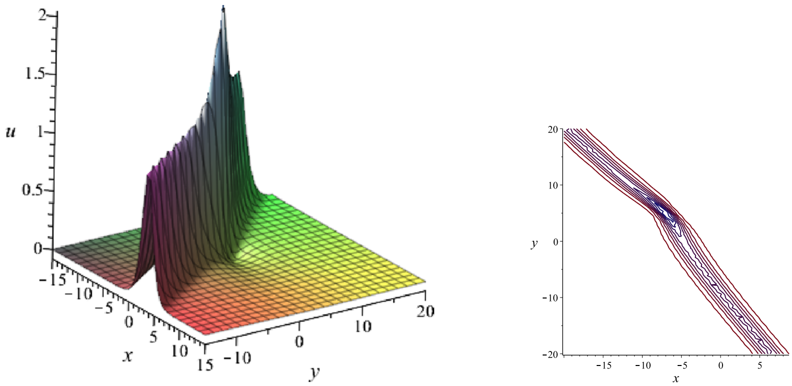
to plot Fig. 3, which are the 3-dimensional plots and contour plots of the solution. From the figures, we can observe that the lump generates from the stripe, and then separates from the stripe.

5 Concluding remarks

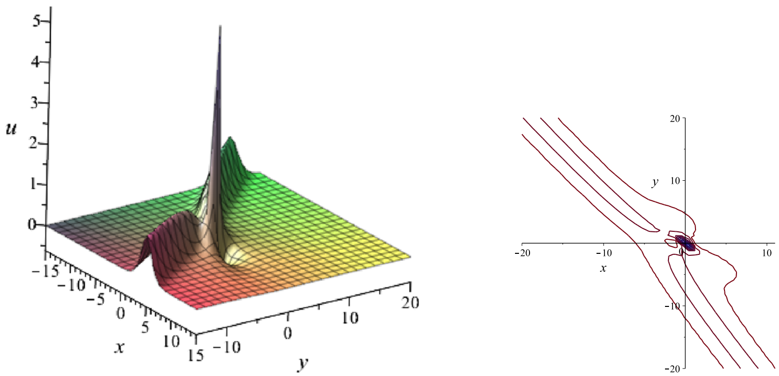
In this paper, based on the Hirota formulation and symbolic computation, a kind of lump solutions and two types of interaction solutions have been studied for the (2 + 1)-dimensional YTSFE, including lump-soliton and lump-kink solutions. With limitation analysis and graphical simulation, we have found the lump appears and moves from one hump of the soliton to the other hump, and finally is swallowed by the soliton



(a) $t = -17$

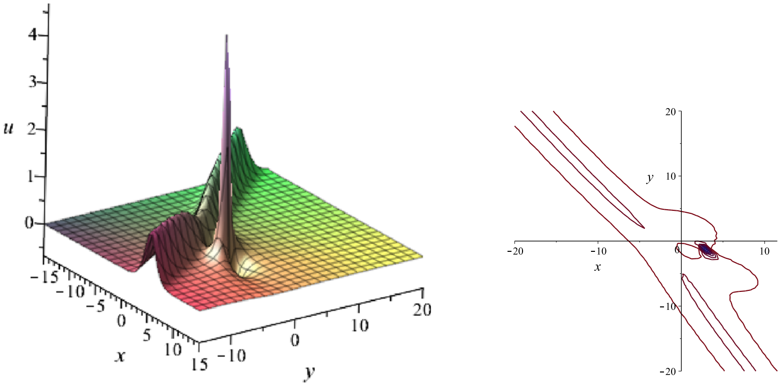


(b) $t = -5$

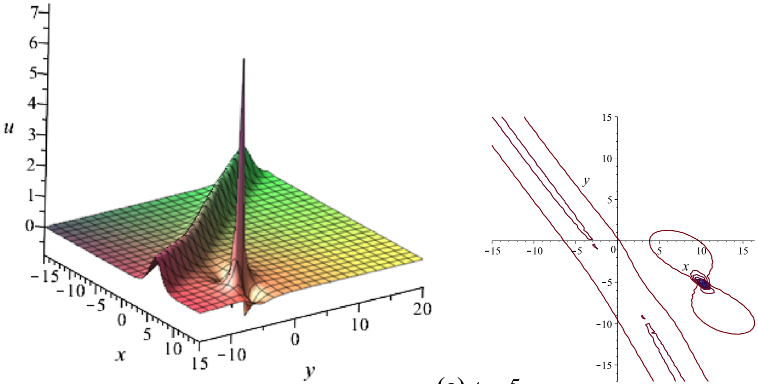


(c) $t = 0$

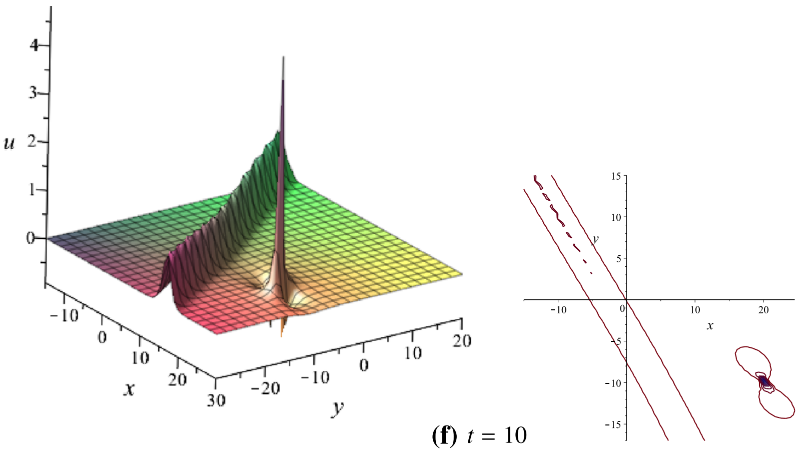
Fig. 3 Interaction between lump and a stripe at different time **a** $t = -17$, **b** $t = -5$, **c** $t = 0$, **d** $t = 1.5$, **e** $t = 5$, **f** $t = 10$



(d) $t = 1.5$



(e) $t = 5$



(f) $t = 10$

Fig. 3 continued

(see Fig. 2); the lump generates from the hump of the one-soliton and finally separates from it (see Fig. 3).

Moreover, the extremum point and maximum of the lump solutions have been given in Sect. 2. To analyze the interaction phenomenon, the interaction solutions w in Sects. 3 and 4 have been properly divided, and the limits of the solutions have been given according to every part. It is shown that the direct method is an effective way to search for lump solutions and interaction solutions, and it can be used for many other nonlinear integrable systems in mathematical physics. The result of this paper is helpful for understanding the propagation of nonlinear waves and enrich the nonlinear dynamics.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest concerning the publication of this manuscript.

Appendix A

For simplicity, we consider three cases with $a_1 = 0$, $a_2 = 0$ or $a_{10} = 0$, respectively:

Case 1

$$\Big\{a_1 = a_1, a_2 = 0, a_3 = -\frac{3(a_1^2 + a_5^2)a_9^2}{4a_1}, a_4 = a_4, a_5 = a_5, a_6 = \varepsilon\frac{(a_1^2 + a_5^2)a_9}{a_1},$$
$$a_7 = \frac{3a_5a_9^2(a_1^2 + a_5^2)}{4a_1^2}, a_8 = a_8, a_9 = a_9, a_{10} = \varepsilon\frac{a_5a_9^2}{a_1}, a_{11} = -\frac{a_9^3(a_1^2 - 3\,a_5^2)}{4a_1^2},$$
$$a_{12} = a_{12}, a_{13} = \frac{4\,a_1^2 + 8\,a_1^2a_5^2 + 4\,a_5^4 + a_9^4}{4a_9^2(a_1^2 + a_5^2)}\Big\},$$

Case 2

$$\Big\{a_1 = 0, a_2 = \varepsilon\, a_5a_9, a_3 = \varepsilon\,\frac{3}{2}\,a_6a_9, a_4 = a_4, a_5 = a_5, a_6 = a_6,$$
$$a_7 = -\frac{3(a_5^2a_9^2 - a_6^2)}{4a_5}, a_8 = a_8, a_9 = a_9, a_{10} = \frac{a_6a_9}{a_5},$$
$$a_{11} = -\frac{a_9(a_5^2a_9^2 - 3\,a_6^2)}{4a_5^2}, a_{12} = a_{12}, a_{13} = \frac{4\,a_5^4 + a_9^4}{4a_5^2a_9^2}\Big\},$$

Case 3

$$\left\{a_1 = a_1, a_2 = \varepsilon\, a_5 a_9, a_3 = -\frac{3}{4}\, a_1 a_9^2, a_4 = a_4, a_5 = a_5,\right.$$
$$a_6 = -\varepsilon a_1 a_9, a_7 = -\frac{3}{4}\, a_5 a_9^2, a_8 = a_8, a_9 = a_9, a_{10} = 0,$$
$$\left. a_{11} = -\frac{1}{4}\, a_9^3, a_{12} = a_{12}, a_{13} = \frac{4\, a_1^4 + 8\, a_1^2 a_5^2 + 4\, a_5^4 + a_9^4}{4 a_9^2 (a_1^2 + a_5^2)}\right\},$$

where $\varepsilon = \pm 1$. Actually, these three sets of solutions contain six cases corresponding to different values of ε.

Appendix B

Case 1

$$\left\{a_1 = a_1, a_2 = \frac{a_5 k_1^2 + a_1 k_2}{k_1}, a_3 = -\frac{3}{4}\, \frac{a_1 k_1^4 - 2\, a_5 k_1^2 k_2 - a_1 k_2^2}{k_1^2}, a_4 = a_4, a_5 = a_5,\right.$$
$$a_6 = -\frac{a_1 k_1^2 - a_5 k_2}{k_1}, a_7 = -\frac{3}{4}\, \frac{a_5 k_1^4 + 2\, a_1 k_1^2 k_2 - a_5 k_2^2}{k_1^2},$$
$$\left. a_8 = a_8, a_9 = \frac{a_1^2 + a_5^2}{k_1^2}, k_1 = k_1, k_2 = k_2, k_3 = -\frac{k_1^4 - 3\, k_2^2}{4 k_1}\right\},$$

Case 2

$$\left\{a_1 = 0, a_2 = \varepsilon a_5 k_1, a_3 = \varepsilon \frac{3}{2}\, a_6 k_1, a_4 = a_4, a_5 = a_5, a_6 = a_6,\right.$$
$$a_7 = -\frac{3}{4}\, \frac{a_5^2 k_1^2 - a_6^2}{a_5}, a_8 = a_8, a_9 = \frac{a_5^2}{k_1^2},$$
$$\left. k_1 = k_1, k_2 = \frac{a_6 k_1}{a_5}, k_3 = -\frac{k_1 (a_5^2 k_1^2 - 3\, a_6^2)}{4 a_5^2}\right\},$$

Case 3

$$\left\{a_1 = a_1, a_2 = \frac{-a_5 k_1^2 + a_1 k_2}{k_1}, a_3 = -\frac{3}{4}\, \frac{a_1 k_1^4 + 2\, a_5 k_1^2 k_2 - a_1 k_2^2}{k_1^2}, a_4 = a_4, a_5 = a_5,\right.$$
$$a_6 = \frac{a_1 k_1^2 + a_5 k_2}{k_1}, a_7 = \frac{3}{4}\, \frac{-a_5 k_1^4 + 2\, a_1 k_1^2 k_2 + a_5 k_2^2}{k_1^2},$$
$$\left. a_8 = a_8, a_9 = \frac{a_1^2 + a_5^2}{k_1^2}, k_1 = k_1, k_2 = k_2, k_3 = -\frac{k_1^4 - 3\, k_2^2}{4 k_1}\right\},$$

where $\varepsilon = \pm 1$.

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