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Higher-Order Matrix Spectral Problems and Their Integrable Hamiltonian Hierarchies

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Abstract: Starting from a kind of higher-order matrix spectral problems, we generate integrable Hamiltonian hierarchies through the zero-curvature formulation. To guarantee the Liouville integrability of the obtained hierarchies, the trace identity is used to establish their Hamiltonian structures. Illuminating examples of coupled nonlinear Schrödinger equations and coupled modified Korteweg–de Vries equations are worked out.

Keywords: Lax pair; zero-curvature equation; integrable hierarchy NLS equations; mKdV equations

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1. Introduction

Constructing integrable equations is a challenging and active area of research in mathematical physics, requiring a combination of mathematical insight and technical expertise. Integrable equations often have a rich array of analytic and geometric structures, and the study of their solutions can reveal new and unexpected connections to other areas of mathematical physics.

There are various approaches to constructing integrable equations. One common method is to apply Lax pairs of matrix spectral problems [1–3]. The existence of recursion structures behind matrix spectral problems leads to integrable hierarchies, the members of which commute with each other and so present integrable equations.

Let us consider a vector potential: $u = (u_1, \cdots, u_q)^T$ and assume that λ is the spectral parameter. The standard procedure for using Lax pairs is to start from a loop algebra $\tilde{g}$ to formulate a spatial spectral matrix:

$$U = U(u, \lambda) = e_0(\lambda) + u_1 e_1(\lambda) + \cdots + u_q e_q(\lambda), \tag{1}$$

where $e_1, \cdots, e_q$ are linear independent, and e_0 is a pseudoregular element in the loop algebra $\tilde{g}$, which satisfies

$$\mathrm{Ker}\,\mathrm{ad}_{e_0} \oplus \mathrm{Im}\,\mathrm{ad}_{e_0} = \tilde{g}, \ [\mathrm{Ker}\,\mathrm{ad}_{e_0}, \mathrm{Ker}\,\mathrm{ad}_{e_0}] = 0.$$

These properties guarantee that there will be a Laurent series solution $Z = \sum_{s \geq 0} \lambda^{-s} Z^{[s]}$ to the stationary zero-curvature equation:

$$Z_x = i[U, Z]. \tag{2}$$

Once a spatial matrix spectral problem is determined, an integrable hierarchy can be derived via a hierarchy of zero-curvature equations:

$$U_t - V_x^{[r]} + i[U, V^{r]}] = 0,\ r \geq 0. \tag{3}$$

These zero-curvature equations are the compatibility conditions between the spatial and temporal matrix spectral problems:

$$-i\phi_x = U\phi,\ -i\phi_t = V^{[r]}\phi,\ r \geq 0, \tag{4}$$

and so they will present integrable equations. A basic tool to show the Liouville integrability is the trace identity [4,5]:

$$\frac{\delta}{\delta u}\int \mathrm{tr}(Z\frac{\partial U}{\partial \lambda})dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\mathrm{tr}(Z\frac{\partial U}{\partial u}), \tag{5}$$

where $\frac{\delta}{\delta u}$ denotes the variational derivative with respect to u and γ is the constant:

$$\gamma = -\frac{\lambda}{2}\frac{\partial}{\partial \lambda}\ln|\mathrm{tr}(Z^2)|.$$

Various integrable hierarchies are constructed through the zero-curvature formulation, based on special linear algebras (see, e.g., [2,6–15]) and special orthogonal algebras (see, e.g., [16–19]). Bi-Hamiltonian structures can often be found, which show the Liouville integrability of the associated zero-curvature equations [20]. There are many integrable hierarchies with two scalar potentials, p and q. The following four spectral matrices:

$$U = \begin{bmatrix} \lambda & p \\ q & -\lambda \end{bmatrix},\ U = \begin{bmatrix} \lambda^2 & \lambda p \\ \lambda q & -\lambda^2 \end{bmatrix},\ U = \begin{bmatrix} \lambda & \lambda p \\ \lambda q & -\lambda \end{bmatrix},\ U = \begin{bmatrix} \lambda v & \lambda p \\ \lambda q & -\lambda v \end{bmatrix},$$

where $pq + v^2 = 1$, generate the Ablowitz–Kaup–Newell–Segur hierarchy [2], the Kaup–Newell hierarchy [21], the Wadati–Konno–Ichikawa hierarchy [22] and the Heisenberg hierarchy [23], respectively. The four counterparts of spectral matrices associated with so(3,$\mathbb{R}$) are

$$U = \begin{bmatrix} 0 & q & -\lambda \\ q & 0 & -p \\ \lambda & p & 0 \end{bmatrix},\ U = \begin{bmatrix} 0 & -\lambda q & -\lambda^2 \\ \lambda q & 0 & -\lambda p \\ \lambda^2 & \lambda p & 0 \end{bmatrix},$$

$$U = \begin{bmatrix} 0 & -\lambda q & -\lambda \\ \lambda q & 0 & -\lambda p \\ \lambda & \lambda p & 0 \end{bmatrix},\ U = \begin{bmatrix} 0 & -\lambda q & -\lambda v \\ \lambda q & 0 & -\lambda p \\ \lambda v & \lambda p & 0 \end{bmatrix},$$

where $p^2 + q^2 + v^2 = 1$.

This paper aims to construct integrable hierarchies of four-component equations through the zero-curvature formulation. Hamiltonian structures of the resulting hierarchies are provided by using the trace identity. Two illuminating examples are four-component nonlinear Schrödinger type equations and four-component modified Korteweg–de Vries type equations. A conclusion is provided, together with concluding remarks, in the final section.

2. Seventh-Order Matrix Spectral Problems and an Integrable Hierarchy

Within the zero-curvature formulation, let us introduce a seventh-order matrix spatial spectral problem:

$$-i\phi_x = U\phi = U(u,\lambda)\phi,\ U = \begin{bmatrix} \lambda & p_1 & p_1 & p_2 & p_1 & p_1 & 0 \\ q_1 & 0 & 0 & 0 & 0 & 0 & p_1 \\ q_1 & 0 & 0 & 0 & 0 & 0 & p_1 \\ q_2 & 0 & 0 & 0 & 0 & 0 & p_2 \\ q_1 & 0 & 0 & 0 & 0 & 0 & p_1 \\ q_1 & 0 & 0 & 0 & 0 & 0 & p_1 \\ 0 & q_1 & q_1 & q_2 & q_1 & q_1 & -\lambda \end{bmatrix}, \tag{6}$$

where u is the four-dimensional potential

$$u = u(x,t) = (p_1, p_2, q_1, q_2)^T. \tag{7}$$

This spectral problem is different from the matrix Ablowitz–Kaup–Newell–Segur spectral problem and its reductions (see, e.g., [24,25]), and it was determined by a machine learning process with Maple symbolic computation.

In order to construct an associated integrable hierarchy, we start by solving the stationary zero-curvature equation (2) by looking for a Laurent series solution:

$$Z = \begin{bmatrix} a & b_1 & b_1 & b_2 & b_1 & b_1 & 0 \\ c_1 & 0 & 0 & d & 0 & 0 & b_1 \\ c_1 & 0 & 0 & d & 0 & 0 & b_1 \\ c_2 & -d & -d & 0 & -d & -d & b_2 \\ c_1 & 0 & 0 & d & 0 & 0 & b_1 \\ c_1 & 0 & 0 & d & 0 & 0 & b_1 \\ 0 & c_1 & c_1 & c_2 & c_1 & c_1 & -a \end{bmatrix} = \sum_{s\geq 0} \lambda^{-s} Z^{[s]}, \tag{8}$$

with

$$a = \sum_{s\geq 0} \lambda^{-s} a^{[s]},\ b_j = \sum_{s\geq 0} \lambda^{-s} b_j^{[s]},\ c_j = \sum_{s\geq 0} \lambda^{-s} c_j^{[s]},\ d = \sum_{s\geq 0} \lambda^{-s} d^{[s]}, \tag{9}$$

where $j = 1, 2$. Obviously, the corresponding stationary zero-curvature equation (2) engenders the initial conditions:

$$a_x^{[0]} = 0,\ b_1^{[0]} = b_2^{[0]} = c_1^{[0]} = c_2^{[0]} = 0,\ d_x^{[0]} = 0, \tag{10}$$

and the recursion relation:

$$\begin{cases} b_1^{[s+1]} = -ib_{1,x}^{[s]} + p_1 a^{[s]} + p_2 d^{[s]}, \\ b_2^{[s+1]} = -ib_{2,x}^{[s]} + p_2 a^{[s]} - 4p_1 d^{[s]}, \end{cases} \tag{11}$$

$$\begin{cases} c_1^{[s+1]} = ic_{1,x}^{[s]} + q_1 a^{[s]} - q_2 d^{[s]}, \\ c_2^{[s+1]} = ic_{2,x}^{[s]} + q_2 a^{[s]} + 4q_1 d^{[s]}, \end{cases} \tag{12}$$

and

$$\begin{cases} d_x^{[s+1]} = i(q_1 b_2^{[s+1]} - q_2 b_1^{[s+1]} + p_1 c_2^{[s+1]} - p_2 c_1^{[s+1]}), \\ a_x^{[s+1]} = i(-4q_1 b_1^{[s+1]} - q_2 b_2^{[s+1]} + 4p_1 c_1^{[s+1]} + p_2 c_2^{[s+1]}) \\ \qquad = -(4q_1 b_{1,x}^{[s]} + q_2 b_{2,x}^{[s]} + 4p_1 c_{1,x}^{[s]} + p_2 c_{2,x}^{[s]}), \end{cases} \tag{13}$$

where $s \geq 0$. We take the initial values,

$$a^{[0]} = 1,\ d^{[0]} = 0, \tag{14}$$

and choose the constant of integration as zero,

$$a^{[s]}|_{u=0} = 0,\ d^{[s]}|_{u=0} = 0,\ s \geq 1. \tag{15}$$

Then, we can work out the first four sets of $a^{[s]}, b_1^{[s]}, b_2^{[s]}, c_1^{[s]}, c_2^{[s]}$ and $d^{[s]}$:

$$b_1^{[1]} = p_1,\ b_2^{[1]} = p_2,\ c_1^{[1]} = q_1,\ c_2^{[1]} = q_2,\ a^{[1]} = 0,\ d^{[1]} = 0;$$

$$\begin{cases} b_1^{[2]} = -ip_{1,x},\ b_2^{[2]} = -ip_{2,x},\ c_1^{[2]} = iq_{1,x},\ c_2^{[2]} = iq_{2,x}, \\ a^{[2]} = -4p_1q_1 - p_2q_2,\ d^{[2]} = -p_1q_2 + p_2q_1; \end{cases}$$

$$\begin{cases} b_1^{[3]} = -p_{1,xx} - 4p_1^2q_1 - 2p_1p_2q_2 + p_2^2q_1, \\ b_2^{[3]} = -p_{2,xx} + 4p_1^2q_2 - 8p_1p_2q_1 - p_2^2q_2, \end{cases}$$

$$\begin{cases} c_1^{[3]} = -q_{1,xx} - 4p_1q_1^2 + p_1q_2^2 - 2p_2q_1q_2, \\ c_2^{[3]} = -q_{2,xx} - 8p_1q_1q_2 + 4p_2q_1^2 - p_2q_2^2, \end{cases}$$

$$\begin{cases} a^{[3]} = -i(4p_1q_{1,x} - 4p_{1,x}q_1 + p_2q_{2,x} - p_{2,x}q_2), \\ d^{[3]} = -i(p_1q_{2,x} - p_2q_{1,x} - p_{1,x}q_2 + p_{2,x}q_1); \end{cases}$$

and

$$\begin{cases} b_1^{[4]} = i(p_{1,xxx} + 12p_1p_{1,x}q_1 + 3p_1p_{2,x}q_2 - 3p_2p_{2,x}q_1 + 3p_{1,x}p_2q_2), \\ b_2^{[4]} = i(p_{2,xxx} + 12p_1p_{2,x}q_1 - 12p_1p_{1,x}q_2 + 12p_{1,x}p_2q_1 + 3p_2p_{2,x}q_2), \end{cases}$$

$$\begin{cases} c_1^{[4]} = -i(q_{1,xxx} + 12p_1q_1q_{1,x} - 3p_1q_2q_{2,x} + 3p_2q_1q_{2,x} + 3p_2q_{1,x}q_2), \\ c_2^{[4]} = -i(q_{2,xxx} + 12p_1q_1q_{2,x} + 12p_1q_{1,x}q_2 - 12p_2q_1q_{1,x} + 3p_2q_2q_{2,x}), \end{cases}$$

$$\begin{cases} a^{[4]} = 24p_1^2q_1^2 - 6p_1^2q_2^2 + 24p_1p_2q_1q_2 - 6p_2^2q_1^2 + \frac{3}{2}p_2^2q_2^2 \\ \qquad +4p_1q_{1,xx} + 4p_{1,x,x}q_1 + p_2q_{2,xx} + p_{2,xx}q_2 - 4p_{1,x}q_{1,x} - p_{2,x}q_{2,x}, \\ d^{[4]} = 12(p_1q_1 + \frac{1}{4}p_2q_2)(p_1q_2 - p_2q_1) + p_{1,xx}q_2 - p_{2,xx}q_1 \\ \qquad -p_2q_{1,xx} + p_1q_{2,xx} - p_{1,x}q_{2,x} + p_{2,x}q_{1,x}. \end{cases}$$

Now, we can introduce the temporal matrix spectral problems:

$$-i\phi_t = V^{[r]}\phi = V^{[r]}(u,\lambda)\phi,\ V^{[r]} = (\lambda^r Z)_+ = \sum_{s=0}^{r} \lambda^s Z^{[r-s]},\ r \geq 0, \quad (16)$$

which are the other parts of the Lax pairs of matrix spectral problems in the zero-curvature formulation. The compatibility conditions of the spatial and temporal matrix spectral problems, (6) and (16), are the zero-curvature equations (3). Those equations yield a four-component integrable hierarchy:

$$u_{t_r} = K^{[r]} = (ib_1^{[r+1]}, ib_2^{[r+1]}, -ic_1^{[r+1]}, -ic_2^{[r+1]})^T,\ r \geq 0, \quad (17)$$

or precisely,

$$p_{1,t_r} = ib_1^{[r+1]},\ p_{2,t_r} = ib_2^{[r+1]},\ q_{1,t_r} = -ic_1^{[r+1]},\ q_{2,t_r} = -ic_2^{[r+1]},\ r \geq 0. \quad (18)$$

The first two nonlinear examples in the above integrable hierarchy are the coupled nonlinear Schrödinger equations

$$\begin{cases} ip_{1,t_2} = p_{1,xx} + 4p_1^2q_1 + 2p_1p_2q_2 - p_2^2q_1, \\ ip_{2,t_2} = p_{2,xx} - 4p_1^2q_2 + 8p_1p_2q_1 + p_2^2q_2, \\ iq_{1,t_2} = -q_{1,xx} - 4p_1q_1^2 + p_1q_2^2 - 2p_2q_1q_2, \\ iq_{2,t_2} = -q_{2,xx} - 8p_1q_1q_2 + 4p_2q_1^2 - p_2q_2^2, \end{cases} \quad (19)$$

and the coupled modified Korteweg–de Vries equations

$$\begin{cases} p_{1,t_3} = -p_{1,xxx} - 12p_1p_{1,x}q_1 - 3p_1p_{2,x}q_2 + 3p_2p_{2,x}q_1 - 3p_{1,x}p_2q_2, \\ p_{2,t_3} = -p_{2,xxx} - 12p_1p_{2,x}q_1 + 12p_1p_{1,x}q_2 - 12p_{1,x}p_2q_1 - 3p_2p_{2,x}q_2, \\ q_{1,t_3} = -q_{1,xxx} - 12p_1q_1q_{1,x} + 3p_1q_2q_{2,x} - 3p_2q_1q_{2,x} - 3p_2q_{1,x}q_2, \\ q_{2,t_3} = -q_{2,xxx} - 12p_1q_1q_{2,x} - 12p_1q_{1,x}q_2 + 12p_2q_1q_{1,x} - 3p_2q_2q_{2,x}. \end{cases} \tag{20}$$

They provide supplements to the classes of integrable nonlinear Schrödinger equations and modified Korteweg–de Vries equations.

To furnish Hamiltonian structures for the integrable hierarchy (18), we apply the trace identity (5) to the matrix spectral problem (6). Noting that the solution Z is given by (8), we can directly compute

$$\mathrm{tr}\big(Z\frac{\partial U}{\partial \lambda}\big) = 2a,\ \mathrm{tr}\big(Z\frac{\partial U}{\partial u}\big) = (8c_1, 2c_2, 8b_1, 2b_2)^T,$$

and thus, an application of the trace identity (5) yields

$$\frac{\delta}{\delta u}\int \lambda^{-s-1}a^{[s+1]}\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma-s}(4c_1^{[s]}, c_2^{[s]}, 4b_1^{[s]}, b_2^{[s]})^T,\ s\geq 0.$$

Considering the case with $s=2$, we see $\gamma=0$, and therefore, we arrive at

$$\frac{\delta\mathcal{H}^{[s]}}{\delta u} = (4c_1^{[s+1]}, c_2^{[s+1]}, 4b_1^{[s+1]}, b_2^{[s+1]})^T,\ s\geq 0, \tag{21}$$

where the Hamiltonian functionals are given by

$$\mathcal{H}^{[s]} = -\int \frac{a^{[s+2]}}{s+1}\,dx,\ s\geq 0. \tag{22}$$

Based on (21), we can find the Hamiltonian structures for the integrable hierarchy (18):

$$u_{t_r} = K^{[r]} = J\frac{\delta\mathcal{H}^{[r]}}{\delta u},\ J = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{0} & \frac{1}{4}i & 0 \\ & & 0 & i \\ \hline -\frac{1}{4}i & 0 & \multicolumn{2}{c}{0} \\ 0 & -i & & \end{array}\right],\ r\geq 0, \tag{23}$$

where J is a Hamiltonian operator and the Hamiltonian functionals $\mathcal{H}^{[r]}$, $r\geq 0$, are defined by (22). The associated Hamiltonian structures exhibit a connection $S = J\frac{\delta\mathcal{H}}{\delta u}$ from a conserved functional $\mathcal{H}$ to a symmetry S. Further, we can explore basic integrable properties of the hierarchy (18). The commuting property of those vector fields $K^{[r]}$, $r\geq 0$:

$$[\![K^{[s_1]}, K^{[s_2]}]\!] = K^{[s_1]\prime}(u)[K^{[s_2]}] - K^{[s_2]\prime}(u)[K^{[s_1]}] = 0,\ s_1, s_2\geq 0, \tag{24}$$

is guaranteed by exploring a Lax operator algebra:

$$[\![V^{[s_1]}, V^{[s_2]}]\!] = V^{[s_1]\prime}(u)[K^{[s_2]}] - V^{[s_2]\prime}(u)[K^{[s_1]}] + [V^{[s_1]}, V^{[s_2]}] = 0,\ s_1, s_2\geq 0, \tag{25}$$

which is a consequence of the isospectral zero-curvature equations. Moreover, it follows from a recursion structure $K^{[r+1]} = \Phi K^{[r]}$ and the skew-symmetric property of ΦJ that the conserved functionals also commute under the corresponding Poisson bracket:

$$\{\mathcal{H}^{[s_1]}, \mathcal{H}^{[s_2]}\}_J = \int \big(\frac{\delta\mathcal{H}^{[s_1]}}{\delta u}\big)^T J\frac{\delta\mathcal{H}^{[s_2]}}{\delta u}\,dx = 0,\ s_1, s_2\geq 0. \tag{26}$$

A combination of the Hamiltonian operator J with the recursion operator Φ [26] can generate a bi-Hamiltonian structure [20] for each integrable equation in the hierarchy (18).

3. Cases of Bigger Matrix Spectral Problems

Let us fix an arbitrary natural number m. We take a generalization of the matrix spatial spectral problem (6):

$$-i\phi_x = U\phi,\ U = \left[\begin{array}{c|ccc|c} \lambda & \mathbf{p}_1 & p_2 & \mathbf{p}_1 & 0 \\ \hline \mathbf{q}_1 & & & & \mathbf{p}_1^T \\ q_2 & & 0 & & p_2 \\ \mathbf{q}_1 & & & & \mathbf{p}_1^T \\ \hline 0 & \mathbf{q}_1^T & q_2 & \mathbf{q}_1^T & -\lambda \end{array}\right]_{(2m+3)\times(2m+3)}, \tag{27}$$

where

$$\mathbf{p}_1 = (\underbrace{p_1, \cdots, p_1}_{m}),\ \mathbf{q}_1 = (\underbrace{q_1, \cdots, q_1}_{m})^T,$$

and we assume a Laurent series solution to the stationary zero-curvature equation (2):

$$Z = \left[\begin{array}{c|ccc|c} a & \mathbf{b}_1 & b_2 & \mathbf{b}_1 & 0 \\ \hline \mathbf{c}_1 & 0 & \mathbf{d} & 0 & \mathbf{b}_1^T \\ c_2 & -\mathbf{d}^T & 0 & -\mathbf{d}^T & b_2 \\ \mathbf{c}_1 & 0 & \mathbf{d} & 0 & \mathbf{b}_1^T \\ \hline 0 & \mathbf{c}_1^T & c_2 & \mathbf{c}_1^T & -a \end{array}\right]_{(2m+3)\times(2m+3)} = \sum_{s\geq 0} \lambda^{-s} Z^{[s]}, \tag{28}$$

where the entries of Z are defined by (9) and

$$\mathbf{b}_1 = (\underbrace{b_1, \cdots, b_1}_{m}),\ \mathbf{c}_1 = (\underbrace{c_1, \cdots, c_1}_{m})^T, \mathbf{d} = (\underbrace{d, \cdots, d}_{m})^T.$$

Then, the corresponding stationary zero-curvature equation (2) yields

$$\begin{cases} b_{1,x} = i(\lambda b_1 - p_1 a - p_2 d), \\ b_{2,x} = i(\lambda b_2 - p_2 a + 2m p_1 d), \\ c_{1,x} = -i(\lambda c_1 - q_1 a + q_2 d), \\ c_{2,x} = -i(\lambda c_2 - q_2 a - 2m q_1 d), \\ d_x = i(q_1 b_2 - q_2 b_1 + p_1 c_2 - p_2 c_1), \\ a_x = i(2m p_1 c_1 + p_2 c_2 - 2m q_1 b_1 - q_2 b_2) \\ \quad = -\lambda^{-1}(2m q_1 b_{1,x} + q_2 b_{2,x} + 2m p_1 c_{1,x} + p_2 c_{2,x}), \end{cases}$$

and an application of the trace identity (5) leads to

$$\frac{\delta}{\delta u}\int a\,dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}(2mc_1, c_2, 2mb_1, b_2)^T.$$

Following these equalities, we obtain the Hamiltonian structures for the associated integrable equations:

$$u_{t_r} = K^{[r]} = (ib_1^{[r+1]}, ib_2^{[r+1]}, -ic_1^{[r+1]}, -ic_2^{[r+1]})^T = J\frac{\delta\mathcal{H}^{[r]}}{\delta u},\ r\geq 1, \tag{29}$$

where

$$J = \left[\begin{array}{c|c} 0 & \begin{matrix} \frac{1}{2m}i & 0 \\ 0 & i \end{matrix} \\ \hline \begin{matrix} -\frac{1}{2m}i & 0 \\ 0 & -i \end{matrix} & 0 \end{array}\right],\ \mathcal{H}^{[r]} = -\int \frac{a^{[r+2]}}{r+1},\ r\geq 0. \tag{30}$$

Again, we have the two Abelian algebras of symmetries and conserved functionals, as depicted in (24) and (26).

Upon taking the initial values in (14) and zero constants of integration, we can work out the first two nonlinear examples in the generalized hierarchy (29), which are the coupled nonlinear Schrödinger equations

$$\begin{cases} ip_{1,t_2} = p_{1,xx} + 2mp_1^2 q_1 + 2p_1p_2q_2 - p_2^2 q_1, \\ ip_{2,t_2} = p_{2,xx} - 2mp_1^2 q_2 + 4mp_1p_2q_1 + p_2^2 q_2, \\ iq_{1,t_2} = -q_{1,xx} - 2mp_1q_1^2 + p_1q_2^2 - 2p_2q_1q_2, \\ iq_{2,t_2} = -q_{2,xx} - 4mp_1q_1q_2 + 2mp_2q_1^2 - p_2q_2^2, \end{cases} \tag{31}$$

and the coupled modified Korteweg–de Vries equations

$$\begin{cases} p_{1,t_3} = -p_{1,xxx} - 6mp_1p_{1,x}q_1 - 3p_1p_{2,x}q_2 + 3p_2p_{2,x}q_1 - 3p_{1,x}p_2q_2, \\ p_{2,t_3} = -p_{2,xxx} - 6mp_1p_{2,x}q_1 + 6mp_1p_{1,x}q_2 - 6mp_{1,x}p_2q_1 - 3p_2p_{2,x}q_2, \\ q_{1,t_3} = -q_{1,xxx} - 6mp_1q_1q_{1,x} + 3p_1q_2q_{2,x} - 3p_2q_1q_{2,x} - 3p_2q_{1,x}q_2, \\ q_{2,t_3} = -q_{2,xxx} - 6mp_1q_1q_{2,x} - 6mp_1q_{1,x}q_2 + 6mp_2q_1q_{1,x} - 3p_2q_2q_{2,x}. \end{cases} \tag{32}$$

where m is an arbitrary natural number. We point out that some of the coefficients in the above two integrable equations depend on the number of copies of p_1 and q_1 in the spatial spectral matrix U defined by (27).

4. Concluding Remarks

Kinds of higher-order matrix spectral problems were introduced and their associated integrable Hamiltonian hierarchies were generated through the zero-curvature formulation. A crucial step was to formulate a Laurent series solution to the corresponding stationary zero-curvature equations. All integrable equations in the resulting hierarchies were shown to be Liouville integrable via the trace identity.

We remark that one can generalize the considered matrix spectral problems further by involving more copies of p_2 as we did for p_1. Of course, we can also introduce more dependent variables in matrix spectral problems so that bigger systems of integrable Hamiltonian equations, consisting of more components, can be generated (see, e.g., [27,28]). If we take non-semisimple Lie algebras to begin with, then more general integrable structures, such as integrable couplings, could be explored (see, e.g., [18,29,30]).

It would be interesting to determine structures of soliton solutions for the resulting integrable equations by incorporating and integrating a wide variety of techniques in soliton theory, such as the Riemann–Hilbert technique [31], the Zakharov–Shabat dressing method [32], the Darboux transformation [33,34] and the determinant approach [35]. Other interesting solutions can often be worked out by taking wave number reductions of soliton solutions (see, e.g., [36–40]). Upon imposing nonlocal group reductions for the considered matrix spectral problems, novel nonlocal reduced integrable equations can also be computed (see, e.g., [41,42]), and their soliton solutions need further investigation.

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References

1. Das, A. *Integrable Models*; World Scientific: Teaneck, NJ, USA, 1989.
2. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **1974**, *53*, 249–315. [CrossRef]
3. Drinfel'd, V.; Sokolov, V.V. Lie algebras and equations of Korteweg–de Vries type. *Sov. J. Math.* **1985**, *30*, 1975–2036. [CrossRef]
4. Tu, G.Z. On Liouville integrability of zero-curvature equations and the Yang hierarchy. *J. Phys. A Math. Gen.* **1989**, *22*, 2375–2392.
5. Ma, W.X. A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction. *Chin. Ann. Math. Ser. A* **1992**, *13*, 115–123.
6. Antonowicz, M.; Fordy, A.P. Coupled KdV equations with multi-Hamiltonian structures. *Phys. D* **1987**, *28*, 345–357. [CrossRef]
7. Xia, T.C.; Yu, F.J.; Zhang, Y. The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions. *Phys. A Stat. Mech. Appl.* **2004**, *343*, 238–246. [CrossRef]
8. Manukure, S. Finite-dimensional Liouville integrable Hamiltonian systems generated from Lax pairs of a bi-Hamiltonian soliton hierarchy by symmetry constraints. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *57*, 125–135. [CrossRef]
9. Liu, T.S.; Xia, T.C. Multi-component generalized Gerdjikov-Ivanov integrable hierarchy and its Riemann-Hilbert problem. *Nonlinear Anal. Real World Appl.* **2022**, *68*, 103667. [CrossRef]
10. Wang, H.F.; Zhang, Y.F. Application of Riemann-Hilbert method to an extended coupled nonlinear Schrödinger equations. *J. Comput. Appl. Math.* **2023**, *420*, 114812. [CrossRef]
11. Ma, W.X. Matrix integrable fourth-order nonlinear Schrödinger equations and their exact soliton solutions. *Chin. Phys. Lett.* **2022**, *39*, 100201. [CrossRef]
12. Ma, W.X. Matrix integrable fifth-order mKdV equations and their soliton solutions. *Chin. Phys. B* **2023**, *32*, 020201. [CrossRef]
13. Wu, Y.T.; Geng, X.G.; Hu, X.B.; Zhu, S.M. A generalized Hirota-Satsuma coupled Korteweg-de Vries equation and Miura transformations. *Phys. Lett. A* **1999**, *255*, 259–264. [CrossRef]
14. Zhu, X.M.; Zhang, J.B. The integrability of a new fractional soliton hierarchy and its application. *Adv. Math. Phys.* **2022**, *2022*, 2200092. [CrossRef]
15. Zhang, Y.F.; Wang, H.F.; Bai, N. Schemes for generating different nonlinear Schrödinger integrable equations and their some properties. *Acta Math. Appl. Sin. Engl. Ser.* **2022**, *38*, 579–600. [CrossRef]
16. Yao, Y.Q.; Shen, S.F.; Ma, W.X. A soliton hierarchy associated with a spectral problem of 2nd degree in a spectral parameter and its bi-Hamiltonian structure. *Adv. Math. Phys.* **2016**, *2016*, 3589704. [CrossRef]
17. Ma, W.X. Integrable nonlocal nonlinear Schrödinger equations associated with so(3,ℝ). *Proc. Am. Math. Soc. Ser. B* **2022**, *9*, 1–11. [CrossRef]
18. Wang, Z.B.; Wang, H.F. Integrable couplings of two expanded non-isospectral soliton hierarchies and their bi-Hamiltonian structures. *Int. J. Geom. Methods Mod. Phys.* **2022**, *19*, 2250160. [CrossRef]
19. Ma, W.X. A multi-component integrable hierarchy and its integrable reductions. *Phys. Lett. A* **2023**, *457*, 128575. [CrossRef]
20. Magri, F. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.* **1978**, *19*, 1156–1162. [CrossRef]
21. Kaup, D.J.; Newell, A.C. An exact solution for a derivative nonlinear Schrödinger equation. *J. Math. Phys.* **1978**, *19*, 798–801. [CrossRef]
22. Wadati, M.; Konno, K.; Ichikawa, Y.H. New integrable nonlinear evolution equations. *J. Phys. Soc. Jpn.* **1979**, *47*, 1698–1700. [CrossRef]
23. Takhtajan, L.A. Integration of the continuous Heisenberg spin chain through the inverses scattering method. *Phys. Lett. A* **1977**, *64*, 235–237. [CrossRef]
24. Ma, W.X. Reduced non-local integrable NLS hierarchies by pairs of local and non-local constraints. *Int. J. Appl. Comput. Math.* **2022**, *8*, 206. [CrossRef]
25. Ma, W.X. Sasa-Satsuma type matrix integrable hierarchies and their Riemann-Hilbert problems and soliton solutions. *Phys. D Nonlinear Phenom.* **2023**, *446*, 133672. [CrossRef]
26. Fuchssteiner, B.; Fokas, A.S. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D Nonlinear Phenom.* **1981**, *4*, 47–66. [CrossRef]
27. Zhang, Y.; Dong, H.H. Multi-component Gerdjikov-Ivanov system and its Riemann-Hilbert problem under zero boundary conditions. *Nonlinear Anal. Real World Appl.* **2021**, *60*, 103279. [CrossRef]
28. Ye R.S.; Zhang, Y. Two-component complex modified Korteweg–de Vries equations: New soliton solutions from novel binary Darboux transformation. *Theor. Math. Phys.* **2023**, *214*, 183–193. [CrossRef]
29. Shen, S.F.; Li, C.X.; Jin, Y.Y.; Yu, S.M. Multi-component integrable couplings for the Ablowitz-Kaup-Newell-Segur and Volterra hierarchies. *Math. Methods Appl. Sci.* **2015**, *38*, 4345–4356. [CrossRef]
30. Wang, H.F.; Zhang, Y.F. A new multi-component integrable coupling and its application to isospectral and nonisospectral problems. *Commun. Nonlinear Sci. Numer. Simul.* **2022**, *105*, 106075. [CrossRef]
31. Novikov, S.P.; Manakov, S.V.; Pitaevskii, L.P.; Zakharov, V.E. *Theory of Solitons: The Inverse Scattering Method*; Consultants Bureau: New York, NY, USA, 1984.
32. Doktorov, E.V.; Leble, S.B. *A Dressing Method in Mathematical Physics*; Springer: Dordrecht, The Netherland, 2007.
33. Matveev, V.B.; Salle, M.A. *Darboux Transformations and Solitons*; Springer: Berlin, Germany, 1991.

34. Geng, X.G.; Li, R.M.; Xue, B. A vector general nonlinear Schrödinger equation with $(m+n)$ components. *J. Nonlinear Sci.* **2020**, *30*, 991–1013. [CrossRef]
35. Aktosun, T.; Busse, T.; Demontis, F.; van der Mee, C. Symmetries for exact solutions to the nonlinear Schrödinger equation. *J. Phys. A Math. Theor.* **2010**, *43*, 025202. [CrossRef]
36. Chen, S.T.; Ma, W.X. Lump solutions of a generalized Calogero-Bogoyavlenskii-Schiff equation. *Comput. Math. Appl.* **2018**, *76*, 1680–1685. [CrossRef]
37. Cheng, L.; Zhang, Y.; Lin, M.J. Lax pair and lump solutions for the (2+1)-dimensional DJKM equation associated with bilinear Bäcklund transformations. *Anal. Math. Phys.* **2019**, *9*, 1741–1752. [CrossRef]
38. Sulaiman, T.A.; Yusuf, A.; Abdeljabbar, A.; Alquran, M. Dynamics of lump collision phenomena to the (3+1)-dimensional nonlinear evolution equation. *J. Geom. Phys.* **2021**, *169*, 104347. [CrossRef]
39. Manukure, S.; Chowdhury, A.; Zhou, Y. Complexiton solutions to the asymmetric Nizhnik-Novikov-Veselov equation. *Int. J. Mod. Phys. B* **2019**, *33*, 1950098. [CrossRef]
40. Zhou, Y.; Manukure, S.; McAnally, M. Lump and rogue wave solutions to a (2+1)-dimensional Boussinesq type equation. *J. Geom. Phys.* **2021**, *167*, 104275. [CrossRef]
41. Ma, W.X. Reduced nonlocal integrable mKdV equations of type $(-\lambda, \lambda)$ and their exact soliton solutions. *Commun. Theor. Phys.* **2022**, *74*, 065002. [CrossRef]
42. Ma, W.X. Integrable non-local nonlinear Schrödinger hierarchies of type $(-\lambda^*, \lambda)$ and soliton solutions. *Rep. Math. Phys.* 2023, *91*, *to appear*.