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Integrable nonlocal PT-symmetric generalized so$(3, \mathbb{R})$-mKdV equations

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Abstract

Based on a soliton hierarchy associated with so(3, $\mathbb{R}$), we construct two integrable nonlocal PT-symmetric generalized mKdV equations. The key step is to formulate two nonlocal reverse-spacetime similarity transformations for the involved spectral matrix, and therefore, integrable nonlocal complex and real reverse-spacetime generalized so(3, $\mathbb{R}$)-mKdV equations of fifth-order are presented. The resulting reduced nonlocal integrable equations inherit infinitely many commuting symmetries and conservation laws.

Keywords: integrable equation, lax pair, nonlocal reduction, PT-symmetry, zero curvature equation

1. Introduction

Matrix spectral problems associated with matrix Lie algebras are used to study integrable equations [1, 2], whose Hamiltonian structures are often furnished by the trace identity [3, 4], and whose Riemann–Hilbert problems can be formulated to establish inverse scattering transforms [5]. The well-known integrable equations associated with simple Lie algebras include the KdV equation [6], the NLS equation [7], the derivative NLS equation [8], higher-order NLS and mKdV equations [9, 10], the nonlocal NLS equation [11] and the nonlocal mKdV equation [12].

If we build matrix spectral problems by using non-semi-simple matrix Lie algebras, the so-called integrable couplings, both continuous and discrete, can be generated, and the variational identity [13] helps furnish their Hamiltonian structures, which lead to novel hereditary recursion operators in block matrix form [14]. Darboux transformations are also presented to solve integrable couplings [15].

We will apply the special orthogonal Lie algebra $\mathfrak{g} = \mathrm{so}(3, \mathbb{R})$. This Lie algebra can be realized by all 3×3 trace-free, skew-symmetric real matrices. Thus, a basis can be taken as

$$h_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad h_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

with the corresponding structure equations being given by

$$[h_1, h_2] = h_3, \ [h_2, h_3] = h_1, \ [h_3, h_1] = h_2. \tag{2}$$

We can take other representations of so(3, $\mathbb{R}$) to start to study integrable equations. The Lie algebra so(3, $\mathbb{R}$) is one of the only two three-dimensional real Lie algebras, whose derived algebra is equal to itself. The other such Lie algebra is the special linear algebra sl(2, $\mathbb{R}$), which has been widely used to study integrable equations [2]. It is worth noting that the two complex Lie algebras, sl(2, $\mathbb{C}$) and so(3, $\mathbb{C}$), are isomorphic to each other over the complex field. The following matrix loop algebra

$$\tilde{\mathfrak{g}} = \widetilde{\mathrm{so}}(3, \mathbb{R}) = \{A \in \mathrm{so}(3, \mathbb{R}) |\ \text{entries of } A \ -\text{Laurent series in } \lambda\}, \tag{3}$$

λ being a spectral parameter, will be used in our construction. This matrix loop algebra has already been used to construct

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integrable equations [16]. Based on the perturbation-type loop algebras of $\widetilde{\mathrm{so}}(3, \mathbb{R})$, we can also construct integrable couplings [14].

In this paper, starting from matrix spectral problems, we would first like to revisit an application of so(3, $\mathbb{R}$) to integrable equations [16], with a slightly modified spectral matrix. We will then make two pairs of nonlocal integrable reductions for the spectral matrix to generate two fifth-order scalar nonlocal reverse-spacetime equations, which are Liouville integrable, i.e. possess infinitely many commuting symmetries and conservation laws. The presented scalar nonlocal integrable equations are a nonlocal complex reverse-spacetime generalzied so(3, $\mathbb{R}$)-mKdV equation:

$$\begin{aligned} r_t = &\frac{15}{8}r_x(r^*(-x,-t))^4 + \frac{15}{4}r^2r_x(r^*(-x,-t))^2 \\ &+ \frac{15}{8}r^4r_x - \frac{5}{2}r_x^3 \\ &- 5r_xr^*(-x,-t)r_{xx}^*(-x,-t) - 10rr_xr_{xx} \\ &- \frac{5}{2}r_x(r_x^*(-x,-t))^2 \\ &- \frac{5}{2}r_{xxx}(r^*(-x,-t))^2 + 5r_{xx}r^* \\ &\times(-x,-t)r_x^*(-x,-t) - \frac{5}{2}r^2r_{xxx} + r_{5x}, \end{aligned}$$

where r^* denotes the complex conjugate of r, and a nonlocal real reverse-spacetime generalzied so(3, $\mathbb{R}$)-mKdV equation:

$$\begin{aligned} r_t = &\frac{15}{8}r_x(r(-x,-t))^4 + \frac{15}{4}r^2r_x(r(-x,-t))^2 \\ &+ \frac{15}{8}r^4r_x - \frac{5}{2}r_x^3 \\ &- 5r_xr(-x,-t)r_{xx}(-x,-t) - 10rr_xr_{xx} \\ &- \frac{5}{2}r_x(r_x(-x,-t))^2 \\ &- \frac{5}{2}r_{xxx}(r(-x,-t))^2 + 5r_{xx}r(-x,-t)r_x \\ &\times(-x,-t) - \frac{5}{2}r^2r_{xxx} + r_{5x}. \end{aligned}$$

It is easy to see that both nonlocal integrable equations are PT-symmetric. Namely, they are invariant under the parity-time transformation: $x \to -x$, $t \to -t$, $\mathrm{i} \to -\mathrm{i}$.

2. A fifth-order integrable system

2.1. Matrix spectral problems

Let i denote the unit imaginary number. We consider a Lax pair of matrix spectral problems:

$$-\mathrm{i}\phi_x = \mathcal{M}\phi = \mathcal{M}(\boldsymbol{u},\lambda)\phi,\ -\mathrm{i}\phi_t = \mathcal{N}\phi = \mathcal{N}(\boldsymbol{u},\lambda)\phi, \tag{4}$$

with

$$\mathcal{M} = \mathcal{M}(\boldsymbol{u},\lambda) = \begin{bmatrix} 0 & -s & -\lambda \\ s & 0 & -r \\ \lambda & r & 0 \end{bmatrix}, \tag{5}$$

and

$$\mathcal{N} = \mathcal{N}(\boldsymbol{u},\lambda) = \sum_{l=0}^{5}\begin{bmatrix} 0 & -g_l & -e_l \\ g_l & 0 & -f_l \\ e_l & f_l & 0 \end{bmatrix}\lambda^l. \tag{6}$$

In the above spectral problems, λ is a spectral parameter, $\boldsymbol{u} = (r, s)^{\mathrm{T}}$ is a potential, $\phi = (\phi_1, \phi_2, \phi_3)^{\mathrm{T}}$ is a column eigenfunction, and e_l, f_l, g_l are determined by

$$\begin{aligned}
f_0 &= g_0 = 0,\ e_0 = -1; \\
f_1 &= -r,\ g_1 = -s,\ e_1 = 0; \\
f_2 &= \mathrm{i}s_x,\ g_2 = -\mathrm{i}r_x,\ e_2 = \frac{1}{2}(r^2+s^2); \\
f_3 &= -r_{xx} + \frac{1}{2}r^3 + \frac{1}{2}rs^2,\ g_3 = -s_{xx} \\
&\quad + \frac{1}{2}r^2s + \frac{1}{2}s^3,\ e_3 = \mathrm{i}(r_xs - rs_x); \\
f_4 &= \mathrm{i}\left(s_{xxx} - \frac{3}{2}r^2s_x - \frac{3}{2}s^2s_x\right), \\
g_4 &= \mathrm{i}\left(-r_{xxx} + \frac{3}{2}r^2r_x + \frac{3}{2}r_xs^2\right), \\
e_4 &= rr_{xx} + ss_{xx} - \frac{1}{2}r_x^2 - \frac{1}{2}s_x^2 - \frac{3}{8}(r^2+s^2)^2; \\
f_5 &= -r_{xxxx} + \frac{5}{2}r^2r_{xx} + \frac{5}{2}rr_x^2 + \frac{3}{2}r_{xx}s^2 + 3r_xss_x \\
&\quad + rss_{xx} - \frac{1}{2}rs_x^2 - \frac{3}{8}r(r^2+s^2)^2, \\
g_5 &= -s_{xxxx} + \frac{5}{2}s^2s_{xx} + \frac{5}{2}ss_x^2 + \frac{3}{2}r^2s_{xx} + 3rr_xs_x \\
&\quad + rr_{xx}s - \frac{1}{2}r_x^2s - \frac{3}{8}s(r^2+s^2)^2, \\
e_5 &= \mathrm{i}\Big(r_{xxx}s - rs_{xxx} - r_{xx}s_x + r_xs_{xx} \\
&\quad - \frac{3}{2}r^2r_xs + \frac{3}{2}rs^2s_x + \frac{3}{2}r^3s_x - \frac{3}{2}r_xs^3\Big); \\
f_6 &= \frac{\mathrm{i}}{8}(15s^4s_x + 30r^2s^2s_x \\
&\quad + 15s_xr^4 - 20s^2s_{xxx} - 80ss_xs_{xx} \\
&\quad - 20s_{xxx}r^2 - 40s_xrr_{xx} - 40s_{xx}rr_x - 20s_xr_x^2 \\
&\quad - 20s_x^3 + 8s_{5x}), \\
g_6 &= \frac{\mathrm{i}}{8}(-15r_xs^4 - 30r^2s^2r_x - 15r^4r_x + 20r_x^3 \\
&\quad + 40r_xss_{xx} + 80ss_xs_{xx} \\
&\quad + 20r_xs_x^2 + 20r_{xxx}s^2 + 40r_{xx}ss_x + 20r^2r_{xxx} \\
&\quad - 8r_{5x}), \\
e_6 &= rr_{4x} + ss_{4x} - r_xr_{xxx} - s_xs_{xxx} \\
&\quad + \frac{1}{2}r_{xx}^2 + \frac{1}{2}s_{xx}^2 \\
&\quad - \frac{5}{2}(r^2+s^2)(rr_{xx}+ss_{xx}) - \frac{5}{4}(r^2-s^2)(r_x^2-s_x^2) \\
&\quad - 5rsr_xs_x + \frac{5}{16}(r^2+s^2)^3.
\end{aligned}$$

The coefficients e_l, f_l, g_l are defined by

$$\begin{cases} f_{l+1} = -\mathrm{i}g_{l,x} + re_l, \\ g_{l+1} = \mathrm{i}f_{l,x} + se_l, \qquad l \geqslant 0. \\ e_{l+1,x} = \mathrm{i}(rg_{l+1} - sf_{l+1}), \end{cases} \tag{7}$$

under the integration conditions

$$e_l|_{u=0} = 0, \, l \geqslant 1,$$

i.e. take the constant of integration as zero, which implies that

$$f_l|_{u=0} = g_l|_{u=0} = 0, \, l \geqslant 1.$$

Such a matrix

$$\mathcal{W} = eh_1 + fh_2 + gh_3 = \sum_{l=0}^{\infty} \begin{bmatrix} 0 & -g_l & -e_l \\ g_l & 0 & -f_l \\ e_l & f_l & 0 \end{bmatrix} \lambda^{-l} \tag{8}$$

solves the stationary zero curvature equation

$$\mathcal{W}_x = \mathrm{i}[\mathcal{M}, \mathcal{W}]. \tag{9}$$

More examples can be found in the literature (see, e.g. [17, 18]).

Now, the zero curvature equation

$$\mathcal{M}_t - \mathcal{N}_x + \mathrm{i}[\mathcal{M}, \mathcal{N}] = 0, \tag{10}$$

leads to a fifth-order integrable system $\boldsymbol{u}_t = \boldsymbol{X}$:

$$\begin{cases} r_t = X_1 = \frac{15}{8} r_x s^4 + \frac{15}{4} r^2 r_x s^2 + \frac{15}{8} r^4 r_x - \frac{5}{2} r_x^3 - 5 r_x s s_{xx} \\ -10 r r_x r_{xx} - \frac{5}{2} r_x s_x^2 - \frac{5}{2} r_{xxx} s^2 - 5 r_{xx} s s_x - \frac{5}{2} r^2 r_{xxx} + r_{5x}, \\ s_t = X_2 = \frac{15}{8} s^4 s_x + \frac{15}{4} r^2 s^2 s_x + \frac{15}{8} r^4 s_x - \frac{5}{2} s^2 s_{xxx} - 10 s s_x s_{xx} \\ -\frac{5}{2} r^2 s_{xxx} - 5 r r_{xx} s_x - 5 r r_x s_{xx} - \frac{5}{2} r_x^2 s_x - \frac{5}{2} s_x^3 + s_{5x}, \end{cases} \tag{11}$$

where $\boldsymbol{X} = (X_1, X_2)^{\mathrm{T}}$.

2.2. An application of the trace identity

We apply the trace identity [3] with our spectral matrix $\mathrm{i}\mathcal{M}$:

$$\frac{\delta}{\delta \boldsymbol{u}} \int \mathrm{tr}\left(\mathcal{W} \frac{\partial \mathcal{M}}{\partial \lambda}\right) \mathrm{d}x = \lambda^{-\gamma} \frac{\lambda}{\partial \lambda} \lambda^{\gamma} \, \mathrm{tr}\left(\mathcal{W} \frac{\partial \mathcal{M}}{\partial \boldsymbol{u}}\right), \tag{12}$$

where the constant γ is given by

$$\gamma = -\frac{\lambda}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln|\langle \mathcal{W}, \mathcal{W} \rangle|. \tag{13}$$

Then, we obtain the following bi-Hamiltonian structure [19] for the integrable system (11):

$$\boldsymbol{u}_t = J_1 \frac{\delta \mathcal{H}_2}{\delta \boldsymbol{u}} = J_2 \frac{\delta \mathcal{H}_1}{\delta \boldsymbol{u}}, \tag{14}$$

where the Hamiltonian pair, J_1 and J_2, is given by

$$J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \, J_2 = \mathrm{i} \begin{bmatrix} -\partial + s\partial^{-1}s & -s\partial^{-1}r \\ -r\partial^{-1}s & -\partial + r\partial^{-1}r \end{bmatrix}, \tag{15}$$

and the Hamiltonian functionals, $\mathcal{H}_1$ and $\mathcal{H}_2$, are determined by

$$\begin{aligned} \mathcal{H}_1 = -\tfrac{\mathrm{i}}{5} \int [& r r_{xxxx} + s s_{xxxx} - r_x r_{xxx} - s_x s_{xxx} \\ & + \tfrac{1}{2} r_{xx}^2 + \tfrac{1}{2} s_{xx}^2 \\ & -\tfrac{5}{2} r (r^2 + s^2) r_{xx} - \tfrac{5}{2} s (r^2 + s^2) s_{xx} \\ & -5 r s r_x s_x \\ & -\tfrac{5}{4} (r^2 - s^2) r_x^2 + \tfrac{5}{4} (r^2 - s^2) s_x^2 \\ & + \tfrac{5}{16} (r^2 + s^2)^3] \, \mathrm{d}x, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mathcal{H}_2 = \int \Big\{ & \tfrac{1}{6} s r_{5x} - \tfrac{1}{6} r s_{5x} - \tfrac{1}{6} s_x r_{xxxx} + \tfrac{1}{6} r_x s_{xxxx} \\ & + \tfrac{1}{12} (2 s_{xx} - 5 s r^2 - 5 s^3) r_{xxx} - \tfrac{1}{12} (2 r_{xx} - 5 r^3 - 5 r s^2) s_{xxx} \\ & + \tfrac{1}{12} [5(3r^2 - s^2) s_x - 20 r s r_x] r_{xx} \\ & - \tfrac{1}{12} [5(3s^2 - r^2) r_x - 20 r s s_x] s_{xx} \\ & + \tfrac{5}{16} (r s_x - s r_x) [\tfrac{4}{3} (r_x^2 + s_x^2) - (r^2 + s^2)] \Big\} \, \mathrm{d}x. \end{aligned} \tag{17}$$

The Hamiltonian formulation leads to infinitely many symmetries and conservation laws for the integrable system (11), which can often be generated through symbolic computation by computer algebra systems (see, e.g. [20, 21]). The operator

$$\Phi = J_2 J_1^{-1} = \mathrm{i} \begin{bmatrix} s\partial^{-1}r & -\partial + s\partial^{-1}s \\ \partial - r\partial^{-1}r & -r\partial^{-1}s \end{bmatrix}, \, \partial = \frac{\partial}{\partial x}, \tag{18}$$

is a hereditary recursion operator [22, 23] for the integrable system (11).

3. Nonlocal generalized so(3, $\mathbb{R}$)-mKdV equations

3.1. Integrable complex reverse-spacetime reductions

Firstly, we consider a pair of specific complex reverse-spacetime similarity transformations for the spectral matrix:

$$\begin{aligned} & \mathcal{M}^{\dagger}(-x, -t, -\lambda^*) = -\Theta \mathcal{M}(x, t, \lambda) \Theta^{-1}, \\ & \Theta = \begin{bmatrix} 0 & 0 & \sigma \\ 0 & 1 & 0 \\ \sigma & 0 & 0 \end{bmatrix}, \sigma = \pm 1, \end{aligned} \tag{19}$$

where $\dagger$ and $*$ stand for the Hermitian transpose and the complex conjugate, respectively. They lead to the potential reductions

$$r^*(-x, -t) = -\sigma s(x, t), \, \sigma = \pm 1. \tag{20}$$

Under these potential reduction, one has

$$\begin{aligned} & e_l^*(-x, -t) = (-1)^l e_l(x, t), f_l^*(-x, -t) \\ & = (-1)^l \sigma g_l(x, t), \, l \geqslant 1. \end{aligned} \tag{21}$$

We can prove these results by the mathematical induction. Actually, under the induction assumption for $l = n$ and using the recursion relation (7), we can compute

$$
\begin{aligned}
f_{n+1}^*(-x,-t) &= -\mathrm{i}g_{n,x}^*(-x,-t)+r^*(-x,-t)e_n^*(-x,-t)\\
&=(-1)^{n+1}\sigma[\,\mathrm{i}f_{n,x}(x,t)+s(x,t)e_n(x,t)]\\
&=(-1)^{n+1}\sigma g_{n+1}(x,t),\\
e_{n+1,x}^*(-x,-t) &= -\mathrm{i}[r^*(-x,-t)g_{n+1}^*(-x,-t)\\
&\quad -s^*(-x,-t)f_{n+1}^*(-x,-t)]\\
&=\mathrm{i}(-1)^{n+1}[r(x,t)g_{n+1}(x,t)-s(x,t)f_{n+1}(x,t)]\\
&=(-1)^{n+1}e_{n+1,x}(x,t).
\end{aligned}
$$

Therefore, one obtains

$$\mathcal{N}^\dagger(-x,-t,-\lambda^*)=-\Theta\mathcal{N}(x,t,\lambda)\Theta^{-1}, \tag{22}$$

and then

$$
\begin{aligned}
&((\mathcal{M}_t-\mathcal{N}_x+\mathrm{i}[\mathcal{M},\mathcal{N}])(-x,-t,-\lambda^*))^\dagger\\
&=\Theta(\mathcal{M}_t-\mathcal{N}_x+\mathrm{i}[\mathcal{M},\mathcal{N}])(x,t,\lambda)\Theta^{-1}.
\end{aligned} \tag{23}
$$

This tells that the potential reductions defined by (20) are compatible with the zero curvature equation of the integrable system (11). Then, one obtains two reduced scalar nonlocal integrable equations associated with so(3, $\mathbb{R}$):

$$r_t = X_1|_{s(x,t)=-\sigma r^*(-x,-t)}, \tag{24}$$

where $\boldsymbol{X}=(X_1,X_2)^{\mathrm{T}}$ is defined as in (11). The infinitely many symmetries and conservation laws for the integrable system (11) will be reduced to infinitely many ones for the above nonlocal integrable equations in (24), under (20).

With $\sigma=1$, the third-order nonlinear reduced scalar integrable equation presents a nonlocal complex reverse-spacetime PT-symmetric generalized so(3, $\mathbb{R}$)-mKdV equation of fifth-order:

$$
\begin{aligned}
r_t &= \frac{15}{8}r_x(r^*(-x,-t))^4+\frac{15}{4}r^2r_x(r^*(-x,-t))^2\\
&\quad+\frac{15}{8}r^4r_x-\frac{5}{2}r_x^3\\
&\quad-5r_xr^*(-x,-t)r_{xx}^*(-x,-t)-10rr_xr_{xx}\\
&\quad-\frac{5}{2}r_x(r_x^*(-x,-t))^2\\
&\quad-\frac{5}{2}r_{xxx}(r^*(-x,-t))^2+5r_{xx}r^*\\
&\quad\times(-x,-t)r_x^*(-x,-t)-\frac{5}{2}r^2r_{xxx}+r_{5x},
\end{aligned} \tag{25}
$$

where r^* denotes the complex conjugate of r. Note that the first component of $\boldsymbol{X}$ is even with respect to s and odd with respect to r. Therefore, the fifth-order reduced scalar nonlocal integrable equation with $\sigma=-1$ in (24) is exactly the same as the complex nonlocal reverse-space generalized so(3, $\mathbb{R}$)-mKdV equation (25).

Let us define

$$
\begin{aligned}
\boldsymbol{Y}^{[n]} &= (Y_1^{[n]},Y_2^{[n]})^{\mathrm{T}}\\
&=(\Phi^{n-1}\boldsymbol{X})|_{s(x,t)=-\sigma r^*(-x,-t)},\ n\geqslant 1,
\end{aligned}
$$

where $\boldsymbol{X}=(X_1,X_2)^{\mathrm{T}}$ and Φ are defined by (11) and (18), respectively. Then, through applying the hierarchy of symmetries $Y_1^{[n]}$, $n\geqslant 1$, a kind of specific solutions to the nonlocal integrable equation (25) can be determined by

$$
\begin{aligned}
r &= \exp(\varepsilon_1Y_1^{[1]})\exp(\varepsilon_2Y_1^{[2]})\cdots\\
&\quad\exp(\varepsilon_mY_1^{[m]})r_0,\ m\geqslant 1,
\end{aligned}
$$

where exp is the exponential map, ε_j, $1\leqslant j\leqslant m$, are small parameters, and r_0 is an arbitrarily given solution. In particular, we can take $r_0=a\mathrm{e}^{\mathrm{i}\theta}x$, where a is an arbitrary constant and θ is a constant angle determined by $\mathrm{e}^{4\mathrm{i}\theta}+1=0$.

3.2. Integrable real reverse-spacetime reductions

Secondly, we consider another pair of specific real reverse-spacetime similarity transformations for the spectral matrix:

$$
\begin{aligned}
&\mathcal{M}^{\mathrm{T}}(-x,-t,\lambda)=\Theta\mathcal{M}(x,t,\lambda)\Theta^{-1},\\
&\Theta=\begin{bmatrix}0&0&\sigma\\0&1&0\\\sigma&0&0\end{bmatrix},\sigma=\pm1,
\end{aligned} \tag{26}
$$

where T means taking the matrix transpose as before. They generate the potential reductions

$$r(-x,-t)=\sigma s(x,t),\ \sigma=\pm1. \tag{27}$$

Under these potential reductions, one can have

$$e_l(-x,-t)=e_l(x,t),f_l(-x,-t)=\sigma g_l(x,t),\ l\geqslant 1. \tag{28}$$

These results can be verified by the mathematical induction. A direct computation can be made as follows. Under the induction assumption for $l=n$ and applying the recursion relation (7), we can have

$$
\begin{aligned}
f_{n+1}(-x,-t) &= \mathrm{i}g_{n,x}(-x,-t)+r(-x,-t)e_n(-x,-t)\\
&=\sigma[\,\mathrm{i}f_{n,x}(x,t)+s(x,t)e_n(x,t)]\\
&=\sigma g_{n+1}(x,t),\\
e_{n+1,x}(-x,-t) &= -\,\mathrm{i}[r(-x,-t)g_{n+1}(-x,-t)\\
&\quad -s(-x,-t)f_{n+1}(-x,-t)]\\
&=\mathrm{i}[r(x,t)g_{n+1}(x,t)-s(x,t)f_{n+1}(x,t)]\\
&=e_{n+1,x}(x,t).
\end{aligned}
$$

Then, we arrive at

$$\mathcal{N}^{\mathrm{T}}(-x,-t,\lambda)=\Theta\mathcal{N}(x,t,\lambda)\Theta^{-1}, \tag{29}$$

and therefore, we obtain

$$
\begin{aligned}
&((\mathcal{M}_t-\mathcal{N}_x+\mathrm{i}[\mathcal{M},\mathcal{N}])(-x,-t,\lambda))^{\mathrm{T}}\\
&=-\Theta(\mathcal{M}_t-\mathcal{N}_x+i[\mathcal{M},\mathcal{N}])(x,t,\lambda)\Theta^{-1}.
\end{aligned} \tag{30}
$$

This guarantees that the potential reductions in (27) are compatible with the zero curvature equation of the integrable system (11).

In this way, one obtains two reduced scalar nonlocal integrable equations associated with so(3, $\mathbb{R}$):

$$r_t = X_1|_{s(x,t)=\sigma r(-x,-t)}, \tag{31}$$

where $\boldsymbol{X}=(X_1,X_2)^{\mathrm{T}}$ is given as in (11). Moreover, the infinitely many symmetries and conservation laws for the integrable system (11) are reduced to infinitely many ones for the above nonlocal integrable equations in (31), under (27).

With $\sigma = 1$, the fifth-order nonlinear reduced scalar integrable equation presents a nonlocal real reverse-spacetime PT-symmetric generalized so(3, $\mathbb{R}$)-mKdV equation:

$$\begin{aligned} r_t = &\frac{15}{8} r_x (r(-x,-t))^4 + \frac{15}{4} r^2 r_x (r(-x,-t))^2 \\ &+ \frac{15}{8} r^4 r_x - \frac{5}{2} r_x^3 \\ &- 5 r_x r(-x,-t) r_{xx}(-x,-t) - 10 r r_x r_{xx} \\ &- \frac{5}{2} r_x (r_x(-x,-t))^2 \\ &- \frac{5}{2} r_{xxx} (r(-x,-t))^2 + 5 r_{xx} r \\ &\times (-x,-t) r_x(-x,-t) - \frac{5}{2} r^2 r_{xxx} + r_{5x}. \end{aligned} \tag{32}$$

It is easy to see that even and odd properties with respect to r and s in the two components of X implies that the fifth-order nonlinear reduced scalar integrable equation with $\sigma = -1$ in (31) is exactly the same as the nonlocal real reverse-spacetime generalized so(3, $\mathbb{R}$)-mKdV equation (32).

Similarly, define

$$\boldsymbol{Z}^{[n]} = (Z_1^{[n]}, Z_2^{[n]})^{\mathrm{T}} = (\Phi^{n-1}\boldsymbol{X})|_{s(x,t)=\sigma r(-x,-t)},\ n \geqslant 1,$$

where again, $\boldsymbol{X} = (X_1, X_2)^{\mathrm{T}}$ and Φ are defined by (11) and (18), respectively. Then, by using the hierarchy of symmetries $Z_1^{[n]}$, $n \geqslant 1$, a kind of specific solutions to the nonlocal integrable equation (32) can be presented as follows:

$$r = \exp(\varepsilon_1 Z_1^{[1]}) \exp(\varepsilon_2 Z_1^{[2]}) \cdots \exp(\varepsilon_m Z_1^{[m]}) r_0,\ m \geqslant 1,$$

where again, exp is the exponential map, ε_j, $1 \leqslant j \leqslant m$, are small parameters, and r_0 is an arbitrarily given solution. Particularly, we can take $r_0 = a\sin(cx + \mathrm{d}t) + b\cos(cx + \mathrm{d}t)$, where a, b, c, d are constants satisfying $a^2 = b^2$ and $d = \frac{15}{2} b^4 c + 10 b^2 c^3 + c^5$.

4. Conclusion and remarks

We have presented two fifth-order nonlocal integrable equations from a pair of matrix spectral problems associated with the special orthogonal Lie algebra so(3, $\mathbb{R}$). The presented nonlocal integrable equations inherit the common integrable characteristic: the existence of infinitely many symmetries and conservation laws.

Each pair of nonlocal integrable reductions generates the two same reduced nonlocal integrable equations. This phenomenon for integrable equations associated with so(3, $\mathbb{R}$) is different from the one for integrable equations associated with sl(2, $\mathbb{R}$). In the case of sl(2, $\mathbb{R}$), there are two inequivalent focusing and defocusing integrable reductions, both local and nonlocal.

For integrable equations associated with the special orthogonal Lie algebra so(3, $\mathbb{R}$), there are still many interesting questions. Particularly, we would like to know how to formulate Riemann–Hilbert problems so that the associated inverse scattering transforms [11] could be presented and N-soliton solutions [24, 25] could be worked out, which might lead to lump wave solutions [26–28] or rogue wave solutions [29, 30] to their higher-dimensional counterparts.

We remark that in general, establishing the global existence of solutions for nonlocal differential equations can be very challenging compared to local existence results. It is important to note that the global existence of solutions is not guaranteed for general Cauchy problems of nonlocal differential equations. Soliton solutions are explicitly presented only for particular integrable equations in the nonlocal case (see, e.g. [11, 12, 31, 32]). Analyzing mathematical properties of nonlocal differential equations, even nonlocal linear ordinary differential equations, and establishing their global existence results often requires careful analysis and applications of specialized techniques. Very little is known so far about integrable equations generated from matrix spectral problems associated with so(3, $\mathbb{R}$), in both local and nonlocal cases.

To summarize, nonlocal integrable equations are a challenging and actively researched field. While progress has been made in understanding specific classes of nonlocal integrable equations, there is still much to learn about their dynamical behavior, mathematical properties, and solution techniques. Continued research and exploration are necessary to advance our knowledge in this area.

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