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RESEARCH ARTICLE

Lump solutions to a generalized Hietarinta-type equation via symbolic computation

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Abstract Lump solutions are one of important solutions to partial differential equations, both linear and nonlinear. This paper aims to show that a Hietarinta-type fourth-order nonlinear term can create lump solutions with second-order linear dispersive terms. The key is a Hirota bilinear form. Lump solutions are constructed via symbolic computations with Maple, and specific reductions of the resulting lump solutions are made. Two illustrative examples of the generalized Hietarinta-type nonlinear equations and their lumps are presented, together with three-dimensional plots and density plots of the lump solutions.

Keywords Soliton equation, lump solution, symbolic computation, Hirota derivative, dispersion relation
MSC 35Q51, 35Q53, 37K40

1 Introduction

Soliton solutions to integrable equations are analytic and usually exponentially localized in space and time [1,46]. The Hirota bilinear method [3,15] is among the most effective approaches to soliton solutions. Suppose that P is a polynomial in x, y, and t. Then a Hirota bilinear differential equation in $(2+1)$-dimensions can be defined by

$$P(D_x, D_y, D_t) f \cdot f = 0,$$

where D_x, D_y, and D_t are Hirota's bilinear derivatives [15]. An associated partial differential equation (PDE) with a dependent variable u is often

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determined by some logarithmic transformation of

$$u = 2(\log f)_x, \quad u = 2(\log f)_{xx}.$$

Within the Hirota bilinear formulation, an N-soliton solution (see, e.g., [14]) is presented via

$$f = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}\Big),$$

where $\sum_{\mu=0,1}$ is the sum over all possibilities for $\mu_1, \mu_2, \ldots, \mu_N$ taking either 0 or 1, and the wave variables and the phase shifts are given by

$$\xi_i = k_i x + l_i y - \omega_i t + \xi_{i,0}, \quad 1 \leqslant i \leqslant N,$$

and

$$\mathrm{e}^{a_{ij}} = -\frac{P(k_i - k_j, l_i - l_j, \omega_j - \omega_i)}{P(k_i + k_j, l_i + l_j, \omega_j + \omega_i)}, \quad 1 \leqslant i < j \leqslant N,$$

respectively. Here, the wave numbers k_i, l_i and the frequencies ω_i, $1 \leqslant i \leqslant N$, need to satisfy the associated dispersion relations

$$P(k_i, l_i, -\omega_i) = 0, \quad 1 \leqslant i \leqslant N,$$

but the phases shifts $\xi_{i,0}$, $1 \leqslant i \leqslant N$, are arbitrary constants.

It has been shown recently that lump solutions to integrable equations are remarkably varied, which can describe diverse wave phenomena. Lumps are rational solutions, which are analytic and localized in all directions in space (see, e.g., [42,43,51]), and they can also be derived from computing long wave limits of soliton equations (see, e.g., [49]). The KPI equation has abundant lump solutions (see, e.g., [27]), and its special lump solutions are constructed from its soliton solutions [44]. Other integrable equations which possess lump solutions contain the three-dimensional three-wave resonant interaction [18], the Davey-Stewartson II equation [49], the Ishimori-I equation [17], the BKP equation [11,59], and the KP equation with a self-consistent source [63]. Furthermore, nonintegrable equations can possess lump solutions, among which are a few generalized KP, BKP, KP-Boussinesq, Sawada-Kotera, Calogero-Bogoyavlenskii-Schiff and Bogoyavlensky-Konopelchenko equations in $(2+1)$-dimensions [6,7,24,31,37,39,65]. The crucial step in finding lump solutions is to construct positive quadratic function solutions to Hirota bilinear equations [42]. Then based on positive quadratic function solutions, the logarithmic transformations yield lump solutions to nonlinear PDEs.

In this paper, we would like to discuss a generalized Hietarinta-type fourth-order equation in $(2+1)$-dimensional dispersive waves and determine its diverse lump solutions. The key is a Hirota bilinear form in the solution process (see, e.g., [26,42,43]). The considered Hietarinta-type nonlinear equation contains two fourth-order nonlinear terms and five second-order linear terms. Lump solutions will be determined via symbolic computation with Maple. Two illustrative examples of the considered model equation will be made, together with specific lump solutions and their three-dimensional plots and density plots. Concluding remarks will be given finally in the last section.

2 A generalized Hietarinta-type equation

We would like to consider a generalized Hietarinta-type equation:

$$\begin{aligned} P(u) &= \alpha_1(6u_xu_{xx} + u_{xxxx}) + \alpha_2(3u_tu_{tt} + 3u_{xt}v_{tt} + u_{xttt}) \\ &\quad + \gamma_1u_{yt} + \gamma_2u_{xx} + \gamma_3u_{xt} + \gamma_4u_{xy} + \gamma_5u_{yy} \\ &= 0, \end{aligned} \tag{2.1}$$

where $v_x = u$, and the constants α_1, α_2, and γ_i, $1 \leqslant i \leqslant 5$, are generally arbitrary. The coefficient α_2 corresponds to a Hietarinta-type nonlinear term studied in [13]. We will see that this nonlinear term creates the complexity of presenting lump solutions, and the corresponding constant term in the solution of the associated Hirota bilinear equation is very complicated.

It is straightforward to check that through the logarithmic transformations

$$u = 2(\log f)_x, \quad v = 2\log f, \tag{2.2}$$

the above generalized Hietarinta-type nonlinear equation (2.1) is linked with the following Hirota bilinear equation:

$$\begin{aligned} B(f) &= (\alpha_1D_x^4 + \alpha_2D_xD_t^3 + \gamma_1D_yD_t + \gamma_2D_x^2 \\ &\quad + \gamma_3D_xD_t + \gamma_4D_xD_y + \gamma_5D_y^2)f\cdot f \\ &= 0, \end{aligned} \tag{2.3}$$

where D_x, D_y, and D_t are three Hirota bilinear derivatives. In fact, the connection between the nonlinear and bilinear equations reads

$$P(u) = \Big(\frac{B(f)}{f^2}\Big)_x,$$

when u, v, and f are determined by (2.2). The generalized Hietarinta-type equation (2.3) contains two types of fourth-order derivative terms and five second-order derivative terms. We will show that there exist abundant lump solutions to the generalized Hietarinta-type equation (2.3).

If we take

$$\alpha_1 = \gamma_1 = \gamma_2 = 0,$$

then the generalized Hietarinta-type equation (2.1) presents a reduced nonlinear equation:

$$\alpha_2(3u_tu_{tt} + 3u_{xt}v_{tt} + u_{xttt}) + \gamma_3u_{xt} + \gamma_4u_{xy} + \gamma_5u_{yy} = 0,$$

which possesses a Hirota bilinear form

$$(\alpha_2 D_x D_t^3 + \gamma_3 D_x D_t + \gamma_4 D_x D_y + \gamma_5 D_y^2) f \cdot f = 0,$$

under (2.2), and an explicit lump solution in one case of this equation will be presented later.

If we take

$$\gamma_4 = \gamma_5 = 0,$$

then the generalized Hietarinta-type equation (2.1) gives another reduced nonlinear equation:

$$\alpha_1(6u_x u_{xx} + u_{xxxx}) + \alpha_2(3u_t u_{tt} + 3u_{xt} v_{tt} + u_{xttt}) + \gamma_1 u_{yt} + \gamma_2 u_{xx} + \gamma_3 u_{xt} = 0,$$

whose Hirota bilinear form is given by

$$(\alpha_1 D_x^4 + \alpha_2 D_x D_t^3 + \gamma_1 D_y D_t + \gamma_2 D_x^2 + \gamma_3 D_x D_t) f \cdot f = 0,$$

under (2.2). An example of lump solutions in one case of this equation will be presented later as well.

3 Lump solutions via symbolic computation

In this section, we would like to compute lump solutions to the generalized Hietarinta-type fourth-order nonlinear equation (2.1) via symbolic computations with Maple.

Using a general ansatz on lump solutions in (2+1)-dimensions [27], we start to determine positive quadratic solutions

$$f = (a_1 x + a_2 y + a_3 t + a_4)^2 + (a_5 x + a_6 y + a_7 t + a_8)^2 + a_9, \tag{3.1}$$

to the corresponding Hirota bilinear equation (2.3). The task is to conduct symbolic computations to determine the involved constant parameters a_i, $1 \leqslant i \leqslant 9$.

A direct computation with a Maple code can determine a set of solutions for the parameters:

$$\begin{aligned} a_3 &= -\frac{b_1}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}, \\ a_7 &= -\frac{b_2}{(a_2\gamma_1 + a_1\gamma_3)^2 + (a_6\gamma_1 + a_5\gamma_3)^2}, \\ a_9 &= \frac{\alpha_1 b_3 + \alpha_2(b_{4,1} + b_{4,2} + b_{4,3})}{q}, \end{aligned} \tag{3.2}$$

and all other a_i's are arbitrary. The involved seven constants of b_i, $b_{4,i}$, $1 \leqslant i \leqslant 3$, and q are given by

$$\begin{aligned} b_1 = {} & [(a_1^2 a_2 + 2\, a_1 a_5 a_6 - a_2 a_5^2)\gamma_2 + a_1(a_2^2 + a_6^2)\gamma_4 + a_2(a_2^2 + a_6^2)\gamma_5]\gamma_1 \\ & + [a_1(a_1^2 + a_5^2)\gamma_2 + a_2(a_1^2 + a_5^2)\gamma_4 + (a_1 a_2^2 + 2\, a_2 a_5 a_6 - a_1 a_6^2)\gamma_5]\gamma_3, \end{aligned}$$

$$
\begin{aligned}
b_2 =& [(-a_1^2a_6 + 2\,a_1a_2a_5 + a_5^2a_6)\gamma_2 + a_5(a_2^2 + a_6^2)\gamma_4 + a_6(a_2^2 + a_6^2)\gamma_5]\gamma_1 \\
&+ [a_5(a_1^2 + a_5^2)\gamma_2 + a_6(a_1^2 + a_5^2)\gamma_4 + (-a_2^2a_5 + 2\,a_1a_2a_6 + a_5a_6^2)\gamma_5]\gamma_3, \\
b_3 =& -3(a_1^2 + a_5^2)^4\gamma_3^4 - 12(a_1^2 + a_5^2)^3(a_1a_2 + a_5a_6)\gamma_1\gamma_3^3 - 6(a_1^2 + a_5^2)^2p_1\gamma_1^2\gamma_3^2 \\
&- 12(a_2^2 + a_6^2)(a_1^2 + a_5^2)^2(a_1a_2 + a_5a_6)\gamma_1^3\gamma_3 - 3(a_2^2 + a_6^2)^2(a_1^2 + a_5^2)^2\gamma_1^4, \\
b_{4,1} =& 3(a_2^2 + a_6^2)^2p_2\gamma_3\gamma_5^3 + 3({a_2}^2 + {a_6}^2)^3(a_1a_2 + a_5a_6)\gamma_1\gamma_5^3 \\
&+ 3(a_2^2 + a_6^2)(a_1a_2 + a_5a_6)p_3\gamma_3\gamma_4\gamma_5^2 + 3(a_2^2 + a_6^2)^2p_1\gamma_1\gamma_4\gamma_5^2 \\
&+ p_4\gamma_2\gamma_3\gamma_5^2 + 3(a_2^2 + a_6^2)(a_1a_2 + a_5a_6)p_3\gamma_1\gamma_2\gamma_5^2, \\
b_{4,2} =& 3(a_2^2 + a_6^2)(a_1^2 + a_5^2)p_3\gamma_3\gamma_4^2\gamma_5 + 9(a_2^2 + a_6^2)^2(a_1^2 + a_5^2) \\
&\cdot (a_1a_2 + a_5a_6)\gamma_1\gamma_4^2\gamma_5 + 6(a_1^2 + a_5^2)(a_1a_2 + a_5a_6)p_3\gamma_2\gamma_3\gamma_4\gamma_5 \\
&+ 6(a_2^2 + a_6^2)(a_1^2 + a_5^2)p_3\gamma_1\gamma_2\gamma_4\gamma_5 + 9(a_1^2 + a_5^2)^2p_2\gamma_2^2\gamma_3\gamma_5 \\
&+ 3(a_1^2 + a_5^2)(a_1a_2 + a_5a_6)p_3\gamma_1\gamma_2^2\gamma_5, \\
b_{4,3} =& 3(a_2^2 + a_6^2)(a_1^2 + a_5^2)^2(a_1a_2 + a_5a_6)\gamma_3\gamma_4^3 + 3(a_2^2 + a_6^2)^2(a_1^2 + a_5^2)^2\gamma_1\gamma_4^3 \\
&+ 3(a_1^2 + a_5^2)^2p_1\gamma_2\gamma_3\gamma_4^2 + 9(a_2^2 + a_6^2)(a_1^2 + a_5^2)^2(a_1a_2 + a_5a_6)\gamma_1\gamma_2\gamma_4^2 \\
&+ 9(a_1^2 + a_5^2)^3(a_1a_2 + a_5a_6)\gamma_2^2\gamma_3\gamma_4 + 3(a_1^2 + a_5^2)^2p_1\gamma_1\gamma_2^2\gamma_4 \\
&+ 3(a_1^2 + a_5^2)^4\gamma_2^3\gamma_3 + 3(a_1^2 + a_5^2)^3(a_1a_2 + a_5a_6)\gamma_1\gamma_2^3,
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
q =& (a_1^2 + a_5^2)(a_1a_6 - a_2a_5)^2\gamma_3^4\gamma_5 + 2(a_1a_2 + a_5a_6)(a_1a_6 - a_2a_5)^2\gamma_1\gamma_3^3\gamma_5 \\
&+ (a_2^2 + a_6^2)(a_1a_6 - a_2a_5)^2\gamma_1^2\gamma_3^2\gamma_5 - (a_1^2 + a_5^2)(a_1a_6 - a_2a_5)^2\gamma_1\gamma_3^3\gamma_4 \\
&+ p_5\gamma_1^2\gamma_3^2\gamma_4 - (a_2^2 + a_6^2)(a_1a_6 - a_2a_5)^2\gamma_1^3\gamma_3\gamma_4 \\
&+ (a_1^2 + a_5^2)(a_1a_6 - a_2a_5)^2\gamma_1^2\gamma_2\gamma_3^2 + 2(a_1a_2 + a_5a_6)(a_1a_6 - a_2a_5)^2\gamma_1^3\gamma_2\gamma_3 \\
&+ (a_2^2 + a_6^2)(a_1a_6 - a_2a_5)^2\gamma_1^4\gamma_2,
\end{aligned}
\tag{3.4}
$$

where for brevity, we define five polynomials $p_i,\ 1 \leqslant i \leqslant 5$, as follows:

$$
\begin{aligned}
p_1 &= 3a_1^2a_2^2 + a_1^2a_6^2 + 4a_1a_2a_5a_6 + a_2^2a_5^2 + 3a_5^2a_6^2, \\
p_2 &= (a_1a_2 - a_1a_6 + a_2a_5 + a_5a_6)(a_1a_2 + a_1a_6 - a_2a_5 + a_5a_6), \\
p_3 &= 3{a_1}^2a_2^2 - a_1^2a_6^2 + 8a_1a_2a_5a_6 - a_2^2a_5^2 + 3a_5^2a_6^2, \\
p_4 &= 9a_1^4a_2^4 - 6a_1^4a_2^2a_6^2 + 9a_1^4a_6^4 + 48a_1^3a_2^3a_5a_6 - 48a_1^3a_2a_5a_6^3 \\
&\quad - 6a_1^2a_2^4a_5^2 + 132a_1^2a_2^2a_5^2a_6^2 - 6a_1^2a_5^2a_6^4 - 48a_1a_2^3a_5^3a_6 \\
&\quad + 48a_1a_2a_5^3a_6^3 + 9a_2^4a_5^4 - 6a_2^2a_5^4a_6^2 + 9a_5^4a_6^4,
\end{aligned}
\tag{3.5}
$$

$$
p_5 = -2a_1^3a_2a_6^2 + 4a_1^2a_2^2a_5a_6 - 2a_1^2a_5a_6^3 - 2a_1a_2^3a_5^2 + 4a_1a_2a_5^2a_6^2 - 2a_2^2a_5^3a_6.
$$

The constant $b_{4,1}$ consists of terms involving γ_5^3 and γ_5^2; the constant $b_{4,2}$, γ_5; and the constant $b_{4,3}$, γ_5^0. The above solutions for a_3 and a_7 represent abundant dispersion relations in (2 + 1)-dimensional dispersive waves, and the solution for a_9 exhibits a very complicated coefficient in quadratic solutions f to Hirota bilinear equations, special reductions of which will be made in the nest section.

We point out that all the above expressions for the wave frequencies and the constant term in (3.2)–(3.5) have been presented through direct simplifications with Maple. To generate lump solutions, besides $a_9 > 0$ to guarantee the analyticity of rational solutions, we require only one basic condition:

$$a_1a_6 - a_2a_5 \neq 0,$$

which implies the localization of rational solutions in all spatial directions.

4 Specific reductions

4.1 Case of $\gamma_1 = \gamma_2 = 0$

We consider the case of

$$\gamma_1 = \gamma_2 = 0, \quad \gamma_3 = \gamma_4 = \gamma_5 = 1.$$

The corresponding generalized Hietarinta-type nonlinear equation and bilinear equation read

$$\alpha_1(6u_xu_{xx} + u_{xxxx}) + \alpha_2(3u_tu_{tt} + 3u_{xt}v_{tt} + u_{xttt})u_{xt} + u_{xy} + u_{yy} = 0,$$

where $v_x = u$, and

$$(\alpha_1 D_x^4 + \alpha_2 D_x D_t^3 + D_x D_t + D_x D_y + D_y^2) f \cdot f = 0,$$

respectively. The reduced frequencies and constant coefficient are

$$\begin{aligned} a_3 &= -\frac{a_1^2a_2 + a_1a_2^2 - a_1a_6^2 + a_2a_5^2 + 2a_2a_5a_6}{a_1^2 + a_5^2}, \\ a_7 &= -\frac{a_1^2a_6 - a_2^2a_5 + a_5^2a_6 + a_5a_6^2 + 2a_1a_2a_6}{a_1^2 + a_5^2}, \\ a_9 &= -\frac{3(a_1^2 + a_5^2)^4\alpha_1 - b_4\alpha_2}{(a_1^2 + a_5^2)(a_1a_6 - a_2a_5)^2}, \end{aligned}$$

where

$$\begin{aligned} b_4 = 3(a_2^2 + a_6^2)[(a_1 + a_2)^2 + (a_5 + a_6)^2][a_1^3a_2 + (a_2^2 + a_5a_6 - a_6^2)a_1^2 \\ + a_2a_5(a_5 + 4a_6)a_1 - a_5^2(a_2^2 - a_5a_6 - a_6^2)]. \end{aligned}$$

4.2 Case of $\gamma_1 = \gamma_4 = 0$

We consider the case of

$$\gamma_1 = \gamma_4 = 0, \quad \gamma_2 = \gamma_3 = \gamma_5 = 1.$$

The corresponding generalized Hietarinta-type nonlinear equation and bilinear equation read

$$\alpha_1(6u_xu_{xx}+u_{xxxx})+\alpha_2(3u_tu_{tt}+3u_{xt}v_{tt}+u_{xttt})u_{xx}+u_{xt}+u_{yy}=0,$$

where $v_x=u$, and

$$(\alpha_1D_x^4+\alpha_2D_xD_t^3+D_x^2+D_xD_t+D_y^2)f\cdot f=0,$$

respectively. The reduced frequencies and constant coefficient are

$$a_3=-\frac{a_1^3+a_1a_2^2+a_1a_5^2-a_1a_6^2+2a_2a_5a_6}{a_1^2+a_5^2},$$

$$a_7=-\frac{a_1^2a_5-a_2^2a_5+a_5^3+a_5a_6^2+2a_1a_2a_6}{a_1^2+a_5^2},$$

$$a_9=-\frac{3(a_1^2+a_5^2)^4\alpha_1-b_4\alpha_2}{(a_1^2+a_5^2)(a_1a_6-a_2a_5)^2},$$

where

$$b_4=3[a_1^4+(a_2^2+2a_5^2-a_6^2)a_1^2+4a_2a_5a_6a_1-a_5^2(a_2^2-a_5^2-a_6^2)]\cdot[(a_1-a_6)^2+(a_2+a_5)^2][(a_1+a_6)^2+(a_2-a_5)^2].$$

4.3 Case of $\gamma_2=\gamma_4=0$

We consider the case of

$$\gamma_2=\gamma_4=0,\quad \gamma_1=\gamma_3=\gamma_5=1.$$

The corresponding generalized Hietarinta-type nonlinear equation and bilinear equation read

$$\alpha_1(6u_xu_{xx}+u_{xxxx})+\alpha_2(3u_tu_{tt}+3u_{xt}v_{tt}+u_{xttt})u_{yt}+u_{xt}+u_{yy}=0,$$

where $v_x=u$, and

$$(\alpha_1D_x^4+\alpha_2D_xD_t^3+D_yD_t+D_xD_t+D_y^2)f\cdot f=0,$$

respectively. The reduced frequencies and constant coefficient are

$$a_3=-\frac{a_1a_2^2-a_1a_6^2+a_2^3+2a_2a_5a_6+a_2a_6^2}{(a_1+a_2)^2+(a_5+a_6)^2},$$

$$a_7=-\frac{2a_1a_2a_6-a_2^2a_5+a_2^2a_6+a_5a_6^2+a_6^3}{(a_1+a_2)^2+(a_5+a_6)^2},$$

$$a_9=\frac{b_3\alpha_1+b_4\alpha_2}{[(a_1+a_2)^2+(a_5+a_6)^2](a_1a_6-a_2a_5)^2},$$

where

$$
\begin{aligned}
b_3 = & -3a_1^8 - 12a_1^7a_2 - 18\Big(a_2^2 + \frac{2}{3}\,a_5^2 + \frac{2}{3}\,a_5a_6 + \frac{1}{3}\,a_6^2\Big)a_1^6 \\
& - 12a_2(a_2^2 + 3a_5^2 + 2a_5a_6 + a_6^2)a_1^5 - 3[6a_5^4 + 12a_5^3a_6 + (14a_2^2 + 10a_6^2)a_5^2 \\
& + (4a_6a_2^2 + 4a_6^3)a_5 + (a_2^2 + a_6^2)^2]a_1^4 - 24a_5^2\Big(a_2^2 + \frac{3}{2}\,a_5^2 + 2a_5a_6 + a_6^2\Big)a_2a_1^3 \\
& - [12a_5^6 + 36a_6a_5^5 + (30a_2^2 + 42a_6^2)a_5^4 + 24a_6(a_2^2 + a_6^2)a_5^3 \\
& + 6(a_2^2 + a_6^2)^2a_5^2]a_1^2 - 3[4a_5^6 + 8a_6a_5^5 + 4(a_2^2 + a_6^2)a_5^4]a_2a_1 \\
& - 3a_5[a_5^7 + 4a_6a_5^6 + 2(a_2^2 + 3a_6^2)a_5^5 + 4a_6(a_2^2 + a_6^2)a_5^4 + (a_2^2 + a_6^2)^2a_5^3],
\end{aligned}
$$

$$
\begin{aligned}
b_4 = {} & 3(a_2^2 - a_6^2)(a_2^2 + a_6^2)^2a_1^2 + 3[4a_6(a_2^2 + a_6^2)^2a_5 + (a_2^2 + a_6^2)^3]a_2a_1 \\
& - 3a_5[(a_2^2 - a_6^2)(a_2^2 + a_6^2)^2a_5 - a_6(a_2^2 + a_6^2)^3].
\end{aligned}
$$

4.4 Case of $\gamma_2 = \gamma_5 = 0$

We consider the case of

$$\gamma_2 = \gamma_5 = 0, \quad \gamma_1 = \gamma_3 = \gamma_4 = 1.$$

The corresponding generalized Hietarinta-type nonlinear equation and bilinear equation read

$$\alpha_1(6u_xu_{xx} + u_{xxxx}) + \alpha_2(3u_tu_{tt} + 3u_{xt}v_{tt} + u_{xttt})u_{yt} + u_{xt} + u_{xy} = 0,$$

where $v_x = u$, and

$$(\alpha_1 D_x^4 + \alpha_2 D_x D_t^3 + D_y D_t + D_x D_t + D_x D_y) f \cdot f = 0,$$

respectively. The reduced frequencies and constant coefficient are

$$
\begin{aligned}
a_3 &= -\frac{(a_1^2 + a_5^2)a_2 + (a_2^2 + a_6^2)a_1}{(a_1 + a_2)^2 + (a_5 + a_6)^2}, \\
a_7 &= -\frac{(a_1^2 + a_5^2)a_6 + (a_2^2 + a_6^2)a_5}{(a_1 + a_2)^2 + (a_5 + a_6)^2},
\end{aligned}
$$

$$
\begin{aligned}
a_9 = {} & 3\{[(a_1 + a_2)^2 + (a_5 + a_6)^2]^2\alpha_1 - (a_2^2 + a_6^2)[(a_1 + a_2)a_2 + (a_5 + a_6)a_6]\alpha_2\} \\
& \cdot (a_1^2 + a_5^2)^2\{[(a_1 + a_2)^2 + (a_5 + a_6)^2](a_1a_6 - a_2a_5)^2\}^{-1}.
\end{aligned}
$$

4.5 Case of $\gamma_3 = \gamma_4 = 0$

We consider the case of

$$\gamma_3 = \gamma_4 = 0, \quad \gamma_1 = \gamma_2 = \gamma_5 = 1.$$

The corresponding generalized Hietarinta-type nonlinear equation and bilinear equation read

$$\alpha_1(6u_xu_{xx} + u_{xxxx}) + \alpha_2(3u_tu_{tt} + 3u_{xt}v_{tt} + u_{xttt})u_{yt} + u_{xx} + u_{yy} = 0,$$

where $v_x = u$, and

$$(\alpha_1 D_x^4 + \alpha_2 D_x D_t^3 + D_y D_t + D_x^2 + D_y^2) f \cdot f = 0,$$

respectively. The reduced frequencies and constant coefficient are

$$a_3 = -\frac{a_2^3 + a_1^2 a_2 - a_2 a_5^2 + a_2 a_6^2 + 2a_1 a_5 a_6}{a_2^2 + a_6^2},$$
$$a_7 = -\frac{a_6^3 - a_1^2 a_6 + a_2^2 a_6 + a_5^2 a_6 + 2a_1 a_2 a_5}{a_2^2 + a_6^2},$$
$$a_9 = -\frac{3(a_1^2 + a_5^2)^2 (a_2^2 + a_6^2)^2 \alpha_1 - b_4 \alpha_2}{(a_2^2 + a_6^2)(a_1 a_6 - a_2 a_5)^2},$$

where

$$b_4 = 3(a_1 a_2 + a_5 a_6)[(a_1 + a_6)^2 + (a_2 - a_5)^2] \cdot [(a_1 - a_6)^2 + (a_2 + a_5)^2](a_1^2 + a_2^2 + a_5^2 + a_6^2).$$

4.6 Case of $\gamma_3 = \gamma_5 = 0$

We consider the case of

$$\gamma_3 = \gamma_5 = 0, \quad \gamma_1 = \gamma_2 = \gamma_4 = 1.$$

The corresponding generalized Hietarinta-type nonlinear equation and bilinear equation read

$$\alpha_1(6u_x u_{xx} + u_{xxxx}) + \alpha_2(3u_t u_{tt} + 3u_{xt} v_{tt} + u_{xttt}) u_{yt} + u_{xx} + u_{xy} = 0,$$

where $v_x = u$, and

$$(\alpha_1 D_x^4 + \alpha_2 D_x D_t^3 + D_y D_t + D_x^2 + D_x D_y) f \cdot f = 0,$$

respectively. The reduced frequencies and constant coefficient are

$$a_3 = -\frac{(a_1^2 - a_5^2)a_2 + (a_2^2 + a_6^2)a_1 + 2a_1 a_5 a_6}{a_2^2 + a_6^2},$$
$$a_7 = -\frac{(a_5^2 - a_1^2)a_6 + (a_2^2 + a_6^2)a_5 + 2a_1 a_2 a_5}{a_2^2 + a_6^2},$$
$$a_9 = -\frac{3(a_1^2 + a_5^2)^2 (a_2^2 + a_6^2)^2 \alpha_1 - b_4 \alpha_2}{(a_2^2 + a_6^2)(a_1 a_6 - a_2 a_5)^2},$$

where

$$b_4 = 3[(a_1 + a_2)^2 + (a_5 + a_6)^2][(a_1 + a_2)a_2 + (a_5 + a_6)a_6](a_1^2 + a_5^2)^2.$$

4.7 Case of $\gamma_4 = \gamma_5 = 0$

We consider the case of

$$\gamma_4 = \gamma_5 = 0, \quad \gamma_1 = \gamma_2 = \gamma_3 = 1.$$

The corresponding generalized Hietarinta-type nonlinear equation and bilinear equation read

$$\alpha_1(6u_x u_{xx} + u_{xxxx}) + \alpha_2(3u_t u_{tt} + 3u_{xt} v_{tt} + u_{xttt})u_{yt} + u_{xx} + u_{xt} = 0,$$

where $v_x = u$, and

$$(\alpha_1 D_x^4 + \alpha_2 D_x D_t^3 + D_y D_t + D_x^2 + D_x D_t) f \cdot f = 0,$$

respectively. The reduced frequencies and constant coefficient are

$$a_3 = -\frac{(a_1 + a_2)a_1^2 + (a_1 - a_2)a_5^2 + 2a_1 a_5 a_6}{(a_1 + a_2)^2 + (a_5 + a_6)^2},$$
$$a_7 = -\frac{(a_5 + a_6)a_5^2 + (a_5 - a_6)a_1^2 + 2a_1 a_2 a_5}{(a_1 + a_2)^2 + (a_5 + a_6)^2},$$

$$\begin{aligned} a_9 = & -3\{[(a_1 + a_2)^2 + (a_5 + a_6)^2]^2 \alpha_1 - (a_1^2 + a_5^2)[a_1(a_1 + a_2) \\ & + a_5(a_5 + a_6)]\alpha_2\}(a_1^2 + a_5^2)^2\{[(a_1 + a_2)^2 + (a_5 + a_6)^2](a_1 a_6 - a_2 a_5)^2\}^{-1}. \end{aligned}$$

5 Two illustrative examples

Let us first choose

$$\alpha_1 = \gamma_1 = \gamma_2 = 0, \quad \alpha_2 = \gamma_3 = \gamma_4 = \gamma_5 = 1,$$

which leads to a specific generalized Hietarinta-type nonlinear equation

$$3u_t u_{tt} + 3u_{xt} v_{tt} + u_{xttt} + u_{xt} + u_{xy} + u_{yy} = 0, \tag{5.1}$$

where $v_x = u$. This has a Hirota bilinear form

$$(D_x D_t^3 + D_x D_t + D_x D_y + D_y^2) f \cdot f = 0,$$

under the logarithmic transformations in (2.2).

Based on the previous computation in Subsection 4.1, we know that there are lump solutions if we guarantee

$$a_1^3 a_2 + a_5^3 a_6 + (a_1^2 - a_5^2)(a_2^2 - a_6^2) + a_1(a_1 a_6 + a_2 a_5 + 4a_2 a_6)a_5 > 0,$$

so that $a_9 > 0$.

Upon further taking

$$a_1 = 3, \quad a_2 = 2, \quad a_4 = a_6 = 1, \quad a_5 = -1, \quad a_8 = -3,$$

the transformations in (2.2) with (3.1) present a pair of lump solutions to the first specific generalized Hietarinta-type nonlinear equation (5.1):

$$u_1 = \frac{4(-5t + 10x + 5y + 6)}{(-\frac{5}{2}t + 3x + 2y + 1)^2 + (-\frac{5}{2}t - x + y - 3)^2 + 75}.$$

Three three-dimensional plots and density plots of the lump solution u_1 at three different times are made by using Maple in Figure 1.

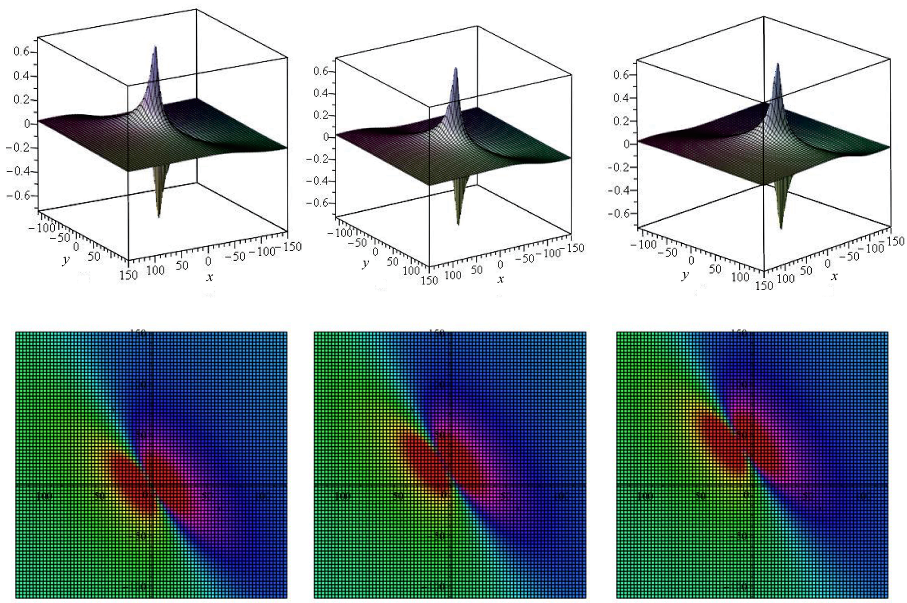

Fig. 1 Profiles of u_1 when $t = 0, 10, 20$: 3d plots (top) and density plots (bottom)

Let us second choose

$$\gamma_4 = \gamma_5 = 0, \quad \alpha_1 = \alpha_2 = \gamma_1 = \gamma_2 = \gamma_3 = 1,$$

which leads to another specific generalized Hietarinta-type nonlinear equation

$$u_{xxxx} + 6u_x u_{xx} + 3u_t u_{tt} + 3u_{xt} v_{tt} + u_{xttt} + u_{yt} + u_{xx} + u_{xt} = 0, \tag{5.2}$$

where $v_x = u$. This has a Hirota bilinear form

$$(D_x^4 + D_x D_t^3 + D_y D_t + D_x^2 + D_x D_t) f \cdot f = 0,$$

under the logarithmic transformations in (2.2).

Based on the previous computation in Subsection 4.7, we know that there are lump solutions if we guarantee

$$[(a_1+a_2)^2+(a_5+a_6)^2]^2-(a_1^2+a_5^2)[a_1(a_1+a_2)+a_5(a_5+a_6)]<0,$$

so that $a_9>0$.

Upon further taking

$$a_1=-3,\quad a_2=2,\quad a_4=-2,\quad a_5=-1,\quad a_6=1,\quad a_8=6,$$

the transformations in (2.2) with (3.1) present a pair of lump solutions to the second specific generalized Hietarinta-type nonlinear equation (5.2):

$$u_2=\frac{4(-30t+10x-7y)}{(8t-3x+2y-2)^2+(6t-x+y+6)^2+8700}.$$

Similarly, three three-dimensional plots and density plots of the lump solution u_2 at three different times are made through Maple in Figure 2.

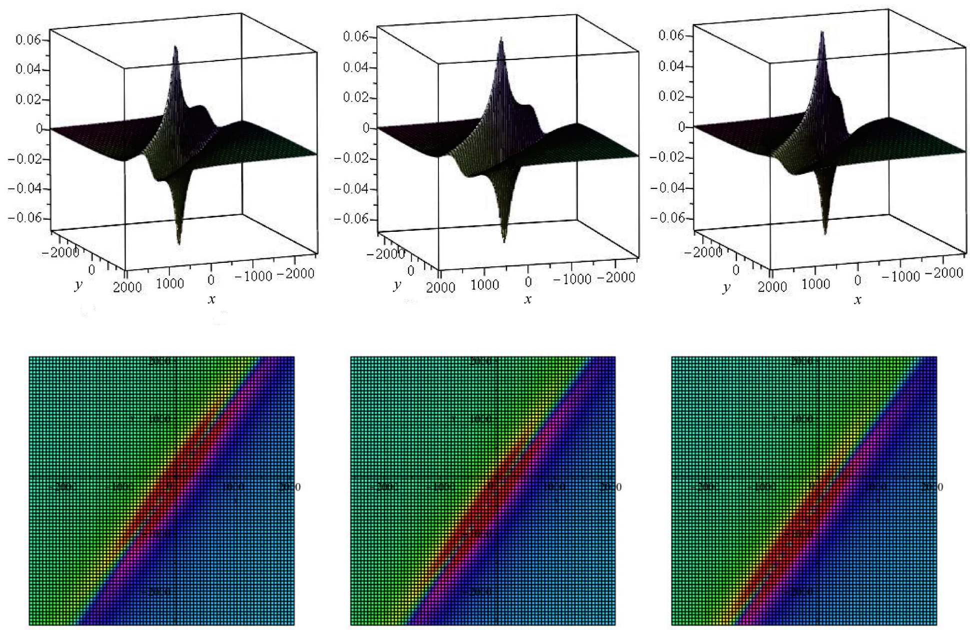

Fig. 2 Profiles of u_2 when $t=0,50,100$: 3d plots (top) and density plots (bottom)

6 Concluding remarks

With Maple symbolic computation, we have shown that the Hietarinta-type fourth-order nonlinear term can create lump solutions, together with second-order dispersive terms. The resulting lump solutions were explicitly presented in terms of the coefficients in the considered model equation. Our analysis provides another example of nonlinear partial differential equations in dispersive waves, which possess lump solutions. Specific reductions were also made and two illustrative examples were given, together with their 3d plots and density plots at three different times.

We point out that the adopted ansatz on lump solutions is increasingly being used in computations of exact solutions (see, e.g., [4,5,16,57]), and all such solutions obtained this way provide valuable insights into other solution methods in soliton theory, which include the Wronskian technique (see, e.g., [40,56]), Darboux transformations (see, e.g., [58,61,68]), the generalized bilinear approach (see, e.g., [25]), the multiple-wave expansion approach (see, e.g., [22,30]), the Riemann-Hilbert technique (see, e.g., [29]), symmetry reductions (see, e.g., [9,23,48,54]), and symmetry constraints (see, e.g., [20,21,38] for the continuous case and [8,35] for the discrete case).

We also mention that on one hand, various recent studies exhibit the striking richness of lump solutions to both linear PDEs [30,33,34], and nonlinear PDEs in $(2+1)$-dimensions (see, e.g., [36,45,47,55,64,67], and [41] with higher-order rational dispersion relations) and $(3+1)$-dimensions (see, e.g., [10,12,28,50,60,66]). Based on the Hirota bilinear form and the generalized bilinear forms, some more generic formulations have also been presented for lump solutions [2,42,43]. On the other hand, different classes of homoclinic and heteroclinic interaction solutions between lumps and other kinds of dispersive waves (see, e.g., [19,32,39,52,53,62]) have been generated for integrable equations in $(2+1)$-dimensions.

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