N-Fold Darboux Transformation for the Classical Three-Component Nonlinear Schrödinger Equations and Its Exact Solutions

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Abstract: In this paper, by using the gauge transformation and the Lax pairs, the N-fold Darboux transformation (DT) of the classical three-component nonlinear Schrödinger (NLS) equations is given. In addition, by taking seed solutions and using the DT, exact solutions for the given NLS equations are constructed.

Keywords: classical three-component nonlinear Schrödinger equations; N-fold darboux transformation; exact solutions

1. Introduction

Darboux transformation (DT) is an important technique to construct exact solutions of nonlinear partial differential equations [1]. The main significance of the DT is that infinite sequences of solutions to nonlinear equations can be generated by algebraic procedures. Thus, it has attracted much attention from physicists and mathematicians, it is now playing an important role in mechanics, physics, and differential geometry. The DT for the KdV equation, sine-Gordon equation, and nonlinear Schrödinger (NLS) equations, etc., were widely studied and many great scientific achievements have been derived [2–7].

The NLS equation is widely used in physics [8–15], nonlinear optics [16,17], and soft condensed matter physics [18] and there has been a vast amount of literature involving the NLS equation over the years. The applications of the DT in higher multicomponent NLS equations spatial dimensions and nonlocal equations have attracted much attention over the years [19–22]. Recently there has been an additional interest that has caused a hot topic of research, mainly due to Mark J. Ablowitz and Ziad H. Musslimani developed the nonlocal NLS equations [23,24]. In [25], Zhenya Yan introduced a new integrable nonlocal general vector NLS equation and found its exact solutions (See many other interesting examples about nonlocal NLS equations in [26–29]).

In [30], the classical three-component NLS equations belong to the category of evolutionary equations, for the solution of which the method of the inverse scattering problem is successfully applied.

The rest of the paper is organized as follows: In Section 2, we construct the DT for the three-component NLS equations and give the 1- and 2-fold exact solutions for the three-component NLS equations. In Section 3, we give some conclusions.
2. Darboux Transformation and Exact Solutions for the NLS Equations

2.1. Darboux Transformation

Let $n \in \mathbb{N}$, $I_n$ denote the identity matrix of size; $n$, $\alpha_1$ and $\alpha_2$ are different constants. The matrix spectral problems of the NLS equations read [30]

$$\phi_x = U\phi = U(u, \lambda)\phi, \phi_t = V\phi = V(u, \lambda)\phi,$$

with the Lax pairs as follows:

$$U = i(\lambda \Psi + D), V = i(\lambda^2 Y + G).$$

The four matrices $\Psi, Y, D$ and $G$ as follows:

$$\Psi = \text{diag}(\alpha_1, \alpha_2 I_n), D = D(u) = P = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix},$$

$$Y = \text{diag}(\beta_1, \beta_2 I_n),$$

$$G = G(u, \lambda) = \begin{pmatrix} \beta \lambda & 0 & 0 \\ q & -i q_x & -pq \\ p q & ip_x & \beta \lambda \end{pmatrix},$$

where $p = (p_1, p_2, \ldots, p_n), q = (q_1, q_2, \ldots, q_n)^T, u = (p, q)^T$, and $\beta_1$ and $\beta_2$ are different constants. Note that $G$ can be given by using the potential matrix $D$ as follows:

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -I_n \end{pmatrix} D^2 - \begin{pmatrix} i & 0 \\ 0 & -i I_n \end{pmatrix} D x.$$

when $p_j = q_j = 0$, $2 \leq j \leq n$, the spectral problem in (1) becomes the standard AKNS spectral problem [31].

Now, we consider the system as follows [30]:

$$\begin{cases}
    p_{1,x}(x, t) = \frac{1}{2}i[|p_{1,x}|^2 + 2(|p_1|^2 + |p_2|^2 + |p_3|^2)p_1], \\
    p_{2,x}(x, t) = \frac{1}{2}i[|p_{2,x}|^2 + 2(|p_1|^2 + |p_2|^2 + |p_3|^2)p_2], \\
    p_{3,x}(x, t) = \frac{1}{2}i[|p_{3,x}|^2 + 2(|p_1|^2 + |p_2|^2 + |p_3|^2)p_3],
\end{cases}$$

(7)

where $p_j(x, t)$ $(1 \leq j \leq 3)$ are complex valued functions of the real variables $x$ and $t$. From Equations (2), (3) and (6), we obtain the following spectral problem. Firstly, the corresponding cospectral problems for Equation (7) are introduced

$$\phi_x = U\phi, \phi_t = V\phi, \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T,$$

where $U$ and $V$ are [32–35]:

$$U = \begin{pmatrix} -i\lambda & ip_1 & ip_2 & ip_3 \\ ip_1 & i\lambda & 0 & 0 \\ ip_2 & 0 & i\lambda & 0 \\ ip_3 & 0 & 0 & i\lambda \end{pmatrix},$$

(9)

$$V = \begin{pmatrix} -i\lambda^2 + \frac{1}{2}(|p_1|^2 + |p_2|^2 + |p_3|^2) & i\lambda p_1 - \frac{1}{2} p_{1,x} & i\lambda p_2 - \frac{1}{2} p_{2,x} & i\lambda p_3 - \frac{1}{2} p_{3,x} \\ i\lambda p_1^* + \frac{1}{2} p_{1,x}^* & -\frac{1}{2} |p_1|^2 & -\frac{1}{2} p_{2} p_{1,x}^* & -\frac{1}{2} p_{3} p_{1,x}^* \\ i\lambda p_2^* + \frac{1}{2} p_{2,x}^* & -\frac{1}{2} p_{2} p_{2,x}^* & -\frac{1}{2} |p_2|^2 & -\frac{1}{2} p_{3} p_{2,x}^* \\ i\lambda p_3^* + \frac{1}{2} p_{3,x}^* & -\frac{1}{2} p_{3} p_{3,x}^* & -\frac{1}{2} p_{2} p_{3,x}^* & -\frac{1}{2} |p_3|^2 \end{pmatrix},$$

(10)

here $\lambda$ is a spectral parameter, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ is a column vector solution of Equations (9) and (10) associated with an eigenvalue $\lambda$. 

We introduce a gauge transformation $T$ [36] of Equations (9) and (10), where
\[ \tilde{\phi}_n = T \phi_n. \] (11)

Now, let us define $T$ as follows:
\[
T = \begin{pmatrix}
T_{11}(\lambda) & T_{12}(\lambda) & T_{13}(\lambda) & T_{14}(\lambda) \\
T_{21}(\lambda) & T_{22}(\lambda) & T_{23}(\lambda) & T_{24}(\lambda) \\
T_{31}(\lambda) & T_{32}(\lambda) & T_{33}(\lambda) & T_{34}(\lambda) \\
T_{41}(\lambda) & T_{42}(\lambda) & T_{43}(\lambda) & T_{44}(\lambda)
\end{pmatrix},
\] (12)

\[
\tilde{\phi}_x = \tilde{U} \phi \tilde{U} = (T_x + T U)T^{-1},
\] (13)

\[
\tilde{\phi}_t = \tilde{V} \phi \tilde{V} = (T_t + T V)T^{-1}.
\] (14)

If the $\tilde{U}$, $\tilde{V}$ and $U$, $V$ have the same types, system (10) and (11) are called DT of Equation (7) [26].

Let $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$, $X = (X_1, X_2, X_3, X_4)^T$ and $Y = (Y_1, Y_2, Y_3, Y_4)^T$ be four basic solutions of systems (9) and (10), then we give the following linear algebraic systems [19,20]:

\[
\begin{align*}
\sum_{i=0}^{N-1} (A_{11}^{(i)} + A_{12}^{(i)} M_j^{(1)} + A_{13}^{(i)} M_j^{(2)} + A_{14}^{(i)} M_j^{(3)}) \lambda_j^{N} &= 0, \\
\sum_{i=0}^{N-1} (A_{21}^{(i)} + A_{22}^{(i)} M_j^{(1)} + A_{23}^{(i)} M_j^{(2)} + A_{24}^{(i)} M_j^{(3)}) \lambda_j^{N} &= 0,
\end{align*}
\] (15)

with
\[
\begin{align*}
M_j^{(1)} &= \frac{\phi_2 v^{(k)}_j + \phi_3 v^{(k)}_j + \phi_4 v^{(k)}_j}{\phi_1 + \phi_j + \phi_4 v^{(k)}_j + \phi_4 v^{(k)}_j}, \\
M_j^{(2)} &= \frac{\phi_2 v^{(k)}_j + \phi_3 v^{(k)}_j + \phi_4 v^{(k)}_j}{\phi_1 + \phi_j + \phi_4 v^{(k)}_j + \phi_4 v^{(k)}_j}, \\
M_j^{(3)} &= \frac{\phi_2 v^{(k)}_j + \phi_3 v^{(k)}_j + \phi_4 v^{(k)}_j}{\phi_1 + \phi_j + \phi_4 v^{(k)}_j + \phi_4 v^{(k)}_j}, \quad 0 \leq j \leq 4N,
\end{align*}
\] (16)

where $\lambda_i$ and $v^{(k)}_j, (i \neq k, \lambda_i \neq \lambda_j, v^{(k)}_j \neq v^{(k)}_j, k = 1, 2, 3)$ should be appropriate parameters. Thus, the determinants of coefficients for Equation (15) are nonzero.

From Equation (12), we can define a $4 \times 4$ matrix $T$ as
\[
\begin{align*}
T_{11} &= \lambda^{N} + \sum_{i=0}^{N-1} A_{11}^{(i)} \lambda^i, \\
T_{12} &= \sum_{i=0}^{N-1} A_{12}^{(i)} \lambda^i, \\
T_{13} &= \sum_{i=0}^{N-1} A_{13}^{(i)} \lambda^i, \\
T_{14} &= \sum_{i=0}^{N-1} A_{14}^{(i)} \lambda^i, \\
T_{21} &= \sum_{i=0}^{N-1} A_{21}^{(i)} \lambda^i, \\
T_{22} &= \sum_{i=0}^{N-1} A_{22}^{(i)} \lambda^i, \\
T_{23} &= \sum_{i=0}^{N-1} A_{23}^{(i)} \lambda^i, \\
T_{24} &= \sum_{i=0}^{N-1} A_{24}^{(i)} \lambda^i, \\
T_{31} &= \sum_{i=0}^{N-1} A_{31}^{(i)} \lambda^i, \\
T_{32} &= \sum_{i=0}^{N-1} A_{32}^{(i)} \lambda^i, \\
T_{33} &= \sum_{i=0}^{N-1} A_{33}^{(i)} \lambda^i, \\
T_{34} &= \sum_{i=0}^{N-1} A_{34}^{(i)} \lambda^i, \\
T_{41} &= \sum_{i=0}^{N-1} A_{41}^{(i)} \lambda^i, \\
T_{42} &= \sum_{i=0}^{N-1} A_{42}^{(i)} \lambda^i, \\
T_{43} &= \sum_{i=0}^{N-1} A_{43}^{(i)} \lambda^i, \\
T_{44} &= \sum_{i=0}^{N-1} A_{44}^{(i)} \lambda^i,
\end{align*}
\] (17)

where $N$ is natural number, and the $A_{mn}^{(i)} (m,n = 1,2,3,4, i \geq 0 )$ are the functions of $x$ and $t$. Through calculations, we obtain $T$ as following:
\[ \Delta T = \Pi_{j=1}^{4N} (\lambda - \lambda_j), \] (18)

which proves that $\lambda_j (1 \leq j \leq 4N)$ are $4N$ roots of $T$. Based on Equation (18), we will prove that $\tilde{U}$ and $\tilde{V}$ have the same forms with $U$ and $V$ respectively.
Proposition 1. The matrix $\tilde{U}$ defined by Equation (9) has the same type as $U$, i.e.,

$$
\tilde{U} = \begin{pmatrix}
-\lambda & i\bar{p}_1 & i\bar{p}_2 & i\bar{p}_3 \\
-i\bar{p}_1 & \lambda & 0 & 0 \\
-i\bar{p}_2 & 0 & \lambda & 0 \\
i\bar{p}_3 & 0 & 0 & \lambda 
\end{pmatrix}.
$$

The transformation between old and new potentials are shown by the following formulas:

$$
\begin{align*}
\tilde{p}_1(x, t) &= p_1(x, t) + 2A_{12}, \\
\tilde{p}_2(x, t) &= p_2(x, t) + 2A_{13}, \\
\tilde{p}_3(x, t) &= p_3(x, t) + 2A_{14}.
\end{align*}
$$

The transformations (20) are used to get a DT of the spectral problem (13).

Proof. Let $T^{-1} = \frac{T^\ast}{\Delta T}$ with

$$
(T_x + TU)^T = \begin{pmatrix}
B_{11}(\lambda) & B_{12}(\lambda) & B_{13}(\lambda) & B_{14}(\lambda) \\
B_{21}(\lambda) & B_{22}(\lambda) & B_{23}(\lambda) & B_{24}(\lambda) \\
B_{31}(\lambda) & B_{32}(\lambda) & B_{33}(\lambda) & B_{34}(\lambda) \\
B_{41}(\lambda) & B_{42}(\lambda) & B_{43}(\lambda) & B_{44}(\lambda)
\end{pmatrix}.
$$

It is easy to verify that $B_{s,l}(1 \leq s, l \leq 4)$ are $4N$-order or $(4N + 1)$-order polynomial in $\lambda$.

Through some accurate calculations, $\lambda_j(1 \leq j \leq 4)$ are the roots of $B_{s,l}(1 \leq s, l \leq 4)$. Thus, Equation (21) has the following structure

$$(T_x + TU)^T = (\Delta T)C(\lambda),$$

where

$$
C(\lambda) = \begin{pmatrix}
C_{11}(0) + C_{11}(1) & C_{12}(0) & C_{13}(0) & C_{14}(0) \\
C_{21}(0) & C_{22}(0) + C_{22}(1) & C_{23}(0) & C_{24}(0) \\
C_{31}(0) & C_{32}(0) & C_{33}(0) + C_{33}(1) & C_{34}(0) \\
C_{41}(0) & C_{42}(0) & C_{43}(0) & C_{44}(1) + C_{44}(0)
\end{pmatrix},
$$

and $C_m^{(k)}(m,n = 1,2,3,4, k=0,1)$ satisfy the functions without $\lambda$. Then Equation (22) can be rewritten as

$$(T_x + TU) = C(\lambda)T.
$$

Through comparing the coefficients of $\lambda$ in Equation (24), we obtain
Through some calculations, we know that

In the following subsection, we assume that the new matrix $\tilde{U}$ has the same type of with $U$. It means that $p_j(x,t)$ of $U$ and $\tilde{p}_j(x,t)$ of $\tilde{U}$ have the same structures. From Equations (19) and (25), we know that $\tilde{U} = C(\lambda)$. The proof is completed.

**Proposition 2.** The matrix $\tilde{V}$ defined by Equation (10) has the same types as $V$, i.e.,

$$\tilde{V} = \begin{pmatrix} -i\lambda^2 + \frac{1}{2}(|\tilde{p}_1|^2 + |\tilde{p}_2|^2 + |\tilde{p}_3|^2) & i\lambda \tilde{p}_1 - \frac{1}{2} \tilde{p}_{1,x} & i\lambda \tilde{p}_2 - \frac{1}{2} \tilde{p}_{2,x} & i\lambda \tilde{p}_3 - \frac{1}{2} \tilde{p}_{3,x} \\ i\lambda \tilde{p}_1^* + \frac{1}{2} \tilde{p}_{1,x}^* & -i\lambda^2 - \frac{1}{2} |\tilde{p}_1|^2 & -\frac{1}{2} \tilde{p}_{2}^* \tilde{p}_1 & -\frac{1}{2} \tilde{p}_{3}^* \tilde{p}_1^* \\ i\lambda \tilde{p}_2^* + \frac{1}{2} \tilde{p}_{2,x}^* & -\frac{1}{2} \tilde{p}_{1}^* \tilde{p}_2 & -i\lambda^2 - \frac{1}{2} |\tilde{p}_2|^2 & -\frac{1}{2} \tilde{p}_{3}^* \tilde{p}_2^* \\ i\lambda \tilde{p}_3^* + \frac{1}{2} \tilde{p}_{3,x}^* & -\frac{1}{2} \tilde{p}_{1}^* \tilde{p}_3 & -\frac{1}{2} \tilde{p}_{2}^* \tilde{p}_3 & -i\lambda^2 - \frac{1}{2} |\tilde{p}_3|^2 \end{pmatrix}. \tag{26}$$

**Proof.** Let $T^{-1} = \frac{1}{\lambda} \frac{\partial}{\partial T}$ with

$$\begin{pmatrix} T_i + TV \end{pmatrix} T^* = \begin{pmatrix} E_{11}(\lambda) & E_{12}(\lambda) & E_{13}(\lambda) & E_{14}(\lambda) \\ E_{21}(\lambda) & E_{22}(\lambda) & E_{23}(\lambda) & E_{24}(\lambda) \\ E_{31}(\lambda) & E_{32}(\lambda) & E_{33}(\lambda) & E_{34}(\lambda) \\ E_{41}(\lambda) & E_{42}(\lambda) & E_{43}(\lambda) & E_{44}(\lambda) \end{pmatrix}. \tag{27}$$

It is easy to verify that $E_{ij}(1 \leq i, j \leq 4)$ are $(4N + 1)$-order or $(4N + 2)$-order polynomial in $\lambda$.

Through some calculations, we know that $\lambda_j(1 \leq j \leq 4)$ are the roots of $E_{ij}(1 \leq i, j \leq 4)$. Thus, Equation (27) has the following structure

$$\begin{pmatrix} T_i + TV \end{pmatrix} T^* = (\Delta T) F(\lambda), \tag{28}$$

where

$$F(\lambda) = \begin{pmatrix} F_{11}(\lambda)^2 + F_{11}(\lambda) + F_{11}(0) & F_{11}(\lambda)^2 + F_{12}(\lambda) + F_{12}(0) & F_{11}(\lambda)^2 + F_{13}(\lambda) + F_{13}(0) & F_{11}(\lambda)^2 + F_{14}(\lambda) + F_{14}(0) \\ F_{11}(\lambda) + F_{11}(0) & F_{11}(\lambda) + F_{12}(0) & F_{11}(\lambda) + F_{13}(0) & F_{11}(\lambda) + F_{14}(0) \\ F_{11}(\lambda) + F_{11}(0) & F_{11}(\lambda) + F_{12}(0) & F_{12}(\lambda)^2 + F_{12}(\lambda) + F_{12}(0) & F_{12}(\lambda)^2 + F_{13}(\lambda) + F_{13}(0) \\ F_{11}(\lambda) + F_{11}(0) & F_{11}(\lambda) + F_{12}(0) & F_{11}(\lambda) + F_{12}(0) & F_{13}(\lambda)^2 + F_{13}(\lambda) + F_{13}(0) \end{pmatrix}. \tag{29}$$

and $F_{mn}(m, n = 1, 2, 3, 4, k = 0, 1, 2)$ satisfy the functions without $\lambda$. Then Equation (28) can be as follows:

$$\begin{pmatrix} T_i + TV \end{pmatrix} = F(\lambda) T. \tag{30}$$

$\square$
Through comparing the coefficients of $\lambda$ in Equation (30), we obtain

\[
\begin{align*}
F^{(2)}_{11} &= -i, \quad F^{(1)}_{12} = 0, \quad F^{(1)}_{12} = i(p_1 + 2A_{12}) = i\tilde{p}_1, \\
F^{(2)}_{12} &= -\frac{i}{2}(p_{1x} + 2A_{12}) = -\frac{i}{2}\tilde{p}_{1x}, \\
F^{(1)}_{11} &= \frac{i}{2}[(p_1 + 2A_{12})(p_1^* - 2A_{21}) + (p_2 + 2A_{13})(p_2^* - 2A_{41})] \\
&= \frac{i}{2}(\tilde{p}_1\tilde{p}_1^* + \tilde{p}_2\tilde{p}_2^* + \tilde{p}_3\tilde{p}_3^*), \\
F^{(1)}_{13} &= i(p_2 + 2A_{13}) = i\tilde{p}_2, \\
F^{(1)}_{13} &= -\frac{1}{2}(p_{2x} + 2A_{13}) = -\frac{1}{2}\tilde{p}_{2x}, \\
F^{(1)}_{14} &= i(p_3 + 2A_{14}) = i\tilde{p}_3, \\
F^{(1)}_{14} &= \frac{i}{2}(p_{3x} + 2A_{14}) = \frac{i}{2}\tilde{p}_{3x}, \\
F^{(1)}_{21} &= i(p_1^* - 2A_{21}) = i\tilde{p}_1^*, \\
F^{(1)}_{23} &= \frac{i}{2}(p_1^* - 2A_{21})(p_1^* - 2A_{21}) = \frac{i}{2}\tilde{p}_1\tilde{p}_1^* \\
F^{(2)}_{22} &= i, \quad F^{(1)}_{23} = 0, \quad F^{(1)}_{24} = 0, \\
F^{(1)}_{23} &= -\frac{i}{2}(p_2 + 2A_{13})(p_2^* - 2A_{31}) = -\frac{i}{2}\tilde{p}_2\tilde{p}_2^* \\
F^{(1)}_{24} &= -\frac{i}{2}(p_3 + 2A_{14})(p_3^* - 2A_{31}) = -\frac{i}{2}\tilde{p}_3\tilde{p}_3^* \\
F^{(1)}_{31} &= i(p_2^* + 2A_{31}) = i\tilde{p}_2^*, \quad F^{(1)}_{32} = 0, \\
F^{(1)}_{32} &= i, \quad F^{(1)}_{33} = 0, \\
F^{(1)}_{33} &= -\frac{i}{2}(p_2 + 2A_{13})(p_2^* + 2A_{31}) = -\frac{i}{2}\tilde{p}_2\tilde{p}_2^* \\
F^{(1)}_{34} &= -\frac{i}{2}(p_3 + 2A_{14})(p_3^* + 2A_{31}) = -\frac{i}{2}\tilde{p}_3\tilde{p}_3^* \\
F^{(1)}_{41} &= i(p_3^* - 2A_{41}) = i\tilde{p}_3^*, \\
F^{(1)}_{41} &= \frac{i}{2}(p_{3x}^* - 2A_{41}) = \frac{i}{2}\tilde{p}_{3x}, \\
F^{(1)}_{44} &= 0, \quad F^{(1)}_{44} = i, \quad F^{(1)}_{44} = 0, \\
F^{(1)}_{43} &= -\frac{i}{2}(p_1 + 2A_{12})(p_1^* - 2A_{41}) = -\frac{i}{2}\tilde{p}_1\tilde{p}_1^* \\
F^{(1)}_{44} &= -\frac{i}{2}(p_3 + 2A_{14})(p_3^* - 2A_{41}) = -\frac{i}{2}\tilde{p}_3\tilde{p}_3^*. \\
\end{align*}
\]

(31)

In the above subsection, we assume that the new matrix $\tilde{V}$ has the same type with $V$, which means that they have the same structures only $p_i(x, t)$ of $V$ transformed into $\tilde{p}_i(x, t)$ of $\tilde{V}$. From Equations (20) and (26), we know that $\tilde{V} = F(\lambda)$. The proof is completed.

### 2.2. Exact Solutions for the NLS Equations

In this subsection, we will give some exact solutions of Equation (7) by applying the N-fold DT. Firstly, we give a set of seed solutions $p_1 = p_2 = p_3 = 0$ and substitute the solution into Equations (9) and (10). It is easy to find four basic solutions for these equations:

\[
\psi(\lambda) = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad \phi(\lambda) = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
\]

(32)

\[
X(\lambda) = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad Y(\lambda) = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

(33)
Thus, substituting Equations (32) and (33) into Equation (16), we obtain

\[
\begin{align*}
M^{(1)}_j &= 
\frac{\psi_j^{(1)} e^{iAx + i\lambda_j t}}{e^{iAx} - e^{i\lambda_j t}} = \psi_j^{(1)} e^{2i(\lambda_j x + \lambda_j^2 t)}, \\
M^{(2)}_j &= \frac{\psi_j^{(2)} e^{iAx + i\lambda_j t}}{e^{iAx} - e^{i\lambda_j t}} = \psi_j^{(2)} e^{2i(\lambda_j x + \lambda_j^2 t)}, \\
M^{(3)}_j &= \frac{\psi_j^{(3)} e^{iAx + i\lambda_j t}}{e^{iAx} - e^{i\lambda_j t}} = \psi_j^{(3)} e^{2i(\lambda_j x + \lambda_j^2 t)},
\end{align*}
\]  

(34)

By using Equations (11) and (17), we have the matrix T as following

\[
T = \begin{pmatrix}
\lambda^N + \sum_{i=0}^{N-1} \lambda_i A_{11}^{(i)} & \sum_{i=0}^{N-1} \lambda_i A_{12}^{(i)} & \sum_{i=0}^{N-1} \lambda_i A_{13}^{(i)} & \sum_{i=0}^{N-1} \lambda_i A_{14}^{(i)} \\
\sum_{i=0}^{N-1} \lambda_i A_{21}^{(i)} & \lambda^N + \sum_{i=0}^{N-1} \lambda_i A_{22}^{(i)} & \sum_{i=0}^{N-1} \lambda_i A_{23}^{(i)} & \sum_{i=0}^{N-1} \lambda_i A_{24}^{(i)} \\
\sum_{i=0}^{N-1} \lambda_i A_{31}^{(i)} & \sum_{i=0}^{N-1} \lambda_i A_{32}^{(i)} & \lambda^N + \sum_{i=0}^{N-1} \lambda_i A_{33}^{(i)} & \sum_{i=0}^{N-1} \lambda_i A_{34}^{(i)} \\
\sum_{i=0}^{N-1} \lambda_i A_{41}^{(i)} & \sum_{i=0}^{N-1} \lambda_i A_{42}^{(i)} & \sum_{i=0}^{N-1} \lambda_i A_{43}^{(i)} & \lambda^N + \sum_{i=0}^{N-1} \lambda_i A_{44}^{(i)}
\end{pmatrix}
\]  

(35)

and

\[
\begin{align*}
\sum_{i=0}^{N-1} (A_{11}^{(i)} + A_{12}^{(i)} M^{(1)}_j + A_{13}^{(i)} M^{(2)}_j + A_{14}^{(i)} M^{(3)}_j) \lambda_i + \lambda^N &= 0, \\
\sum_{i=0}^{N-1} (A_{21}^{(i)} + A_{22}^{(i)} M^{(1)}_j + A_{23}^{(i)} M^{(2)}_j + A_{24}^{(i)} M^{(3)}_j) \lambda_i + M^{(1)}_j \lambda^N &= 0, \\
\sum_{i=0}^{N-1} (A_{31}^{(i)} + A_{32}^{(i)} M^{(1)}_j + A_{33}^{(i)} M^{(2)}_j + A_{34}^{(i)} M^{(3)}_j) \lambda_i + M^{(2)}_j \lambda^N &= 0, \\
\sum_{i=0}^{N-1} (A_{41}^{(i)} + A_{42}^{(i)} M^{(1)}_j + A_{43}^{(i)} M^{(2)}_j + A_{44}^{(i)} M^{(3)}_j) \lambda_i + M^{(3)}_j \lambda^N &= 0.
\end{align*}
\]  

(36)

Solving Equation (15) via the Grammer’s rule [21], we have

\[
A^{(N)}_{12} = \frac{\Delta A^{(N)}_{12}}{\Delta^{(N)}}, A^{(N)}_{13} = \frac{\Delta A^{(N)}_{13}}{\Delta^{(N)}}, A^{(N)}_{14} = \frac{\Delta A^{(N)}_{14}}{\Delta^{(N)}},
\]  

(37)

with

\[
\Delta^{(N)} = \begin{vmatrix}
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\end{vmatrix}
\]  

(38)

\[
\Delta A^{(N)}_{12} = \begin{vmatrix}
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\end{vmatrix}
\]  

(39)

\[
\Delta A^{(N)}_{13} = \begin{vmatrix}
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\lambda^0 & \cdots & \lambda^{N-1} & 0 & \cdots & \cdots \\
\end{vmatrix}
\]  

(40)
Using Equations (11), (12), (20) and (37), we can derive the solutions for the NLS

\[ \Delta A_{14}^{(N)} = \begin{vmatrix}
\lambda_1^0 & \ldots & \lambda_1^{N-1} & M_1^{(1)} \lambda_1^0 & \ldots & M_1^{(1)} \lambda_1^{N-1} & M_1^{(2)} \lambda_1^0 & \ldots & -\lambda_1^N \\
\lambda_2^0 & \ldots & \lambda_2^{N-1} & M_2^{(1)} \lambda_2^0 & \ldots & M_2^{(1)} \lambda_2^{N-1} & M_2^{(2)} \lambda_2^0 & \ldots & -\lambda_2^N \\
\lambda_3^0 & \ldots & \lambda_3^{N-1} & M_3^{(1)} \lambda_3^0 & \ldots & M_3^{(1)} \lambda_3^{N-1} & M_3^{(2)} \lambda_3^0 & \ldots & -\lambda_3^N \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\lambda_{4N}^0 & \ldots & \lambda_{4N}^{N-1} & M_{4N}^{(1)} \lambda_{4N}^0 & \ldots & M_{4N}^{(1)} \lambda_{4N}^{N-1} & M_{4N}^{(2)} \lambda_{4N}^0 & \ldots & -\lambda_{4N}^N 
\end{vmatrix}. \tag{41} \]

with \( M_j^{(1)} = v_j^{(1)} e^{2i(\lambda_j x + \lambda_j^2 t)}, M_j^{(2)} = v_j^{(2)} e^{2i(\lambda_j x + \lambda_j^2 t)}, M_j^{(3)} = v_j^{(3)} e^{2i(\lambda_j x + \lambda_j^2 t)} \) (1 \( \leq j \leq 4N \)).

Using Equations (11), (12), (20) and (37), we can derive the solutions for the NLS

Equation (7)

\[
\begin{align*}
\tilde{\rho}_1 &= 2A_{12}^{(N)} = \frac{2 \Delta A_{12}^{(N)}}{\Delta A_{2N}^{(N)}}, \\
\tilde{\rho}_2 &= 2A_{13}^{(N)} = \frac{2 \Delta A_{13}^{(N)}}{\Delta A_{2N}^{(N)}}, \\
\tilde{\rho}_3 &= 2A_{14}^{(N)} = \frac{2 \Delta A_{14}^{(N)}}{\Delta A_{2N}^{(N)}}.
\end{align*} \tag{42} \]

Now, we consider \( N = 1, 2 \) respectively, and draw these plane graphs and the density plots as Figures 1 and 2.

(1) Firstly, we consider \( N = 1, \lambda = \lambda_j (j = 1, 2, 3, 4) \) in Equations (12) and (17), and obtain the matrix \( T \)

\[ T = \begin{pmatrix}
\lambda + A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & \lambda + A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & \lambda + A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & \lambda + A_{44}
\end{pmatrix}, \tag{43} \]

and

\[
\begin{align*}
\lambda_j + A_{11} + M_j^{(1)} A_{12} + M_j^{(2)} A_{13} + M_j^{(3)} A_{14} &= 0, \\
A_{21} + M_j^{(1)} (A_{22} + \lambda_j) + M_j^{(2)} A_{23} + M_j^{(3)} A_{24} &= 0, \\
A_{31} + M_j^{(2)} (A_{33} + \lambda_j) + M_j^{(1)} A_{32} + M_j^{(3)} A_{34} &= 0, \\
A_{41} + M_j^{(3)} (A_{44} + \lambda_j) + M_j^{(1)} A_{42} + M_j^{(2)} A_{43} &= 0.
\end{align*} \tag{44} \]

According to Equations (38)–(41), we get

\[ \Delta^{(1)} = \begin{vmatrix}
1 & M_1^{(1)} & M_1^{(2)} & M_1^{(3)} \\
1 & M_2^{(1)} & M_2^{(2)} & M_2^{(3)} \\
1 & M_3^{(1)} & M_3^{(2)} & M_3^{(3)} \\
1 & M_4^{(1)} & M_4^{(2)} & M_4^{(3)}
\end{vmatrix}, \tag{45} \]

\[ \Delta A_{12}^{(1)} = \begin{vmatrix}
1 & -\lambda_1 & M_1^{(2)} & M_1^{(3)} \\
1 & -\lambda_2 & M_2^{(2)} & M_2^{(3)} \\
1 & -\lambda_3 & M_3^{(2)} & M_3^{(3)} \\
1 & -\lambda_4 & M_4^{(2)} & M_4^{(3)}
\end{vmatrix}, \tag{46} \]

\[ \Delta A_{13}^{(1)} = \begin{vmatrix}
1 & M_1^{(1)} & -\lambda_1 & M_1^{(3)} \\
1 & M_2^{(1)} & -\lambda_2 & M_2^{(3)} \\
1 & M_3^{(1)} & -\lambda_3 & M_3^{(3)} \\
1 & M_4^{(1)} & -\lambda_4 & M_4^{(3)}
\end{vmatrix}, \tag{47} \]

\[ \Delta A_{14}^{(1)} = \begin{vmatrix}
1 & M_1^{(1)} & M_1^{(2)} & -\lambda_1 \\
1 & M_2^{(1)} & M_2^{(2)} & -\lambda_2 \\
1 & M_3^{(1)} & M_3^{(2)} & -\lambda_3 \\
1 & M_4^{(1)} & M_4^{(2)} & -\lambda_4
\end{vmatrix}. \tag{48} \]
Based on Equation (37), we obtain the following systems

\[
A^{(1)}_{12} = \frac{\Delta A^{(1)}_{12}}{\Delta^{(1)}}, A^{(1)}_{13} = \frac{\Delta A^{(1)}_{13}}{\Delta^{(1)}}, A^{(1)}_{14} = \frac{\Delta A^{(1)}_{14}}{\Delta^{(1)}}. \tag{49}
\]

Thus, the exact solutions of Equation (7) are obtained by the DT method as follows

\[
\begin{align*}
\tilde{p}_1 &= 2A^{(1)}_{12} = 2\frac{\Delta A^{(1)}_{12}}{\Delta^{(1)}}, \\
\tilde{p}_2 &= 2A^{(1)}_{13} = 2\frac{\Delta A^{(1)}_{13}}{\Delta^{(1)}}, \\
\tilde{p}_3 &= 2A^{(1)}_{14} = 2\frac{\Delta A^{(1)}_{14}}{\Delta^{(1)}}. \tag{50}
\end{align*}
\]

To illustrate the exact solution (58), we can select some free parameters in the form \(\lambda_1, \lambda_2, \lambda_3, \lambda_4, v_{m}^{(k)} \) \((m = 1, 2, 3, 4, k = 1, 2, 3)\). Figure 1 shows the exact solutions to Equation (50).

\(\text{Figure 1. (a1–a3) are the first-fold exact solutions } \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \text{ of Equation (48); (b1–b3) are the plane graphs of } \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \text{ of Equation (48) at } t = 0; \text{ (c1–c3) are the density plots of } \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \text{ of Equation (48) with } \lambda_1 = 0.1, \lambda_2 = 0.4, \lambda_3 = 0.3, \lambda_4 = 0.2, v_1^{(1)} = 0.01, v_2^{(1)} = 0.3, v_3^{(1)} = 0.4, v_4^{(1)} = 0.05, v_1^{(2)} = 0.1, v_2^{(2)} = 0.02, v_3^{(2)} = 0.04, v_4^{(2)} = 0.6, v_1^{(3)} = 0.03, v_2^{(3)} = 0.2, v_3^{(3)} = 0.5, v_4^{(3)} = 0.06.\)

Nextly, we consider \(N = 2, \lambda = \lambda_j \) \((j = 1, 2, 3, 4, 5, 6, 7, 8)\) in Equations (12) and (17), and obtain the matrix \(T\).
\[ T = \begin{pmatrix}
\lambda^2 + A_{11} \lambda + A_{11}^{(0)} & A_{12}^{(1)} \lambda + A_{12}^{(0)} & A_{13}^{(1)} \lambda + A_{13}^{(0)} & A_{14}^{(1)} \lambda + A_{14}^{(0)} \\
A_{21}^{(1)} \lambda + A_{21}^{(0)} & \lambda^2 + A_{22}^{(1)} \lambda + A_{22}^{(0)} & A_{23}^{(1)} \lambda + A_{23}^{(0)} & A_{24}^{(1)} \lambda + A_{24}^{(0)} \\
A_{31}^{(1)} \lambda + A_{31}^{(0)} & A_{32}^{(1)} \lambda + A_{32}^{(0)} & \lambda^2 + A_{33}^{(1)} \lambda + A_{33}^{(0)} & A_{34}^{(1)} \lambda + A_{34}^{(0)} \\
A_{41}^{(1)} \lambda + A_{41}^{(0)} & A_{42} \lambda + A_{42}^{(0)} & A_{43} \lambda + A_{43}^{(0)} & \lambda^2 + A_{44} \lambda + A_{44}^{(0)}
\end{pmatrix},
\]
and

\[
\begin{align*}
M_1^{(1)} & [A_{12}^{(0)} + A_{12}^{(1)} \lambda_j] + M_2^{(2)} [A_{13}^{(0)} + A_{13}^{(1)} \lambda_j] + M_3^{(3)} [A_{14}^{(0)} + A_{14}^{(1)} \lambda_j] + [A_{11}^{(0)} + A_{11}^{(1)} \lambda_j + \lambda_j^2] = 0, \\
M_4^{(4)} & [\lambda^2 \lambda_j + A_{22}^{(1)} \lambda_j + A_{22}^{(0)}] + M_5^{(5)} [A_{23}^{(1)} \lambda_j + A_{23}^{(0)}] + M_6^{(6)} [A_{24}^{(1)} \lambda_j + A_{24}^{(0)}] + [A_{21}^{(0)} + A_{21}^{(1)} \lambda_j] = 0, \\
M_7^{(1)} & [A_{33}^{(0)} \lambda_j + A_{33}^{(0)}] + M_8^{(2)} [\lambda_j^2 + A_{33}^{(1)} \lambda_j + A_{33}^{(0)}] + M_9^{(3)} [A_{34}^{(0)} + A_{34}^{(1)} \lambda_j] + [A_{31}^{(0)} + A_{31}^{(1)} \lambda_j] = 0, \\
M_{10}^{(1)} & [A_{44}^{(0)} + A_{44}^{(1)} \lambda_j] + M_{11}^{(2)} [A_{43}^{(1)} \lambda_j + A_{43}^{(0)}] + M_{12}^{(3)} [\lambda_j^2 + A_{44}^{(1)} + A_{44}^{(0)} + A_{44}^{(1)} \lambda_j] + [A_{41}^{(0)} + A_{41}^{(1)} \lambda_j] = 0.
\end{align*}
\]

According to Equations (38)–(41), we get

\[ \Delta^{(2)} = \begin{pmatrix}
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_1 & \lambda_1 M_4^{(1)} & \lambda_1 M_5^{(2)} & \lambda_1 M_6^{(3)} & \\
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_2 & \lambda_2 M_4^{(1)} & \lambda_2 M_5^{(2)} & \lambda_2 M_6^{(3)} & \\
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_3 & \lambda_3 M_4^{(1)} & \lambda_3 M_5^{(2)} & \lambda_3 M_6^{(3)} & \\
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_4 & \lambda_4 M_4^{(1)} & \lambda_4 M_5^{(2)} & \lambda_4 M_6^{(3)} & \\
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_5 & \lambda_5 M_4^{(1)} & \lambda_5 M_5^{(2)} & \lambda_5 M_6^{(3)} & \\
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_6 & \lambda_6 M_4^{(1)} & \lambda_6 M_5^{(2)} & \lambda_6 M_6^{(3)} & \\
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_7 & \lambda_7 M_4^{(1)} & \lambda_7 M_5^{(2)} & \lambda_7 M_6^{(3)} & \\
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_8 & \lambda_8 M_4^{(1)} & \lambda_8 M_5^{(2)} & \lambda_8 M_6^{(3)}
\end{pmatrix},
\]

\[ \Delta A_{12}^{(2)} = \begin{pmatrix}
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_1 & -\lambda_1^2 & \lambda_1 M_4^{(1)} & \lambda_1 M_5^{(2)} & \lambda_1 M_6^{(3)} & \\
1 & M_2^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_2 & -\lambda_2^2 & \lambda_2 M_4^{(1)} & \lambda_2 M_5^{(2)} & \lambda_2 M_6^{(3)} & \\
1 & M_3^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_3 & -\lambda_3^2 & \lambda_3 M_4^{(1)} & \lambda_3 M_5^{(2)} & \lambda_3 M_6^{(3)} & \\
1 & M_4^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_4 & -\lambda_4^2 & \lambda_4 M_4^{(1)} & \lambda_4 M_5^{(2)} & \lambda_4 M_6^{(3)} & \\
1 & M_5^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_5 & -\lambda_5^2 & \lambda_5 M_4^{(1)} & \lambda_5 M_5^{(2)} & \lambda_5 M_6^{(3)} & \\
1 & M_6^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_6 & -\lambda_6^2 & \lambda_6 M_4^{(1)} & \lambda_6 M_5^{(2)} & \lambda_6 M_6^{(3)} & \\
1 & M_7^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_7 & -\lambda_7^2 & \lambda_7 M_4^{(1)} & \lambda_7 M_5^{(2)} & \lambda_7 M_6^{(3)} & \\
1 & M_8^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_8 & -\lambda_8^2 & \lambda_8 M_4^{(1)} & \lambda_8 M_5^{(2)} & \lambda_8 M_6^{(3)}
\end{pmatrix},
\]

\[ \Delta A_{13}^{(2)} = \begin{pmatrix}
1 & M_1^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_1 & -\lambda_1^2 & \lambda_1 M_4^{(1)} & \lambda_1 M_5^{(2)} & \lambda_1 M_6^{(3)} & \\
1 & M_2^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_2 & -\lambda_2^2 & \lambda_2 M_4^{(1)} & \lambda_2 M_5^{(2)} & \lambda_2 M_6^{(3)} & \\
1 & M_3^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_3 & -\lambda_3^2 & \lambda_3 M_4^{(1)} & \lambda_3 M_5^{(2)} & \lambda_3 M_6^{(3)} & \\
1 & M_4^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_4 & -\lambda_4^2 & \lambda_4 M_4^{(1)} & \lambda_4 M_5^{(2)} & \lambda_4 M_6^{(3)} & \\
1 & M_5^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_5 & -\lambda_5^2 & \lambda_5 M_4^{(1)} & \lambda_5 M_5^{(2)} & \lambda_5 M_6^{(3)} & \\
1 & M_6^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_6 & -\lambda_6^2 & \lambda_6 M_4^{(1)} & \lambda_6 M_5^{(2)} & \lambda_6 M_6^{(3)} & \\
1 & M_7^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_7 & -\lambda_7^2 & \lambda_7 M_4^{(1)} & \lambda_7 M_5^{(2)} & \lambda_7 M_6^{(3)} & \\
1 & M_8^{(1)} & M_2^{(2)} & M_3^{(3)} & \lambda_8 & -\lambda_8^2 & \lambda_8 M_4^{(1)} & \lambda_8 M_5^{(2)} & \lambda_8 M_6^{(3)}
\end{pmatrix}.
Based on Equation (37), we obtain the following systems

\[
\Delta A^{(2)}_{14} = \begin{vmatrix}
1 & M_1^{(1)} & M_1^{(2)} & M_1^{(3)} & \lambda_1 & \lambda_1 M_1^{(1)} & \lambda_1 M_1^{(2)} & -\lambda_1^2 \\
1 & M_2^{(1)} & M_2^{(2)} & M_2^{(3)} & \lambda_2 & \lambda_2 M_2^{(1)} & \lambda_2 M_2^{(2)} & -\lambda_2^2 \\
1 & M_3^{(1)} & M_3^{(2)} & M_3^{(3)} & \lambda_3 & \lambda_3 M_3^{(1)} & \lambda_3 M_3^{(2)} & -\lambda_3^2 \\
1 & M_4^{(1)} & M_4^{(2)} & M_4^{(3)} & \lambda_4 & \lambda_4 M_4^{(1)} & \lambda_4 M_4^{(2)} & -\lambda_4^2 \\
1 & M_5^{(1)} & M_5^{(2)} & M_5^{(3)} & \lambda_5 & \lambda_5 M_5^{(1)} & \lambda_5 M_5^{(2)} & -\lambda_5^2 \\
1 & M_6^{(1)} & M_6^{(2)} & M_6^{(3)} & \lambda_6 & \lambda_6 M_6^{(1)} & \lambda_6 M_6^{(2)} & -\lambda_6^2 \\
1 & M_7^{(1)} & M_7^{(2)} & M_7^{(3)} & \lambda_7 & \lambda_7 M_7^{(1)} & \lambda_7 M_7^{(2)} & -\lambda_7^2 \\
1 & M_8^{(1)} & M_8^{(2)} & M_8^{(3)} & \lambda_8 & \lambda_8 M_8^{(1)} & \lambda_8 M_8^{(2)} & -\lambda_8^2 \\
\end{vmatrix}
\] (56)

The analytic soliton solutions of the equations are obtained by the DT method as follows

\[
\begin{align*}
\tilde{p}_1 &= 2A_{12}^{(2)} = \frac{2\Delta A_{12}^{(2)}}{\Delta^{(2)}}, \\
\tilde{p}_2 &= 2A_{13}^{(2)} = \frac{2\Delta A_{13}^{(2)}}{\Delta^{(2)}}, \\
\tilde{p}_3 &= 2A_{14}^{(2)} = \frac{2\Delta A_{14}^{(2)}}{\Delta^{(2)}}.
\end{align*}
\] (57)

To illustrate the exact solution (58), we can select some free parameters in the form \(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, v_m^k\) \((m = 1, 2, 3, 4, 5, 6, 7, 8, k = 1, 2, 3)\). Figure 2 shows the exact solutions to Equation (58).
In this paper, we constructed DT for the NLS Equation (7). Through using the Gramers rule [21] and selecting the appropriate parameters, we give the expressions of N-exact solutions, and 1- and 2-fold exact solutions. By solving the three-component NLS equations, we find that it is quite different from the solution of the NLS equation. In the future work, we want to consider the initial solution besides zero, and try to obtain the DT of the coupled nonlocal equation.

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**References**


**Figure 2.** (a1–a3) are the second-fold exact solutions \( \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \) of Equation (56); (b1–b3) are plane graphs \( \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \) of Equation (56) at \( t = 0 \); (c1–c3) are the density plots of \( \tilde{p}_1, \tilde{p}_2, \tilde{p}_3 \) of Equation (56) with \( \lambda_1 = 0.01, \lambda_2 = -0.04, \lambda_3 = -0.03, \lambda_4 = 0.02, \lambda_5 = -0.05, \lambda_6 = 0.06, \lambda_7 = -0.07, \lambda_8 = 0.08, v_1^{(1)} = -0.1, v_2^{(1)} = 0.3, v_3^{(1)} = 0.6, v_4^{(1)} = 0.9, v_5^{(1)} = -0.3, v_6^{(1)} = -0.6, v_7^{(1)} = 0.9, v_8^{(1)} = -0.14, v_1^{(2)} = 0.1, v_2^{(2)} = 0.4, v_3^{(2)} = 0.7, v_4^{(2)} = -0.2, v_5^{(2)} = -0.4, v_6^{(2)} = -0.7, v_7^{(2)} = -0.12, v_8^{(2)} = -0.15, v_1^{(3)} = 0.2, v_2^{(3)} = 0.5, v_3^{(3)} = 0.8, v_4^{(3)} = -0.11, v_5^{(3)} = 0.5, v_6^{(3)} = -0.8, v_7^{(3)} = -0.13, v_8^{(3)} = -0.16. \)


