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Lump solutions of a generalized Calogero–Bogoyavlenskii–Schiff equation

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ABSTRACT

A generalized Calogero–Bogoyavlenskii–Schiff equation is considered, and based on its Hirota bilinear form, a class of lump solutions is explicitly generated via symbolic computations with Maple, together with plots of a specific lump solution. The result enlarges the category of nonlinear partial differential equations which possess lump solutions.

1. Introduction

In the theory of differential equations, the Cauchy problem, one of the fundamental problems, is to find a solution of a differential equation satisfying what are known as initial values. Laplace's method and the Fourier transform method are designed for solving Cauchy problems for linear ordinary and partial differential equations, respectively. In soliton theory, powerful new solution techniques, called the isomonodromic transform method and the inverse scattering transform method, have been developed for attempting Cauchy problems for nonlinear ordinary and partial differential equations, respectively [1,2].

However, it is known that only the simplest differential equations, normally constant-coefficient and linear, are solvable explicitly. It is definitely difficult to determine exact solutions to nonlinear differential equations. Nevertheless, some novel studies have been made on a kind of interesting explicit solutions called lumps. Such solutions are originated from solving soliton equations [3–5], and researchers are motivated by analytical solutions interpretable as lower dimensional branes in open string field theory (see, e.g., [6]). In mathematics, it is known that a Padé approximant is the best approximation of a function by a rational function of given order (see, e.g., [7]), which can be computed through Wynn's epsilon algorithm for a given value of the independent variable [8]. Analytical rational solutions should, therefore, be a kind of solutions one first needs to explore for real wave motion.

Strictly speaking, lumps are a kind of rational function solutions that are localized in all directions in space, and solitons are analytic solutions exponentially localized in all directions in space and time, historically found for nonlinear integrable equations. By taking long wave limits, special lumps can be derived from N-soliton solutions [9]. There also exist positon and complexiton solutions to nonlinear integrable equations, adding to the diversity of solitons [10,11]. Other studies show

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that interaction solutions [12] between two different kinds of solutions exist for (2+1)-dimensional integrable equations, and they can be used to describe plenty of nonlinear phenomena in nature.

The Hirota bilinear method in soliton theory provides a powerful approach to looking for exact solutions [13]. Within the Hirota bilinear formulation, solitons can be usually generated as follows

$$u = 2(\ln f)_{xx},\ f = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^{N} \mu_i\xi_i + \sum_{i<j} \mu_i\mu_j a_{ij}\Big),$$

where

$$\xi_i = k_i x - \omega_i t + \xi_{i,0},\ \ 1 \le i \le N,$$

and

$$e^{a_{ij}} = -\frac{P(k_i - k_j, \omega_j - \omega_i)}{P(k_i + k_j, \omega_j + \omega_i)},\ \ 1 \le i < j \le N,$$

with k_i and ω_i satisfying the corresponding dispersion relation and $\xi_{i,0}$ being arbitrary phase shifts. The polynomial P above determines a Hirota bilinear form

$$P(D_x, D_t)f \cdot f = 0,$$

D_x and D_t being Hirota's bilinear derivatives, for a given partial differential equation with the dependent variable u. It is known that the KPI equation possesses lump solutions [14,15]. Other integrable equations which possess lump solutions include the three-dimensional three-wave resonant interaction [16], the BKP equation [17,18], the Davey–Stewartson equation II [9], the Ishimori-I equation [19] and many others (see, e.g., [4,20]). It will be very interesting to enlarge this category of nonlinear partial differential equations that possess lump solutions.

This letter aims to explore lump solutions, through Maple symbolic computations, to a generalized Calogero–Bogoyavlenskii–Schiff equation, based on the Hirota bilinear formulation. Explicit formulas of the parameters involved in the solutions will be presented and three-dimensional plots and contour plots of a specific example of the presented solutions will be made via Maple plot tools. Concluding remarks will be given in the final section.

2. A search for lump solutions

We consider a generalized Calogero–Bogoyavlenskii–Schiff (gCBS) equation

$$P_{gCBS}(u, v) := u_t + u_{xxy} + 3uu_y + 3u_x v_y + \delta_1 u_y + \delta_2 v_{yy} = 0, \tag{2.1}$$

where $v_x = u$, and δ_1 and δ_2 are two constants, or equivalently,

$$v_{tx} + v_{xxxy} + 3v_x v_{xy} + 3v_{xx} v_y + \delta_1 v_{xy} + \delta_2 v_{yy} = 0. \tag{2.2}$$

This is a generalization of a (2+1)-dimensional CBS equation considered in [21]

$$v_{tx} + v_{xxxy} + 3v_x v_{xy} + 3v_{xx} v_y = 0, \tag{2.3}$$

whose coefficients (3,3) have a different pattern from the original one (4,2) (see, e.g., [22] and references therein). It is direct to see that this gCBS equation (2.1) has a Hirota bilinear form

$$\begin{aligned} B_{gCBS}(f) &:= (D_t D_x + D_x^3 D_y + \delta_1 D_x D_y + \delta_2 D_y^2) f \cdot f \\ &= 2\big[f_{tx} f - f_t f_x + f_{xxxy} f - f_{xxx} f_y - 3f_{xxy} f_x + 3f_{xx} f_{xy} \\ &\quad + \delta_1 (f_{xy} f - f_x f_y) + \delta_2 (f_{yy} f - f_y^2)\big] = 0, \end{aligned} \tag{2.4}$$

under the transformations

$$u = 2(\ln f)_{xx} = \frac{2(f_{xx} f - f_x^2)}{f^2},\ \ v = 2(\ln f)_x = \frac{2f_x}{f}. \tag{2.5}$$

Such characteristic transformations have been adopted in Bell polynomial theories of soliton equations and their generalized counterparts (see, e.g., [23,24]), and precisely, we can have

$$P_{gCBS}(u, v) = \left(\frac{B_{gCBS}(f)}{f^2}\right)_x.$$

Therefore, when f solves the bilinear gCBS equation (2.4), $u = 2(\ln f)_{xx}$ and $v = 2(\ln f)_x$ will solve the (2+1)-dimensional gCBS equation (2.1).

Based on this bilinear form of the gCBS equation (2.1), we look for a class of quadratic function solutions to the (2+1)-dimensional bilinear gCBS equation (2.4), defined by

$$\begin{cases} f = \xi_1^2 + \xi_2^2 + a_9, \\ \xi_1 = a_1 x + a_2 y + a_3 t + a_4, \\ \xi_2 = a_5 x + a_6 y + a_7 t + a_8, \end{cases} \tag{2.6}$$

where $a_i,\ 1 \le i \le 9,$ are constant parameters to be determined. Plugging such a function f into (2.1) leads to a system of algebraic equations on the parameters. Further through symbolic computations with Maple, we can show that the resulting system of algebraic equations has a class of explicit solutions:

$$\left\{ a_3 = -a_2\delta_1 - \frac{a_1(a_2^2 - a_6^2) + 2a_2a_5a_6}{a_1^2 + a_5^2}\delta_2, \quad a_7 = -a_6\delta_1 - \frac{2a_1a_2a_6 - a_5(a_2^2 - a_6^2)}{a_1^2 + a_5^2}\delta_2, \quad a_9 = -\frac{3(a_1^2 + a_5^2)^2(a_1a_2 + a_5a_6)}{(a_1a_6 - a_2a_5)^2\delta_2} \right\}, \tag{2.7}$$

and the other parameters could be arbitrary provided that all the terms in the solutions of u and v will make sense.

By the transformations in (2.5), this yields a large class of lump solutions to the (2+1)-dimensional gCBS equation (2.1), determined by

$$\begin{cases} u = 2(\ln f)_{xx} = \dfrac{2(f_{xx}f - f_x^2)}{f^2} = \dfrac{4(a_1^2 + a_5^2)}{f} - \dfrac{8(a_1\xi_1 + a_5\xi_2)^2}{f^2}, \\ v = 2(\ln f)_x = \dfrac{2f_x}{f} = \dfrac{4(a_1\xi_1 + a_5\xi_2)}{f}. \end{cases} \tag{2.8}$$

The constant δ_2 in the solutions by (2.7) should not be zero, to produce lump solutions, but it could be either positive or negative, which is different from the situation in the KPI equation [15]. We know that the condition

$$a_1a_6 - a_2a_5 \neq 0 \tag{2.9}$$

is necessary and sufficient for a solution f, defined by (2.6), to engender a lump solution in (2+1)-dimensions through (2.5). Under the condition (2.9), we can solve

$$f_x(x(t), y(t), t) = 0,\ f_y(x(t), y(t), t) = 0, \tag{2.10}$$

to obtain all critical points of f:

$$\begin{cases} x = x(t) = \dfrac{(a_2a_7 - a_3a_6)t + (a_2a_8 - a_4a_6)}{a_1a_6 - a_2a_5}, \\ y = y(t) = -\dfrac{(a_1a_7 - a_3a_5)t + (a_1a_8 - a_4a_5)}{a_1a_6 - a_2a_5}, \end{cases} \tag{2.11}$$

where t is arbitrarily fixed. The sum of two squares, i.e., the function $f - a_9$, vanishes at this set of critical points. Therefore, $f > 0$ if and only if $a_9 > 0$, which implies that u and v defined by (2.5) are analytical in $\mathbb{R}^3$, if and only if $a_9 > 0$, and further according to (2.7), if and only if $(a_1a_2 + a_5a_6)\delta_2 < 0$. For any given time t, the point $(x(t), y(t))$ defined by (2.11) is also a critical point of the function $u = 2(\ln f)_{xx}$, and so, by the second derivative test, the lump solution u has a peak at this point $(x(t), y(t))$, because we have

$$\begin{cases} u_{xx} = -\dfrac{24(a_1^2 + a_5^2)^2}{a_9^2} < 0, \\ u_{xx}u_{yy} - u_{xy}^2 = \dfrac{192(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)^2}{a_9^4} > 0 \end{cases} \tag{2.12}$$

at the point $(x(t), y(t))$.

All the solutions generated above add to the solution theories available on soliton solutions and dromion-type solutions, developed through powerful existing approaches such as the Hirota perturbation technique and symmetry constraints (see, e.g., [25–29]).

We take $\delta_1 = 1$ and $\delta_2 = -1$, and then have

$$u_t + u_{xxy} + 3uu_y + 3u_xv_y + u_y - v_{yy} = 0, \tag{2.13}$$

where $u = v_x$. Now, further taking

$$a_1 = 2,\ a_2 = -2,\ a_4 = 1,\ a_5 = 2,\ a_6 = 5,\ a_8 = -1, \tag{2.14}$$

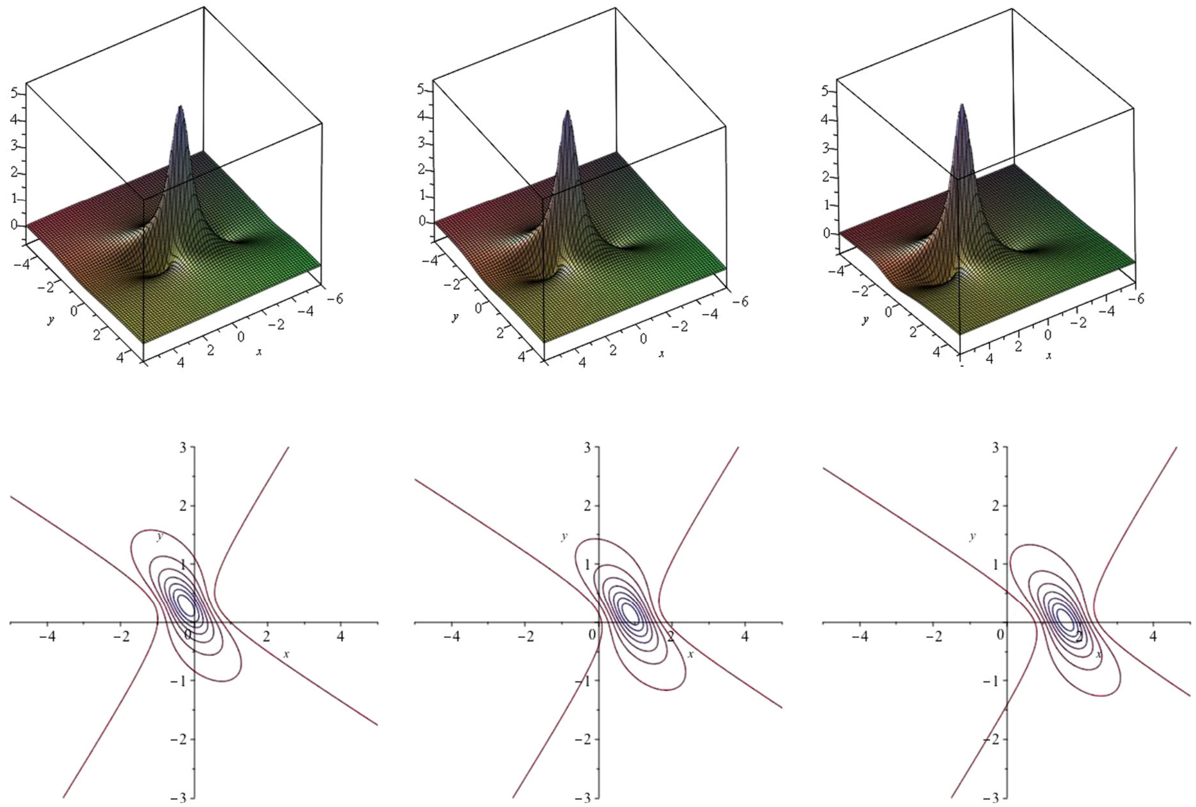

Fig. 1. Profiles of u in (2.15) when $t = 0, 0.3, 0.5$: 3d plots (top) and contour plots (bottom).

which ensures the positivity of the generating function f, we can obtain a specific lump solution as follows:

$$u = \frac{32}{f} - \frac{32(4x+3y-13t)^2}{f^2}, \tag{2.15}$$

where

$$f = (2x - 2y - \frac{33}{4}t + 1)^2 + (2x + 5y - \frac{19}{4}t - 1)^2 + \frac{288}{49}. \tag{2.16}$$

Three three-dimensional plots and contour plots of this lump solution are made, to shed light on the characteristic of the solution, in Fig. 1.

3. Concluding remarks

We have studied a generalized (2+1)-dimensional Calogero–Bogoyavlenskii–Schiff (gCBS) equation to explore lump solutions, via Maple symbolic computations. The result enlarges the class of nonlinear integrable equations which possess lump solutions. Three-dimensional plots and contour plots of a specially chosen solution were made via Maple plot tools.

Through making symbolic computations, it is recognized that many other nonlinear equations possess lump solutions, which include (2+1)-dimensional generalized KP, BKP, KP-Boussinesq and Sawada–Kotera equations [30–31]. Moreover, some recent studies exhibited interaction solutions between lumps and other kinds of exact solutions to nonlinear integrable equation in (2+1)-dimensions, including lump–kink interaction solutions (see, e.g., [32–34]) and lump-soliton interaction solutions (see, e.g., [35–37]). In the (3+1)-dimensional case, lump-type solutions, which are rationally localized in almost all directions in space, were generated for the integrable Jimbo–Miwa equations. Plenty of such solutions were presented for the (3+1)-dimensional Jimbo–Miwa equation (see, e.g., [38,39]) and the (3+1)-dimensional Jimbo–Miwa like equation [40]. It is, of course, interesting to search for lump and interaction solutions to partial differential equations in whatever dimensions.

We remark that we failed to obtain interaction solutions between lumps and kink or soliton solutions for the (2+1)-dimensional gCBS equation (2.1). The existence of such interaction solutions could reflect complete integrability of the partial differential equations under consideration. It is also clear that diverse lump solutions, providing supplements to exact solutions generated from different kinds of combinations (see, e.g., [41]), imply that there exist the corresponding

Lie–Bäcklund symmetries, which may help determine conservation laws [42,43]. Conversely, it should be very interesting to determine what kind of partial differential equations could possess lump solutions.

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References

[1] S. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, Theory of Solitons - The Inverse Scattering Method, Consultants Bureau, New York, 1984.
[2] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.
[3] W.X. Ma, Y. Zhou, R. Dougherty, Lump-type solutions to nonlinear differential equations derived from generalized bilinear equations, Internat. J. Modern Phys. B 30 (2016) 1640018.
[4] W. Tan, H.P. Dai, Z.D. Dai, W.Y. Zhong, Emergence and space-time structure of lump solution to the (2+1)-dimensional generalized KP equation, Pramana –J. Phys. 89 (2017) 77.
[5] W.X. Ma, Y. Zhou, Lump solutions to nonlinear partial differential equations via Hirota bilinear forms, J. Differential Equations 264 (2018) 2633–2659.
[6] L. Bonora, C. Maccaferri, D.D. Tolla, Relevant deformations in open string field theory: a simple solution for lumps, J. High Energy Phys. 1111 (2011) 107.
[7] G.A. Jr. Baker, P. Graves-Morris, Padé Approximants, Cambridge University Press, New York, 1996.
[8] P. Wynn, On the convergence and stability of the epsilon algorithm, SIAM J. Numer. Anal. 3 (1966) 91–122.
[9] J. Satsuma, M.J. Ablowitz, Two-dimensional lumps in nonlinear dispersive systems, J. Math. Phys. 20 (1979) 1496–1503.
[10] W.X. Ma, Wronskian solutions to integrable equations, Discrete Contin. Dyn. Syst. Suppl. (2009) 506–515.
[11] A.-M. Wazwaz, S.A. El-Tantawy, New (3+1)-dimensional equations of Burgers type and Sharma-Tasso-Olver type: multiple-soliton solutions, Nonlinear Dynam. 87 (2017) 2457–2461.
[12] W.X. Ma, Y. You, Solving the Korteweg–de Vries equation by its bilinear form: Wronskian solutions, Trans. Amer. Math. Soc. 357 (2005) 1753–1778.
[13] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, New York, 2004.
[14] S.V. Manakov, V.E. Zakharov, L.A. Bordag, V.B. Matveev, Two-dimensional solitons of the Kadomtsev–Petviashvili equation and their interaction, Phys. Lett. A 63 (1977) 205–206.
[15] W.X. Ma, Lump solutions to the Kadomtsev–Petviashvili equation, Phys. Lett. A 379 (2015) 1975–1978.
[16] D.J. Kaup, The lump solutions and the Bäcklund transformation for the three-dimensional three-wave resonant interaction, J. Math. Phys. 22 (1981) 1176–1181.
[17] C.R. Gilson, J.J.C. Nimmo, Lump solutions of the BKP equation, Phys. Lett. A 147 (1990) 472–476.
[18] J.Y. Yang, W.X. Ma, Lump solutions of the BKP equation by symbolic computation, Internat. J. Modern Phys. B 30 (2016) 1640028.
[19] K. Imai, Dromion and lump solutions of the Ishimori-I equation, Progr. Theoret. Phys. 98 (1997) 1013–1023.
[20] Y. Zhang, H.H. Dong, X.E. Zhang, H.W. Yang, Rational solutions and lump solutions to the generalized (3 + 1)-dimensional shallow water-like equation, Comput. Math. Appl. 73 (2017) 246–252.
[21] A.-M. Wazwaz, The (2+1) and (3+1)-dimensional CBS equations: multiple soliton solutions and multiple singular soliton solutions, Z. Naturforsch. 65A (2010) 173–181.
[22] M.S. Bruzon, M.L. Gandarias, C. Muriel, J. Ramierez, S. Saez, F.R. Romero, The Calogero-Bogoyavlenskii-Schiff equation in 2+1 dimensions, Theoret. Math. Phys. 137 (2003) 1367–1377.
[23] C. Gilson, F. Lambert, J. Nimmo, R. Willox, On the combinatorics of the Hirota D-operators, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 452 (1996) 223–234.
[24] W.X. Ma, Bilinear equations, Bell polynomials and linear superposition principle, J. Phys. Conf. Ser. 411 (2013) 012021.
[25] B. Dorizzi, B. Grammaticos, A. Ramani, P. Winternitz, Are all the equations of the Kadomtsev–Petviashvili hierarchy integrable? J. Math. Phys. 27 (1986) 2848–2852.
[26] X.Y. Li, Q.L. Zhao, Y.X. Li, H.H. Dong, Binary Bargmann symmetry constraint associated with 3×3 discrete matrix spectral problem, J. Nonlinear Sci. Appl. 8 (2015) 496–506.
[27] Q.L. Zhao, X.Y. Li, A Bargmann system and the involutive solutions associated with a new 4-order lattice hierarchy, Anal. Math. Phys. 6 (2016) 237–254.
[28] H.H. Dong, Y. Zhang, X.E. Zhang, The new integrable symplectic map and the symmetry of integrable nonlinear lattice equation, Commun. Nonlinear Sci. Numer. Simul. 36 (2016) 354–365.
[29] X.Y. Li, Q.L. Zhao, A new integrable symplectic map by the binary nonlinearization to the super AKNS system, J. Geom. Phys. 121 (2017) 123–137.
[30] X. Lü, S.T. Chen, W.X. Ma, Constructing lump solutions to a generalized Kadomtsev–Petviashvili-Boussinesq equation, Nonlinear Dynam. 86 (2016) 523–534.
[31] J.P. Yu, Y.L. Sun, Study of lump solutions to dimensionally reduced generalized KP equations, Nonlinear Dynam. 87 (2017) 2755–2763.
[32] H.Q. Zhao, W.X. Ma, Mixed lump-kink solutions to the KP equation, Comput. Math. Appl. 74 (2017) 1399–1405.
[33] T.C. Kofane, M. Fokou, A. Mohamadou, E. Yomba, Lump solutions and interaction phenomenon to the third-order nonlinear evolution equation, Eur. Phys. J. Plus 132 (2017) 465.
[34] J.B. Zhang, W.X. Ma, Mixed lump-kink solutions to the BKP equation, Comput. Math. Appl. 74 (2017) 591–596.
[35] J.Y. Yang, W.X. Ma, Z.Y. Qin, Lump and lump-soliton solutions to the (2+1)-dimensional Ito equation, Anal. Math. Phys. (2017). http://dx.doi.org/10.1007/s13324-017-0181-9.
[36] W.X. Ma, X.L. Yong, H.Q. Zhang, Diversity of interaction solutions to the (2+1)-dimensional Ito equation, Comput. Math. Appl. 75 (2018) 289–295.
[37] J.Y. Yang, W.X. Ma, Z.Y. Qin, Mixed lump-soliton solutions of the BKP equation, East Asian J. Appl. Math. (2017).
[38] J.Y. Yang, W.X. Ma, Abundant lump-type solutions of the Jimbo–Miwa equation in (3+1)-dimensions, Comput. Math. Appl. 73 (2017) 220–225.
[39] Y. Zhang, S.L. Sun, H.H. Dong, Hybrid solutions of (3+1)-dimensional Jimbo–Miwa equation, Math. Probl. Eng. (2017) 5453941.

[40] Harun-Or-Roshid, M.Z. Ali, Lump solutions to a Jimbo–Miwa like equation, 2016, arXiv:1611.04478.
[41] Ö. Ünsal, W.X. Ma, Linear superposition principle of hyperbolic and trigonometric function solutions to generalized bilinear equations, Comput. Math. Appl. 71 (2016) 1242–1247.
[42] N.H. Ibragimov, A new conservation theorem, J. Math. Anal. Appl. 333 (2007) 311–328.
[43] W.X. Ma, Conservation laws by symmetries and adjoint symmetries, Discrete Contin. Dyn. Syst. Ser. S 11 (2018) 707–721.