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Lump solutions to a generalized nonlinear PDE with four fourth-order terms

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Abstract: A combined fourth-order $(2+1)$-dimensional nonlinear partial differential equation which contains four fourth-order nonlinear terms and all second-order linear terms is formulated. This equation covers three generalized KP, Hirota–Satsuma–Ito, and Calogero–Bogoyavlenskii–Schiff equations as examples, which have physical applications in the study of various nonlinear phenomena in nature. In terms of some settings of the coefficients, a class of lump solutions is constructed by the Hirota bilinear method and the solutions are calculated through the symbolic computation system of Maple. Meanwhile, the relation between the coefficients and the solution is explored. Two special lump solutions are generated by taking proper values for the involved coefficients and parameters, and their dynamic behaviors are studied, as illustrative examples. The primary advantage of the Hirota bilinear method is to transform a nonlinear equation into a bilinear one so that the targeted equation can be easily studied.

Keywords: Hirota bilinear method; lump solution; soliton; symbolic computation.

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1 Introduction

In recent years, lump solutions have attracted much attention while finding analytic solutions to nonlinear partial differential equations (PDEs) [1]. Originated from solving certain integrable equations, lump solutions have been introduced, which are a kind of the rational-function solutions localized in all directions in space [1]. Particular examples of lump solutions have been found for many integrable equations like Kadomtev–Petviashviki (KP) equation, which describes nonlinear wave motion in nature [1]. Various efficient approaches have been used to generate lump solutions to PDEs, such as the inverse scattering transform [2], the Hirota bilinear method [3], Darboux transformation [4]. Among them, the Hirota bilinear method plays a prominent role in constructing lump solutions to nonlinear PDEs [5].

Let a polynomial B determine a Hirota bilinear differential equation:

$$B(D_x, D_y, D_t) f \cdot f = 0, \tag{1}$$

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for a given (2 + 1)-dimensional partial differential equation:

$$P(u_t, u_x, u_y, \ldots) = 0 \tag{2}$$

under the logarithmic transformation: $u = 2(\ln f)_x$ or $u = 2(\ln f)_{xx}$. Here D_x, D_y, D_t are the Hirota bilinear derivatives [6], defined by

$$D_x^l D_y^n D_t^m f(x,y,t) \cdot g(x,y,t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m f(x,y,t) \cdot g(x',y',t')|_{x'=x,y'=y,t'=t}. \tag{3}$$

Based on the relation between the bilinear form (1) and the (2 + 1)-dimensional partial differential Eq. (2), if Eq. (1) can be solved, then Eq. (2) can be solved accordingly [1]. In this way, lump solutions to (2 + 1)-dimensional KP equation has been presented via symbolic computation with Maple [1].

Following this line of research, lump solutions for more nonlinear equations have been studied, for example, the B-type Kadomtsev–Petviashvili equation [7, 8], the Calogero–Bogoyavlenskii–Schiff equation [9, 10], the Hirota–Satsuma–Ito (HSI) equation [11] and so on.

A key step in the processes of getting a lump solution is to find positive quadratic function solutions to Hirota bilinear equations. It has been proved that the solution of a general Hirota bilinear equation with even order terms can be constructed by a positive quadratic function, which generates a lump solution to the corresponding nonlinear PDE [5]. This theory has enlarged the category of nonlinear PDEs that process lump solutions, for instance, a generalized KP equation (gKP) [5, 12], a generalized Bogoyavlensky–Konopelchenko (gBK) equation [13], a generalized Hirota–Satsuma–Ito (gHSI) equation [14], a generalized Calogero–Bogoyavlenskii–Schiff (gCBS) equation [10], etc. Most of these generalized forms were formulated by adding some new terms or changing the fixed coefficients into arbitrary ones. Furthermore [15], has studied lump solutions to a nonlinear PDE which contains two fourth-order nonlinear terms and all the possible second-order linear terms with arbitrary constant coefficients. Similarly constructed equations involved three fourth-order terms have been explored in [16, 17].

On the other hand, a lot of nonlinear phenomena can be described by interaction solutions between lump solutions and soliton solutions [18, 19] or other kinds of analytic solutions [20–25], including soliton type solutions to integrable equations [26–30] and various physical model equations (see, for example, [31–33]). There are some related researches even for nonlinear equations in (3 + 1)-dimensions [34–39], linear wave equation [40] and other kinds of lump solutions and solitons (see, for example, [41–44]). Since the interaction properties involve much more complicated mathematical computation, so those properties will not be discussed here.

In this paper, the following generalized nonlinear equation will be studied, which contains four fourth-order nonlinear terms and all second-order linear terms:

$$\begin{aligned} P(u) = {} & \alpha(6u_x u_{xx} + u_{xxxx}) + \beta[3(u_x u_t)_x + u_{xxxt}] + \lambda[3(u_x u_y)_x + u_{xxxy}] + \theta(u_t s_y + 2u_y s_t + u_{tyy})_x \\ & + \gamma_1 u_{yt} + \gamma_2 u_{xx} + \gamma_3 u_{xt} + \gamma_4 u_{xy} + \gamma_5 u_{yy} + \gamma_6 u_{tt} = 0, \end{aligned} \tag{4}$$

where $s_x = u_y$, the coefficients α, β, λ, and θ satisfy $\alpha^2 + \beta^2 + \lambda^2 + \theta^2 \neq 0$, but $\gamma_i, 1 \leq i \leq 6$, are arbitrary constants.

When $\alpha = 1$ and the other coefficients of the fourth-order terms are zero, Eq. (4) is reduced to:

$$(6u_x u_{xx} + u_{xxxx}) + \gamma_1 u_{yt} + \gamma_2 u_{xx} + \gamma_3 u_{xt} + \gamma_4 u_{xy} + \gamma_5 u_{yy} + \gamma_6 u_{tt} = 0. \tag{5}$$

This is a generalized KP (gKP) equation, which possesses a Hirota bilinear form:

$$\left(D_x^4 + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2\right) f \cdot f = 0, \tag{6}$$

under the logarithmic transformation $u = 2(\ln f)_{xx}$. Its lump solutions have been studied in [5].

When $\beta = 1$, $\alpha = \lambda = \theta = 0$, and $\gamma_6 = 0$, Eq. (4) is reduced to a generalized Hirota–Satsuma–Ito (gHSI) equation:

$$[3(u_x u_t)_x + u_{xxxt}] + \gamma_1 u_{yt} + \gamma_2 u_{xx} + \gamma_3 u_{xt} + \gamma_4 u_{xy} + \gamma_5 u_{yy} = 0. \tag{7}$$

It possesses a Hirota bilinear form:

$$\left(D_x^3 D_t + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2\right) f \cdot f = 0, \tag{8}$$

under the logarithmic transformation $u = 2(\ln f)_x$. Its lump solutions have been studied in [14].

When $\lambda = 1$, $\alpha = \beta = \theta = 0$, $\gamma_3 = 1$, and $\gamma_1 = \gamma_2 = \gamma_6 = 0$, Eq. (4) is reduced to a generalized Calogero–Bogoyavlenskii–Schiff (gCBS) equation:

$$[3(u_x u_y)_x + u_{xxxy}] + u_{xt} + \gamma_4 u_{xy} + \gamma_5 u_{yy} = 0. \tag{9}$$

It possesses a Hirota bilinear form:

$$\left(D_x^3 D_y + D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2\right) f \cdot f = 0, \tag{10}$$

under the logarithmic transformation $u = 2(\ln f)_{xx}$. Its lump solutions have been explored in [10].

Under the logarithmic transformation $u = 2(\ln f)_x$, Eq. (4) possesses a Hirota bilinear form:

$$\begin{aligned} B(f) = \Big(&\alpha D_x^4 + \beta D_x^3 D_t + \lambda D_x^3 D_y + \theta D_x D_y^2 D_t + \gamma_1 D_y D_t \\ &+ \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2\Big) f \cdot f = 0. \end{aligned} \tag{11}$$

Because of the relationship: $P(u) = (\frac{B(f)}{f^2})_x$, a solution of the Hirota bilinear Eq. (11) will generate a lump solution to the nonlinear partial differential Eq. (4) accordingly.

When $\alpha = 0$, Eq. (11) becomes:

$$\left(\beta D_x^3 D_t + \lambda D_x^3 D_y + \theta D_x D_y^2 D_t + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2\right) f \cdot f = 0, \tag{12}$$

which has been studied in [17].

When $\theta = 0$, Eq. (11) becomes:

$$\left(\alpha D_x^4 + \beta D_x^3 D_t + \lambda D_x^3 D_y + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2\right) f \cdot f = 0, \tag{13}$$

which has been studied in paper [16].

For the subcase of $\alpha = \theta = 0$, Eq. (11) becomes:

$$\left(\beta D_x^3 D_t + \lambda D_x^3 D_y + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2 + \gamma_6 D_t^2\right) f \cdot f = 0, \tag{14}$$

which has been studied in paper [15].

So, in this article, both $\alpha \neq 0$ and $\theta \neq 0$ are assumed to explore the lump solutions of Eq. (4), which has four fourth-order terms and all the second-order terms, and presents a new model equation that has not been studied before.

In Section 2, lump solutions to Eq. (4) are constructed via a positive quadratic form and calculated through the Maple symbolic system. Meanwhile, the relation between the coefficients and the solutions will be discussed. In Section 3, two specific lump solutions by taking proper values on the involved coefficients and parameters are presented along with their dynamical behaviors. The conclusion is given in Section 4.

2 Lump solutions

In this section, a class of lump solutions of the generalized fourth-order Eq. (4) will be determined under the setting: $\alpha \neq 0$ and $\theta \neq 0$. Actually, the corresponding Hirota bilinear form (11) will be solved by constructing its solution in a positive quadratic form [5] as follows:

$$f = (a_1x + a_2y + a_3t + a_4)^2 + (a_5x + a_6y + a_7t + a_8)^2 + a_9, \tag{15}$$

where $a_i, 1 \leq i \leq 9$, are real constant parameters to be determined.

To get solutions of the above parameters, Eq. (11) is changed into a normal partial differential equation under the definition of Hirota bilinear derivative (3). Then, formula (15) is inserted into this normal PDE to yield a system of algebraic equations on $a_i, 1 \leq i \leq 9$, and the resulting system is solved. Most of these computations are processed through Maple symbolic computation system except for some variable transformation and necessary simplification of the results. In order to facilitate the calculation and expression of the results, some setting on $\gamma_i, 1 \leq i \leq 6$ in Eq. (11) are needed to make, such as $\gamma_5 = 0$, or $\gamma_6 = 0$, to go through the computation process.

Firstly, let $\gamma_6 = 0$. Equation (11) becomes:

$$B(f) = \left(\alpha D_x^4 + \beta D_x^3 D_t + \lambda D_x^3 D_y + \theta D_x D_y^2 D_t + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2\right) f \cdot f = 0. \tag{16}$$

After some calculations, a set of solutions for the parameters is obtained:

$$\begin{cases} a_3 = -\dfrac{b_1}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}, \\ a_7 = -\dfrac{b_2}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}, \\ a_9 = -\dfrac{3\left(a_1^2 + a_5^2\right)(b_3\alpha - b_4\beta + b_5\lambda) - b_6\theta}{(a_1a_6 - a_2a_5)^2\left(\gamma_1^2\gamma_2 - \gamma_1\gamma_3\gamma_4 + \gamma_3^2\gamma_5\right)}. \end{cases} \tag{17}$$

Here, all the other a_i are arbitrary and the constants $b_i, 1 \leq i \leq 6$, are defined as follows:

$$\begin{cases} b_1 = \left[(a_1^2a_2 + 2a_1a_5a_6 - a_2a_5^2)\gamma_2 + a_1\left(a_2^2 + a_6^2\right)\gamma_4 + a_2\left(a_2^2 + a_6^2\right)\gamma_5\right]\gamma_1 \\ \quad + \left[a_1\left(a_1^2 + a_5^2\right)\gamma_2 + a_2\left(a_1^2 + a_5^2\right)\gamma_4 + \left(a_1a_2^2 + 2a_2a_5a_6 - a_1a_6^2\right)\gamma_5\right]\gamma_3, \\ b_2 = \left[(-a_1^2a_6 + 2a_1a_2a_5 + a_5^2a_6)\gamma_2 + a_5\left(a_2^2 + a_6^2\right)\gamma_4 + a_6\left(a_2^2 + a_6^2\right)\gamma_5\right]\gamma_1 \\ \quad + \left[a_5\left(a_1^2 + a_5^2\right)\gamma_2 + a_6\left(a_1^2 + a_5^2\right)\gamma_4 + \left(-a_2^2a_5 + 2a_1a_2a_6 + a_5a_6^2\right)\gamma_5\right]\gamma_3, \\ b_3 = \left(a_1^2 + a_5^2\right)\left[(a_1\gamma_3 + a_2\gamma_1)^2 + (a_5\gamma_3 + a_6\gamma_1)^2\right], \\ b_4 = \left(a_1^2 + a_5^2\right)(a_1a_2 + a_5a_6)(\gamma_1\gamma_2 + \gamma_3\gamma_4) + \left(a_1^2 + a_5^2\right)\left(a_2^2 + a_6^2\right)\gamma_1\gamma_4 \\ \quad + \left(a_1^2 + a_5^2\right)^2\gamma_2\gamma_3 + \left(a_2^2 + a_6^2\right)(a_1a_2 + a_5a_6)\gamma_1\gamma_5 \\ \quad + \left[(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2\right]\gamma_3\gamma_5, \\ b_5 = (a_1a_2 + a_5a_6)\left[(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2\right], \\ b_6 = \left[3(a_1a_2 + a_5a_6)^2 - (a_1a_6 - a_2a_5)^2\right]\left[(a_1a_2 + a_5a_6)\gamma_1\gamma_2 + \left(a_2^2 + a_6^2\right)\gamma_3\gamma_5\right] \\ \quad + \left[3(a_1a_2 + a_5a_6)^2 + (a_1a_6 - a_2a_5)^2\right]\left[\left(a_2^2 + a_6^2\right)\gamma_1\gamma_4 + \left(a_1^2 + a_5^2\right)\gamma_2\gamma_3\right] \\ \quad + 3\left(a_2^2 + a_6^2\right)(a_1a_2 + a_5a_6)\left[\left(a_2^2 + a_6^2\right)\gamma_1\gamma_5 + \left(a_1^2 + a_5^2\right)\gamma_3\gamma_4\right]. \end{cases} \tag{18}$$

To see the affection of the coefficients in the Hirota bilinear form (11) on the parameters of f in the positive quadratic form (15), it turns out that a_3 and a_7 are just affected by $\gamma_i, 1 \leq i \leq 6$, the coefficients of the second-order terms in Eq. (11), while all the coefficients of the fourth-order terms such as α, β, λ, and θ only appear in a_9 (i.e., only a_9 contains the coefficients of the fourth-order terms). This phenomenon also happens in paper [15–17]. Furthermore, because Eqs. (11)–(14) have the same structure in the second-order terms, our a_3 and a_7 are consistent with the corresponding results in [15–17].

As to a_9, its expression has the same structure as the one in the compared papers [15–17], or it can finally be simplified into a fraction with a consistent denominator and a similar numerator. In detail, in these four Eqs. (11)–(14), the same fourth-order term may affect a_9 in the same way. For example, because the term $D_x^3 D_t$ has been involved in all these four equations, the expressions of a_9, which contains the coefficient of $D_x^3 D_t$, are consistent with each other. So, if our result of a_9 is compared with the corresponding result in the paper [16], a new term involved θ is just added, which is the coefficient of the extended term $D_x D_y^2 D_t$, in the numerator of the expression a_9 made in [16].

Secondly, let $\gamma_5 = 0$. Equation (11) becomes:

$$B(f) = \left(\alpha D_x^4 + \beta D_x^3 D_t + \lambda D_x^3 D_y + \theta D_x D_y^2 D_t + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_6 D_t^2\right) f \cdot f = 0. \tag{19}$$

After some calculations, the following expressions are obtained:

$$\begin{cases} a_2 = -\dfrac{c_1}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}, \\ a_6 = -\dfrac{c_2}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}, \\ a_9 = -\dfrac{3\left(a_1^2 + a_5^2\right)(c_3\alpha + c_4\beta - c_5\lambda)}{(a_1a_7 - a_3a_5)^2\left(\gamma_1^2\gamma_2 - \gamma_1\gamma_3\gamma_4 + \gamma_4^2\gamma_6\right)} \\ \qquad - \dfrac{c_6\theta}{(a_1a_7 - a_3a_5)^2\left[(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2\right]\left(\gamma_1^2\gamma_2 - \gamma_1\gamma_3\gamma_4 + \gamma_4^2\gamma_6\right)}, \end{cases} \tag{20}$$

where all the other a_i are arbitrary constants. The involved constants $c_i, 1 \le i \le 6$, are defined as follows:

$$\begin{cases} c_1 = \left[(a_1^2a_3 + 2a_1a_5a_7 - a_3a_5^2)\gamma_2 + a_1\left(a_3^2 + a_7^2\right)\gamma_3 + a_3\left(a_3^2 + a_7^2\right)\gamma_6\right]\gamma_1 \\ \qquad + \left[a_1\left(a_1^2 + a_5^2\right)\gamma_2 + a_3\left(a_1^2 + a_5^2\right)\gamma_3 + \left(a_1a_3^2 + 2a_3a_5a_7 - a_1a_7^2\right)\gamma_6\right]\gamma_4, \\ c_2 = \left[(-a_1^2a_7 + 2a_1a_3a_5 + a_5^2a_7)\gamma_2 + a_5\left(a_3^2 + a_7^2\right)\gamma_3 + a_7\left(a_3^2 + a_7^2\right)\gamma_6\right]\gamma_1 \\ \qquad + \left[a_5\left(a_1^2 + a_5^2\right)\gamma_2 + a_7\left(a_1^2 + a_5^2\right)\gamma_3 + \left(-a_3^2a_5 + 2a_1a_3a_7 + a_5a_7^2\right)\gamma_6\right]\gamma_4, \\ c_3 = \left(a_1^2 + a_5^2\right)\left[(a_1\gamma_4 + a_3\gamma_1)^2 + (a_5\gamma_4 + a_7\gamma_1)^2\right], \\ c_4 = (a_1a_3 + a_5a_7)\left[(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2\right], \\ c_5 = \left(a_1^2 + a_5^2\right)(a_1a_3 + a_5a_7)(\gamma_1\gamma_2 + \gamma_3\gamma_4) + \left(a_1^2 + a_5^2\right)\left(a_3^2 + a_7^2\right)\gamma_1\gamma_3 \\ \qquad + \left(a_1^2 + a_5^2\right)^2\gamma_2\gamma_4 + \left(a_3^2 + a_7^2\right)(a_1a_3 + a_5a_7)\gamma_1\gamma_6 \\ \qquad + \left[(a_1a_3 + a_5a_7)^2 - (a_1a_7 - a_3a_5)^2\right]\gamma_4\gamma_6, \\ c_6 = d_1\left[2d_3^2\gamma_1\gamma_2^2\gamma_4 + d_3d_4\left(\gamma_1^2\gamma_2^2 + 2\gamma_2\gamma_4^2\gamma_6\right) + 2d_3d_5\left(\gamma_1^2\gamma_2\gamma_3 + \gamma_3\gamma_4^2\gamma_6\right)\right. \\ \qquad \left. + d_4d_5\left(2\gamma_1^2\gamma_2\gamma_6 + \gamma_4^2\gamma_6^2\right) + 2d_5^2\gamma_1\gamma_4\gamma_6^2\right] + d_2\left[2d_3^2\gamma_2\gamma_3\gamma_4^2\right. \\ \qquad \left. + 4d_3d_4\gamma_1\gamma_2\gamma_3\gamma_4 + 2d_3d_5\gamma_1\gamma_3^2\gamma_4 + 4d_4d_5\gamma_1\gamma_3\gamma_4\gamma_6 + 2d_5^2\gamma_1^2\gamma_3\gamma_6\right] \\ \qquad + 3d_4\left[d_3^3\gamma_2^2\gamma_4^2 + d_3^2d_5\gamma_3^2\gamma_4^2 + d_3d_5^2\gamma_1^2\gamma_3^2 + d_5^3\gamma_1^2\gamma_6^2\right] \\ \qquad + \left[a_1^4\left(3a_3^4 + a_7^4\right) + 4a_1a_3a_5a_7\left(3a_1^2a_3^2 - a_1^2a_7^2 + 6a_1a_3a_5a_7\right.\right. \\ \qquad \left.\left. - a_3^2a_5^2 + 3a_5^2a_9^2\right) + a_5^4\left(a_3^4 + 3a_7^4\right)\right]\gamma_1\gamma_2\gamma_4\gamma_6. \end{cases} \tag{21}$$

The parameters $d_i, 1 \le i \le 5$, involved in c_6 are defined as follows:

$$\begin{cases} d_1 = \left[3(a_1a_3 + a_5a_7)^2 - (a_1a_7 - a_3a_5)^2\right], \\ d_2 = \left[3(a_1a_3 + a_5a_7)^2 + (a_1a_7 - a_3a_5)^2\right], \\ d_3 = \left(a_1^2 + a_5^2\right), \\ d_4 = (a_1a_3 + a_5a_7), \\ d_5 = \left(a_3^2 + a_7^2\right). \end{cases} \tag{22}$$

Similarly, a_2 and a_6 in the result (20) are just affected by the coefficients of the second-order terms while all the coefficients of the fourth-order terms such as α, β, λ, and θ only affect a_9.

So in each case, $\gamma_6 = 0$ or $\gamma_5 = 0$, upon comparing our resulting parameters (17) or (20) with the related results in [15–17], a change just happens in a_9, the constant term of the positive quadratic form of solution (15), when some new fourth-order terms are added in Eqs. (12)–(14).

For the case of $\gamma_5 = 0$, the expression $a_1a_6 - a_2a_5 \neq 0$ needs to be checked to make sure the solution f defined by (15) with parameters (20) to generate a lump solution. That is:

$$a_1a_6 - a_2a_5 = \frac{(a_1a_7 - a_3a_5)\left[\left(a_1^2 + a_5^2\right)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - \left(a_3^2 + a_7^2\right)\gamma_1\gamma_6 - (a_1a_3 + a_5a_7)\gamma_4\gamma_6\right]}{(a_3\gamma_1 + a_1\gamma_4)^2 + (a_7\gamma_1 + a_5\gamma_4)^2}.$$

So it follows that $a_1a_6 - a_2a_5 \neq 0$ if and only if

$$\begin{cases} a_1a_7 - a_3a_5 \neq 0, \gamma_1^2 + \gamma_4^2 \neq 0 \\ (a_1^2 + a_5^2)(\gamma_1\gamma_2 - \gamma_3\gamma_4) - \left(a_3^2 + a_7^2\right)\gamma_1\gamma_6 - (a_1a_3 + a_5a_7)\gamma_4\gamma_6 \neq 0. \end{cases} \tag{23}$$

Besides, a_9, as the constant term of the positive quadratic function (15), should be positive to guarantee that (15) with the above resulting parameters (20) will present lump solutions.

3 Two specific lump solutions and their graphs

In this section, two specific lump solutions will be presented by taking proper values for the coefficients in Eq. (4) and their graphs will be made through Maple. Most of the following values taken on the involved coefficients and free parameters are found by trial-and-error method to guarantee the positivity of the constant term a_9.

Firstly, the following case is taken:

$$\alpha = \beta = \lambda = \theta = 1, \gamma_3 = \gamma_5 = 1, \gamma_1 = \gamma_2 = \gamma_4 = \gamma_6 = 0. \tag{24}$$

Equation (4) is reduced to

$$6u_xu_{xx} + u_{xxxx} + 3(u_xu_t)_x + u_{xxxt} + 3(u_xu_y)_x + u_{xxxy} + (u_ts_y + 2u_ys_t + u_{tyy})_x + u_{xt} + u_{yy} = 0, \tag{25}$$

which has a Hirota bilinear form:

$$\left(D_x^4 + D_x^3D_t + D_x^3D_y + D_xD_y^2D_t + D_xD_t + D_y^2\right) f \cdot f = 0. \tag{26}$$

Upon substituting the above values (24) into the resulting parameters (17), associated with the special values of the free parameters:

$$a_1 = 2, \quad a_2 = -1, \quad a_4 = 8, \quad a_5 = 1, \quad a_6 = -1, \quad a_8 = 4, \tag{27}$$

it is obtained that $a_3 = -\frac{2}{5}, a_7 = -\frac{4}{5}, a_9 = 22$. Putting all the $a_i, 1 \leq i \leq 9$, into formula (15) leads to the positive quadratic form solution of Eq. (26)

$$f_1 = \left(2x - y - \frac{2}{5}t + 8\right)^2 + \left(x - y - \frac{4}{5}t + 4\right)^2 + 22. \tag{28}$$

Under the logarithmic transformation: $u = 2(\ln f)_x$, the following lump solution of the special fourth-order nonlinear Eq. (25) is generated:

$$u_1 = \frac{2\left(10x - 6y - \frac{16}{5}t + 40\right)}{\left(2x - y - \frac{2}{5}t + 8\right)^2 + \left(x - y - \frac{4}{5}t + 4\right)^2 + 22}. \tag{29}$$

Made by Maple plot tools, the three-dimensional plots and contour plots of this lump solution (29) are showed in Figure 1.

Secondly, the following case is taken:

$$\alpha = \beta = \lambda = \theta = 1, \quad \gamma_4 = \gamma_6 = 1, \quad \gamma_1 = \gamma_2 = \gamma_3 = \gamma_5 = 0. \tag{30}$$

to get another fourth-order nonlinear equation:

$$6u_x u_{xx} + u_{xxxx} + 3(u_x u_t)_x + u_{xxxt} + 3(u_x u_y)_x + u_{xxxy} + (u_t s_y + 2u_y s_t + u_{tyy})_x + u_{xy} + u_{tt} = 0. \tag{31}$$

which has a Hirota bilinear form:

$$\left(D_x^4 + D_x^3 D_t + D_x^3 D_y + D_x D_y^2 D_t + D_x D_y + D_t^2\right) f \cdot f = 0. \tag{32}$$

Substituting the above values (30) into the resulting parameters (20), associated with the special values of the free parameters:

$$a_1 = 1, \quad a_3 = -1, \quad a_4 = 5, \quad a_5 = -1, \quad a_7 = 2, \quad a_8 = -3, \tag{33}$$

tells that $a_2 = -\frac{1}{2}$, $a_6 = \frac{7}{2}$, $a_9 = 255$. The corresponding f defined by (15) reads as follows:

$$f_2 = \left(x - \frac{1}{2}y - t + 5\right)^2 + \left(-x + \frac{7}{2}y + 2t - 3\right)^2 + 255, \tag{34}$$

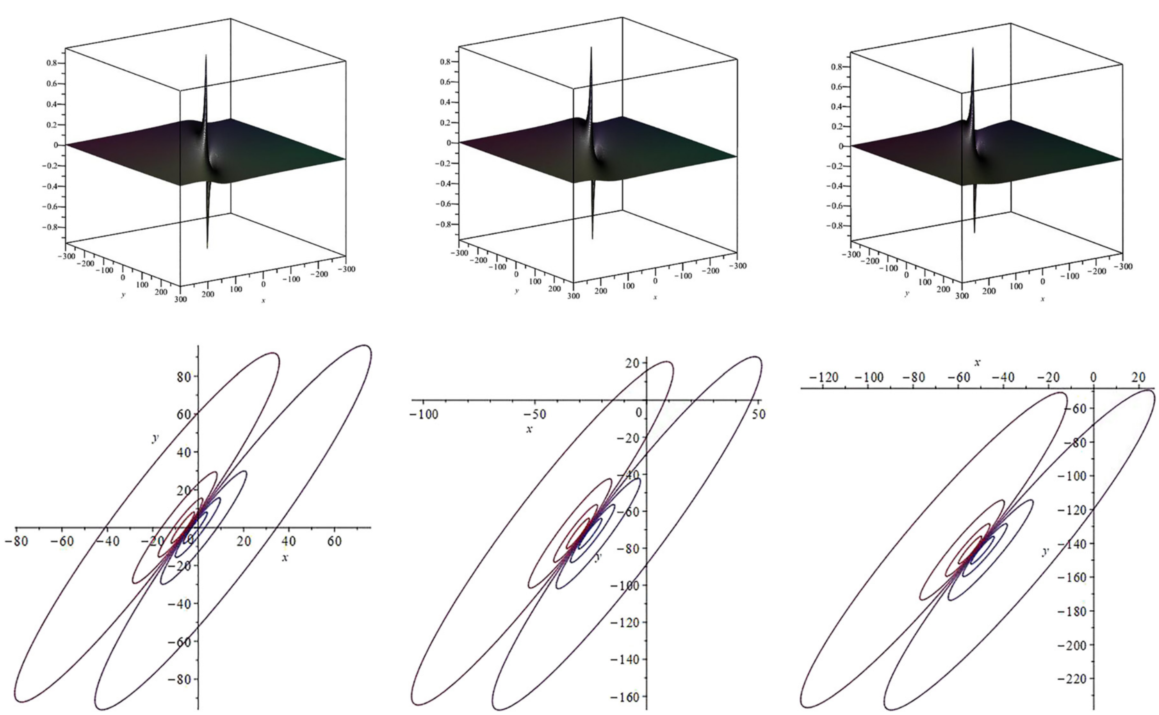

Figure 1: Profiles of u_1 when $t = 0, 60, 120$: 3d plots (top) and contour plots (bottom).

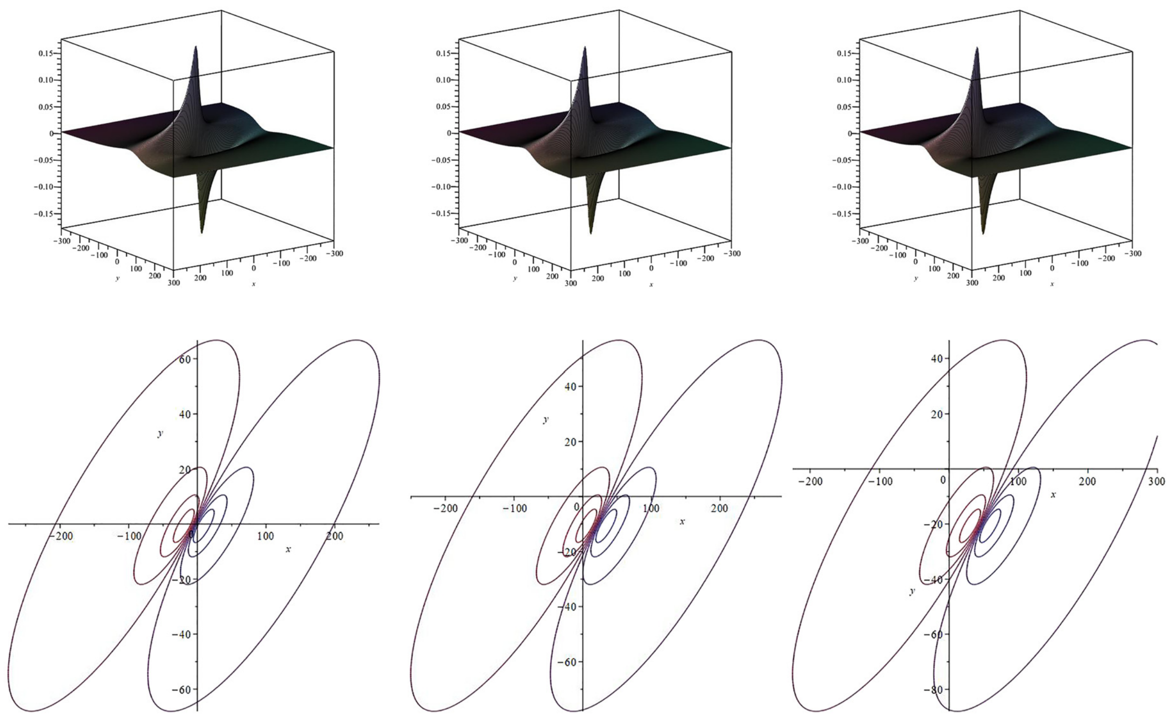

Figure 2: Profiles of u_2 when $t = 0, 30, 60$: 3d plots (top) and contour plots (bottom).

which provides a lump solution of the special fourth-order nonlinear Eq. (32):

$$u_2 = \frac{2(4x - 8y - 6t + 16)}{\left(x - \frac{1}{2}y - t + 5\right)^2 + \left(-x + \frac{7}{2}y + 2t - 3\right)^2 + 255}, \tag{35}$$

under the logarithmic transformation.

Figure 2 displays the three-dimensional plots and contour plots of this lump solution made through Maple.

4 Conclusions

In this article, a new generalized fourth-order (2 + 1)-dimensional nonlinear partial differential equation with four fourth-order terms has been studied. This equation covers several general equations related to the study of various nonlinear phenomena in nature. This is the first time that an equation with 4 fourth-order terms is studied. We analyzed the choices of the parameters to find when a lump solution would occur in this equation. By the Hirota bilinear method, a positive quadratic form of solution (15) has been constructed to generate the lump solutions to this Eq. (4) under the setting of $\gamma_5 = 0$ or $\gamma_6 = 0$. The resulting parameters (17) and (20) turn out that just a_9, the constant term of the positive quadratic form of solution (15), has been affected by the coefficients of the fourth-order terms of this generalized fourth-order nonlinear PDE (4), while the other related parameters are mainly determined by the coefficients of the second-order terms. As illustrative examples, two specific lump solutions have been generated by taking particular values on the involved coefficients in Eq. (4), as well as their dynamical behaviors have been studied.

However, more work needs to be done on the other setting of $\gamma_i, 1 \leq i \leq 6$, in this generalized fourth-order nonlinear PDE (4), because of the difficulty in presenting a simplified expression for the resulting parameters, which could be achieved through the use of Maple or MatLab. Furthermore, if more fourth-order terms are added to the considered nonlinear equation, what effect of the new terms could have on lump solutions? This is another interesting question to be explored in the future.

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