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Lie symmetry analysis, exact solutions, and conservation laws to multi-component nonlinear Schrödinger equations

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Abstract The multi-component nonlinear Schrödinger equations (MNLS) are derived by extending the single-component nonlinear Schrödinger equation to multiple interacting fields. These equations often describe the dynamics of wave packets in quantum mechanics or nonlinear optics. In this paper, we investigate MNLS equations via the Lie symmetry method. The Lie infinitesimal symmetries of the MNLS equations are derived by solving recursive determining equations, and the symmetry reductions of the equations are given by using symmetry variables. Moreover, some interesting explicit solutions for the equations are constructed. Finally, the conservation laws of the MNLS equations are obtained utilizing Ibragimov's method with detailed derivation.

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1 Introduction

Nonlinear Schrödinger equation is a classical field equation. The standard nonlinear Schrödinger equation [1] is

$$ip_t + k_1 p_{xx} + k_2|p|^2 p = 0 \quad (1)$$

for the complex field $p = p(x, t)$, where k_1, and k_2 are nonzero real numbers, representing the coefficients of group velocity dispersion and self-phase modulation, respectively. Various aspects related to the integrability of (1) have been studied in [1–3].

The interaction of waves of different frequencies gives rise to multi-component nonlinear Schrödinger models. The MNLS equations are a very important dynamical system in optics and mathematical physics which are used to describe the simultaneous propagation of multi-nonlinear waves in a uniform medium and have numerous applications in the areas of plasma physics [4], quantum electronics [5], nonlinear optics [6], Bose-Einstein condensates [7], and hydrodynamics [8]. Recently, many studies have also emerged in fields such as rogue oceanic waves [9], atmosphere [10], and matter waves [11].

In this paper, we consider following multi-component nonlinear Schrödinger equations [12,13] generalized from equation (1):

$$p_{j,t} = \frac{1}{2} i \left[p_{j,xx} + 2 \left(\sum_{l=1}^{n} p_l p_l^* \right) p_j \right], \quad 1 \le j \le n, \tag{2}$$

where subscripts x and t denote spatial and temporal partial derivatives. Here we notice that some work has been done for the system (2). The generalized Darboux transformation for system (2) was derived in [13]. A binary Darboux transformation of system (2) has been carried out in [14] (see [12,15] for many other interesting applications about system (2)). As far as we know, Lie symmetries and conservation laws of the system (2) have not been studied.

Exact solutions to nonlinear differential equations play an important role in the proper understanding of many nonlinear phenomena and processes in various areas of natural science. Currently, there are various methods used to solve nonlinear partial differential equations, such as the inverse scattering method [16,17], Darboux transformation [18,19], Hirota bilinear method [20,21], Lie symmetry group method [22–26], and so on. Among them, the Lie symmetry method provides powerful new ways to find exact solutions. The method has applications for both ordinary and partial differential equations and is not restricted to linear equations [24].

Another important application of Lie symmetry to physical problems is the construction of conservation laws [27,28]. For given systems of differential equations arising from a Lagrangian formulation, there exists a fundamental theorem due to E. Noether. Noether proved that for every infinitesimal transformation which is admitted by the action integral of a Lagrangian system, one can constructively find a conservation law. Noether's method is the principal systematic procedure for constructing conservation laws for complicated systems of partial differential equations. However, Noether's theorem can be applied only for differential equations with Lagrangian. Recently, this method has been developed and appears to be fruitful [27,29,30]. In particular, Ibragimov proposed a new method to construct the conservation laws in [31,32] based on the concept of a formal Lagrangian, and gave the formula for constructing conservation laws. In this paper, the Lie symmetry method is used to study the system (2) including constructing explicit solutions, and the conservation laws with Ibragimov's method.

This paper has the following structure. In Sect. 2, we will obtain the Lie point symmetry of the system (2). In Sect. 3, we will derive the symmetry reductions of (2) and construct exact solutions of (2). In Sect. 4, we will give the conservation laws of (2). In Sect. 5, we give summaries and discussions.

2 Lie symmetries

Lie symmetry analysis will be performed on the system (2). We consider the complex-valued function $p(x,t)$ in the following form:

$$p_j(x,t) = u_j(x,t) + v_j(x,t)i, \quad 1 \le j \le n, \tag{3}$$

where $u_j(x,t)$ and $v_j(x,t)$ are real-valued functions. Substituting (3) into (2) and decomposing into real and imaginary parts, we obtain

$$u_{j,t} + \frac{1}{2} v_{j,xx} + \left[\sum_{l=1}^{n} \left(u_l^2 + v_l^2 \right) \right] v_j = 0, \quad 1 \le j \le n, \tag{4}$$

$$v_{j,t} - \frac{1}{2} u_{j,xx} - \left[\sum_{l=1}^{n} \left(u_l^2 + v_l^2 \right) \right] u_j = 0, \quad 1 \le j \le n. \tag{5}$$

To find out the symmetries, we consider the Lie group of point transformations

$$\begin{aligned} x^* &= x + \epsilon\, \xi(x,t,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n) + O(\epsilon^2), \\ t^* &= t + \epsilon\, \tau(x,t,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n) + O(\epsilon^2), \\ u_j^* &= u_j + \epsilon\, \eta_j(x,t,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n) \\ &\quad + O(\epsilon^2),\ 1 \le j \le n, \\ v_j^* &= v_j + \epsilon\, \phi_j(x,t,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n) \\ &\quad + O(\epsilon^2),\ 1 \le j \le n, \end{aligned} \tag{6}$$

where $\epsilon \ll 1$ is a group parameter, and $\xi, \tau, \eta_j, \phi_j$ are the infinitesimals. The vector field corresponding to the transformation group (6) is

$$\begin{aligned} V &= \xi(x,t,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n)\frac{\partial}{\partial x} \\ &\quad + \tau(x,t,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n)\frac{\partial}{\partial t} \\ &\quad + \sum_{j=1}^{n} \eta_j(x,t,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n)\frac{\partial}{\partial u_j} \\ &\quad + \sum_{j=1}^{n} \phi_j(x,t,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n)\frac{\partial}{\partial v_j}. \end{aligned} \tag{7}$$

The vector field (7) will generate the symmetries of Eqs. (4, 5). Equations (4, 5) involve second-order derivatives; therefore, we need to prolong vector field V to the second order. The second prolongation of V is

$$\begin{aligned}\mathrm{pr}^2\, V \ &= \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta_j \frac{\partial}{\partial u_j} \\ &+\phi_j \frac{\partial}{\partial v_j} + \eta^{(1)}_{j,t} \frac{\partial}{\partial u_{j,t}} + \phi^{(1)}_{j,t} \frac{\partial}{\partial v_{j,t}} \\ &+\eta^{(2)}_{j,xx} \frac{\partial}{\partial u_{j,xx}} + \phi^{(2)}_{j,xx} \frac{\partial}{\partial v_{j,xx}}. \end{aligned} \tag{8}$$

To calculate the symmetries of (4)-(5), employ the second prolongation (8) of the vector field (7) along with invariance condition onto (4)-(5):

$$\begin{aligned}&\mathrm{pr}^2\, V\, (\varDelta_{1,j})|_{\varDelta_{1,1}=0,\varDelta_{2,1}=0,\ldots,\varDelta_{1,n}=0,\varDelta_{2,n}=0,} = 0, \\ &1 \le j \le n, \\ &\mathrm{pr}^2\, V\, (\varDelta_{2,j})|_{\varDelta_{1,1}=0,\varDelta_{2,1}=0,\ldots,\varDelta_{1,n}=0,\varDelta_{2,n}=0,} = 0, \\ &1 \le j \le n, \end{aligned} \tag{9}$$

where

$$\begin{aligned}&\varDelta_{1,j} = u_{j,t} + \frac{1}{2}\, v_{j,xx} + \left[\sum_{l=1}^{n} (\, u_l^2 + v_l^2 \,)\right] v_j, \\ &1 \le j \le n, \\ &\varDelta_{2,j} = v_{j,t} - \frac{1}{2}\, u_{j,xx} - \left[\sum_{l=1}^{n} (\, u_l^2 + v_l^2 \,)\right] u_j, \\ &1 \le j \le n. \end{aligned} \tag{10}$$

The system (9) can also be rewritten as

$$\begin{aligned}\mathrm{pr}^2\, V\, (\varDelta_{1,j}) &= \sum_{l=1}^{n} [\eta_l \frac{\partial}{\partial u_l} + \phi_l \frac{\partial}{\partial v_l}] + \eta^{(1)}_{j,t} \frac{\partial}{\partial u_{j,t}} \\ &+\phi^{(2)}_{j,xx} \frac{\partial}{\partial v_{j,xx}}, 1 \le j \le n, \\ \mathrm{pr}^2\, V\, (\varDelta_{2,j}) &= \sum_{l=1}^{n} [\eta_l \frac{\partial}{\partial u_l} + \phi_l \frac{\partial}{\partial v_l}] + \phi^{(1)}_{j,t} \frac{\partial}{\partial v_{j,t}} \\ &+\eta^{(2)}_{j,xx} \frac{\partial}{\partial u_{j,xx}}, 1 \le j \le n, \end{aligned} \tag{11}$$

where

$$\begin{aligned}\eta^{(1)}_{j,t} &= D_t[\eta_j - \xi\, u_{j,x} - \tau\, u_{j,t}] + \xi\, u_{j,xt} + \tau\, u_{j,tt}, \\ &\quad 1 \le j \le n, \\ \phi^{(1)}_{j,t} &= D_t[\phi_j - \xi\, v_{j,x} - \tau\, v_{j,t}] + \xi\, v_{j,xt} + \tau\, v_{j,tt}, \\ &\quad 1 \le j \le n, \\ \eta^{(2)}_{j,xx} &= D_{xx}[\eta_j - \xi\, u_{j,x} - \tau\, u_{j,t}] + \xi\, u_{j,xxx} + \tau\, u_{j,xxt}, \\ &\quad 1 \le j \le n, \\ \phi^{(2)}_{j,xx} &= D_{xx}[\phi_j - \xi\, v_{j,x} - \tau\, v_{j,t}] + \xi\, v_{j,xxx} + \tau\, v_{j,xxt}, \\ &\quad 1 \le j \le n, \end{aligned} \tag{12}$$

and $D_x = \frac{\partial}{\partial x} + u_{j,x}\frac{\partial}{\partial u_j} + v_{j,x}\frac{\partial}{\partial v_j} + u_{j,xt}\frac{\partial}{\partial u_{j,t}} + u_{j,xx}\frac{\partial}{\partial u_{j,x}} + \cdots$, $D_t = \frac{\partial}{\partial t} + u_{j,t}\frac{\partial}{\partial u_j} + v_{j,t}\frac{\partial}{\partial v_j} + u_{j,xt}\frac{\partial}{\partial u_{j,x}} + u_{j,tt}\frac{\partial}{\partial u_{j,t}} + \cdots$ are the total derivative operators.

Combining (9) and (12), we get an equivalent system of (9) as follows

$$\begin{aligned}&2\, v_j[\sum_{l=1}^{n} (u_l\eta_l + v_l\phi_l)] \ +\phi_j \sum_{l=1}^{n} (\, u_l^2 + v_l^2 \,) + \eta^{(1)}_{j,t} \\ &\qquad + \frac{1}{2}\phi^{(2)}_{j,xx} = 0, \quad 1 \le j \le n, \\ &2\, u_j[\sum_{l=1}^{n} (u_l\eta_l + v_l\phi_l)] \ +\eta_j \sum_{l=1}^{n} (\, u_l^2 + v_l^2 \,) - \phi^{(1)}_{j,t} \\ &\qquad + \frac{1}{2}\eta^{(2)}_{j,xx} = 0, \quad 1 \le j \le n. \end{aligned} \tag{13}$$

Substituting (12) into (13), we obtain the determining equations with respect to ξ, τ, η_j and ϕ_j. Solving the determining equations, we find a nontrivial Lie group of transformation which is admitted by the system (4)-(5) with the infinitesimal generators (infinitesimal symmetries) given by

$$\begin{aligned}&V_1 = t\, \frac{\partial}{\partial x} + \sum_{j=1}^{n} \left(u_j x \frac{\partial}{\partial v_j} - v_j x \frac{\partial}{\partial u_j} \right), \\ &V_2 = x\, \frac{\partial}{\partial x} + 2\, t\, \frac{\partial}{\partial t} - \sum_{j=1}^{n} \left(u_j \frac{\partial}{\partial u_j} + v_j \frac{\partial}{\partial v_j} \right), \\ &V_3 = \frac{\partial}{\partial x},\ V_4 = \frac{\partial}{\partial t}, \\ &V_{j+4} = u_j \frac{\partial}{\partial v_j} - v_j \frac{\partial}{\partial u_j}, \quad 1 \le j \le n, \\ &V_{1,lk} = v_l \frac{\partial}{\partial u_k} - u_l \frac{\partial}{\partial v_k} + v_k \frac{\partial}{\partial u_l} - u_k \frac{\partial}{\partial v_l}, \\ &\qquad 1 \le k \le l-1\,;\ 2 \le l \le n, \\ &V_{2,lk} = -u_l \frac{\partial}{\partial u_k} - v_l \frac{\partial}{\partial v_k} + u_k \frac{\partial}{\partial u_l} + v_k \frac{\partial}{\partial v_l}, \\ &\qquad 1 \le k \le l-1\,;\ 2 \le l \le n. \end{aligned} \tag{14}$$

3 Symmetry reductions and Exact solutions

In the following, we will give the symmetry reductions and exact solutions of (4)-(5).

Case 1: For the generator V_1, the characteristic equations are

$$\frac{dx}{t}=\frac{dt}{0}=\frac{du_1}{-v_1x}=\frac{dv_1}{u_1x}=\ldots=\frac{du_n}{-v_nx}=\frac{dv_n}{u_nx}. \quad (15)$$

Integration of the first equation of this system gives the similarity variable $\alpha = t^2$. Using this similarity variable, from (15) we have

$$u_j(x,t)=M_j(\alpha)\cos\left[\frac{x^2}{2t}+N_j(\alpha)\right], 1\le j\le n, \quad (16)$$

$$v_j(x,t)=M_j(\alpha)\sin[\frac{x^2}{2t}+N_j(\alpha)],\ \ 1\le j\le n. \quad (17)$$

Substituting (16) and (17) into (3), we obtain

$$p_j(x,t)=M_j(\alpha)\exp\{i[\frac{x^2}{2t}+N_j(\alpha)]\},\quad 1\le j\le n, \quad (18)$$

where M_j and N_j are fuctons of the similarity variable $\alpha = t^2$.

Then substituting (16)-(17) into (4)-(5) leads to the ordinary differential equations (ODEs)

$$4\alpha M_j^{'}+M_j=0,\quad 1\le j\le n,$$
$$4\alpha M_jN_j^{'}-2\sqrt{\alpha}M_j\left(\sum_{l=1}^{n}M_l^2\right)=0,\quad 1\le j\le n. \quad (19)$$

The general solutions of (19) are

$$M_j=\frac{1}{\alpha^{\frac{1}{4}}}c_j, 1\le j\le n,$$
$$N_j=\frac{ln(\alpha)(\sum_{l=1}^{n}c_l^2)}{2}+\lambda_j, 1\le j\le n, \quad (20)$$

where c_j and λ_j are arbitrary constants.

Thus the exact solutions of the system (2) are given by

$$p_j(x,t)=\frac{c_j}{t^{\frac{1}{2}}}\exp\left\{i\left[\frac{x^2}{2t}+\frac{ln(t^2)\left(\sum_{l=1}^{n}c_l^2\right)}{2}+\lambda_j\right]\right\},\quad 1\le j\le n. \quad (21)$$

For $n = 1$, we obtain exact solutions of (4)-(5)

$$u_1(x,t)=\frac{c_1}{t^{\frac{1}{2}}}\cos\left[\frac{x^2}{2t}+\frac{ln(t^2)c_1^2}{2}+\lambda_1\right], \quad (22)$$

$$v_1(x,t)=\frac{c_1}{t^{\frac{1}{2}}}\sin\left[\frac{x^2}{2t}+\frac{ln(t^2)c_1^2}{2}+\lambda_1\right], \quad (23)$$

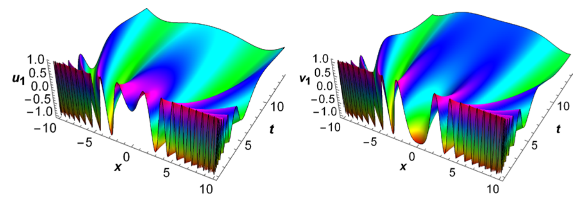

Fig. 1 For $n = 1$, the special exact solutions of (4)-(5) with $c_1 = 1$ and $\lambda_1 = 5$

where c_1 and λ_1 are arbitrary real constants.

Figure 1 shows the dynamic characteristics of the special exact solutions (22) and (23) with $c_1 = 1$ and $\lambda_1 = 5$. And we find that $u_1 \to 0$, $v_1 \to 0 (t \to +\infty)$.

Case 2: For the generator $V_4+\sum_{j=1}^{n}a_jV_{j+4}$, the characteristic euqations are

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du_1}{-a_1v_1}=\frac{dv_1}{a_1u_1}=\cdots=\frac{du_n}{-a_nv_n}=\frac{dv_n}{a_nu_n}. \quad (24)$$

Simple quadrature yields the similarity variable $\alpha = x$. Therefore, invariant solutions are of the form

$$u_j(x,t)=M_j(\alpha)\cos[a_jt+N_j(\alpha)],\quad 1\le j\le n, \quad (25)$$

$$v_j(x,t)=M_j(\alpha)\sin[a_jt+N_j(\alpha)],\quad 1\le j\le n. \quad (26)$$

Substituting (25), (26) into (3), we get

$$p_j(x,t)=M_j(\alpha)\exp\{i[a_jt+N_j(\alpha)]\},\quad 1\le j\le n, \quad (27)$$

where M_j and N_j are functions of the similarity variable $\alpha = x$.

Substituting (27) into system (2), we have

$$2M_j^{'}N_j^{'}+M_jN_j^{''}=0,\quad 1\le j\le n, \quad (28)$$

$$M_j^{''}-2a_jM_j-M_jN_j^{'2}+2M_j\left(\sum_{l=1}^{n}M_l^2\right)=0,\quad 1\le j\le n. \quad (29)$$

Equation (28) leads to

$$N_j^{'}=\frac{\lambda_j}{M_j^2},\quad 1\le j\le n, \quad (30)$$

where λ_j are arbitrary constants. Substituting (30) into (29), we obtain

$$M_j^{''} - 2a_j M_j - M_j(\frac{\lambda_j}{M_j^2})^2 + 2M_j\left(\sum_{l=1}^{n} M_l^2\right) = 0,\quad 1 \le j \le n. \tag{31}$$

Thus the exact solutions of the system (2) are given by

$$p_j(x,t) = M_j(\alpha)\exp\left\{i\left[a_j t + \int \frac{\lambda_j}{M_j^2}d\alpha + \gamma_j\right]\right\},\quad j = 1,\ldots,n, \tag{32}$$

where γ_j are arbitrary constants, M_j are given by (31).

For $n = 1$, if $a_1 = \frac{\lambda_1}{k_1} - \frac{k_1^2}{2}$, we have

$$u_1(x,t) = \sqrt{\frac{\lambda_1}{k_1}}\cos\left[\left(\frac{\lambda_1}{k_1} - \frac{k_1^2}{2}\right)t + k_1 x + \gamma_1\right], \tag{33}$$

$$v_1(x,t) = \sqrt{\frac{\lambda_1}{k_1}}\sin\left[\left(\frac{\lambda_1}{k_1} - \frac{k_1^2}{2}\right)t + k_1 x + \gamma_1\right], \tag{34}$$

where k_1, λ_1 and γ_1 are arbitrary real constants.

Figure 2 shows the periodic waves via solutions (33) and (34) with $k_1 = 2, \lambda_1 = 4, \gamma_1 = 5$.

Figure 3 shows the breather waves via solutions (33) and (34) when taking $a_1 = 1, \lambda_1 = 0$ in (31).

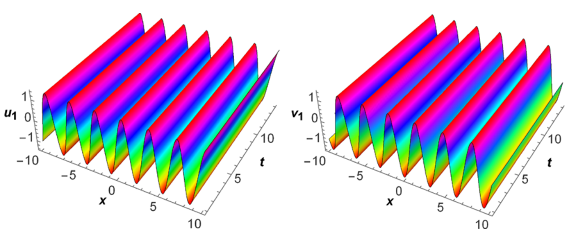

Fig. 2 For $n = 1$, the periodic wave solutions of (4)-(5) with $k_1 = 2, \lambda_1 = 4$ and $\gamma_1 = 5$

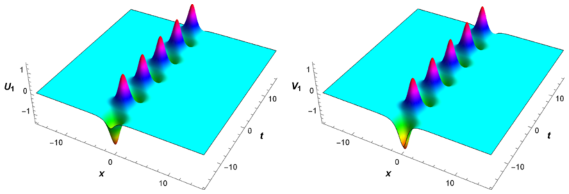

Fig. 3 For $n = 1$, the breather wave solutions of (4)-(5) with $a_1 = 1, \lambda_1 = 0$

Case 3: For the generator $V_2 + \sum_{j=1}^{n} a_j V_{j+4}$, the characteristic equations

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du_1}{-u_1 - a_1 v_1} = \frac{dv_1}{a_1 u_1 - v_1} = \ldots = \frac{du_n}{-u_n - a_n v_n} = \frac{dv_n}{a_n u_n - v_n} \tag{35}$$

provide the similarity variable $\alpha = xt^{-\frac{1}{2}}$.

We seek the invariant solutions in the form

$$u_j(x,t) = t^{-\frac{1}{2}}M_j(\alpha)\cos\left[\frac{a_j}{2}\ln t + N_j(\alpha)\right],\quad 1 \le j \le n, \tag{36}$$

$$v_j(x,t) = t^{-\frac{1}{2}}M_j(\alpha)\quad\sin\left[\frac{a_j}{2}\ln t + N_j(\alpha)\right],\quad 1 \le j \le n. \tag{37}$$

Furthermore, we have

$$p_j(x,t) = t^{-\frac{1}{2}}M_j(\alpha)\exp\left\{i\left[\frac{a_j}{2}\ln t + N_j(\alpha)\right]\right\},\quad 1 \le j \le n, \tag{38}$$

where M_j and N_j are fuctions of the similarity variable $\alpha = xt^{-\frac{1}{2}}$.

Substituting (38) into (2), we have

$$M_j + \alpha M_j^{'} - 2M_j^{'}N_j^{'} - M_j N_j^{''} = 0,\quad 1 \le j \le n, \tag{39}$$

$$M_j^{''} - M_j N_j^{'2} + \alpha M_j N_j^{'} - a_j M_j + 2M_j\left(\sum_{l=1}^{n} M_l^2\right) = 0,\ 1 \le j \le n. \tag{40}$$

From (39), we get

$$M_j = \exp\left\{-\int \frac{1 - N_j^{''}}{\alpha - 2N_j^{'}}d\alpha + \lambda_j\right\},\quad 1 \le j \le n, \tag{41}$$

where λ_j are arbitrary constants. And then (40) leads to N_j.

Thus the exact solutions of the system (2) are given by

$$p_j(x,t) = t^{-\frac{1}{2}}\exp\left\{i\left[\frac{a_j}{2}\ln t + N_j(\alpha)\right] - \int \frac{1 - N_j^{''}}{\alpha - 2N_j^{'}}d\alpha + \lambda_j\right\},\ 1 \le j \le n, \tag{42}$$

where λ_j are arbitrary constants, N_j are given by (40).

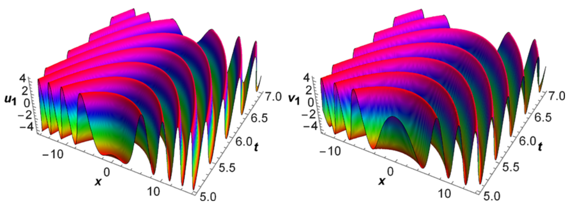

Fig. 4 For $n = 1$, the special exact solutions of (4)-(5) with $e^{k_1} = 10$ and $\lambda_1 = 5$

For $n = 1$, if $a_1 = 2e^{2k_1}$, we get the explicit solutions of (4)-(5)

$$u_1(x,t) = t^{-\frac{1}{2}} e^{k_1} \cos\left[e^{2k_1}\ln t + \frac{(xt^{-\frac{1}{2}})^2}{2} + \lambda_1\right], \quad (43)$$

$$v_1(x,t) = t^{-\frac{1}{2}} e^{k_1} \sin\left[e^{2k_1}\ln t + \frac{(xt^{-\frac{1}{2}})^2}{2} + \lambda_1\right], \quad (44)$$

where k_1, λ_1 and γ_1 are arbitrary real constants.

Figure 4 shows the dynamic characteristics of the solutions (43) and (44). It is clear that $u_1 \to 0$, $v_1 \to 0 (t \to +\infty)$.

Case 4: For the generator $V_3 + aV_4 + \sum_{j=1}^{n} b_j V_{j+4}$, the characteristic system

$$\frac{dx}{1} = \frac{dt}{a} = \frac{du_1}{-b_1 v_1} = \frac{dv_1}{b_1 u_1} = \ldots = \frac{du_n}{-b_n v_n} = \frac{dv_n}{b_n u_n} \quad (45)$$

provides the similarity variable $\alpha = t - ax$. Then one seeks the solutions in the form

$$u_j(x,t) = M_j(\alpha)\cos[b_j x + N_j(\alpha)], \quad 1 \le j \le n, \quad (46)$$

$$v_j(x,t) = M_j(\alpha)\sin[b_j x + N_j(\alpha)], \quad 1 \le j \le n, \quad (47)$$

where M_j and N_j are functions of α. Furthermore, we obtain

$$p_j(x,t) = M_j(\alpha)\exp\{i[b_j x + N_j(\alpha)]\}, \quad 1 \le j \le n. \quad (48)$$

The after substituting (48) into (2), one obtains

$$\begin{aligned} &2M_j'(-1 + ab_j - a^2 N_j') - a^2 M_j N_j'' = 0, \quad 1 \le j \le n, \\ &a^2 M_j'' - M_j\left[b_j^2 - 2(-1 + ab_j)N_j' + a^2 N_j'^2 - 2\left(\sum_{l=1}^{n} M_l^2\right)\right] = 0, \quad 1 \le j \le n. \end{aligned} \quad (49), (50)$$

From (49)-(50) we obtain

$$N_j' = \frac{(-1 + ab_j)M_j^2 - \lambda_j}{a^2 M_j^2}, \quad 1 \le j \le n, \quad (51)$$

$$a^2 M_j'' - M_j\left[b_j^2 - 2(-1 + ab_j)\frac{(-1 + ab_j)M_j^2 - \lambda_j}{a^2 M_j^2} + a^2\left[\frac{(-1 + ab_j)M_j^2 - \lambda_j}{a^2 M_j^2}\right]^2 - 2\left(\sum_{l=1}^{n} M_l^2\right)\right] = 0, \quad 1 \le j \le n. \quad (52)$$

where λ_j are arbitrary constants. Thus the exect solutions of the system (2) are as follows:

$$p_j(x,t) = M_j(\alpha)\exp\left\{i\left[b_j x + \int\left(\frac{(-1 + ab_j)M_j^2 - \lambda_j}{a^2 M_j^2} d\alpha + \gamma_j\right)\right]\right\}, \quad 1 \le j \le n, \quad (53)$$

where γ_j are arbitrary constants.

Case 5: For the generator $V_3 + aV_4$, we have similarity variable $\alpha = ax - t$, and the invariant solutions of the form

$$\begin{aligned} u_j(x,t) &= M_j(\alpha), \quad 1 \le j \le n, \\ v_j(x,t) &= N_j(\alpha), \quad 1 \le j \le n. \end{aligned} \quad (54)$$

Then substituting the following equation

$$p_j(x,t) = M_j(\alpha) + i\, N_j(\alpha), \quad 1 \le j \le n \quad (55)$$

into systems (2), we get the following reduced system for M_j and N_j:

$$\begin{aligned} &N_j\left[\sum_{l=1}^{n}(M_l^2 + N_l^2)\right] - M_j' + \frac{1}{2}a^2 N_j'' = 0, \quad 1 \le j \le n, \\ &M_j\left[\sum_{l=1}^{n}(M_l^2 + N_l^2)\right] + N_j' + \frac{1}{2}a^2 M_j'' = 0, \quad 1 \le j \le n. \end{aligned} \quad (56)$$

Case 6: For the generator V_3, we have similarity variable $\alpha = t$, and the invariant solutions of the form

$$\begin{aligned} u_j(x,t) &= M_j(\alpha), \quad 1 \le j \le n, \\ v_j(x,t) &= N_j(\alpha), \quad 1 \le j \le n. \end{aligned} \quad (57)$$

Then after substituting the following equations

$$p_j(x,t)=M_j(\alpha)+i\,N_j(\alpha),\quad 1\le j\le n \tag{58}$$

into (2), we get the reduced system for M_j and N_j:

$$\begin{aligned} &M_j\left[\sum_{l=1}^{n}(M_l^2+N_l^2)\right]-N_j^{'}=0,\quad 1\le j\le n,\\ &N_j\left[\sum_{l=1}^{n}(M_l^2+N_l^2)\right]+M_j^{'}=0,\quad 1\le j\le n. \end{aligned} \tag{59}$$

Case 7: For the generator V_4, we have similarity variable $\alpha=x$, and the invariant solutions of the form

$$\begin{aligned} &u_j(x,t)=M_j(\alpha),\quad 1\le j\le n,\\ &v_j(x,t)=N_j(\alpha),\quad 1\le j\le n. \end{aligned} \tag{60}$$

Substituting following equations

$$p_j(x,t)=M_j(\alpha)+i\,N_j(\alpha),\quad 1\le j\le n \tag{61}$$

into (2), then the reduced system for M_j and N_j are given by

$$\begin{aligned} &2\,M_j\left[\sum_{l=1}^{n}(M_l^2+N_l^2)\right]+M_j^{''}=0,\quad 1\le j\le n,\\ &2\,N_j\left[\sum_{l=1}^{n}(M_l^2+N_l^2)\right]+N_j^{''}=0,\quad 1\le j\le n. \end{aligned} \tag{62}$$

4 Conservation laws

Now we use the obtained symmetries to construct the conservation law of (4)-(5). In [31], Ibragimov proposed a new conservation theorem, that is, constructing conservation laws of differential equations that do not require the existence of a Lagrangian.

Ibragimov's method is related to formal Lagrangian, adjoint equations, and Lie symmetry. According to new conservation theorem [31], the Lagrangian of system (4)-(5) can be written as follows

$$\begin{aligned} L=\sum_{j=1}^{n}\Bigg\{&\alpha_j(x,t)\left\{u_{j,t}+\frac{1}{2}v_{j,xx}+\left[\sum_{l=1}^{n}(\,u_l^2+v_l^2)\right]v_j\right\}\\ &+\beta_j(x,t)\left\{v_{j,t}-\frac{1}{2}u_{j,xx}-\left[\sum_{l=1}^{n}(\,u_l^2+v_l^2)\right]u_j\right\}\Bigg\}, \end{aligned} \tag{63}$$

where $\alpha_j(x,t)$ and $\beta_j(x,t)$ are new dependent variables. The adjoint equations of system (4)-(5) have the following form

$$\begin{aligned} &P_j^*=\frac{\delta L}{\delta u_j}=0,\quad 1\le j\le n,\\ &Q_j^*=\frac{\delta L}{\delta v_j}=0,\quad 1\le j\le n, \end{aligned} \tag{64}$$

with

$$\begin{aligned} &\frac{\delta L}{\delta u_j}=\frac{\partial L}{\partial u_j}-D_t\frac{\partial L}{\partial u_{j,t}}+D_x^2\frac{\partial L}{\partial u_{j,xx}},\quad 1\le j\le n,\\ &\frac{\delta L}{\delta v_j}=\frac{\partial L}{\partial v_j}-D_t\frac{\partial L}{\partial v_{j,t}}+D_x^2\frac{\partial L}{\partial v_{j,xx}},\quad 1\le j\le n. \end{aligned} \tag{65}$$

From (63) and (64), we obtain

$$\begin{aligned} P_j^*&=2u_j\sum_{l=1}^{n}(v_l\alpha_l-u_l\beta_l)-\beta_j\left[\sum_{l=1}^{n}(\,u_l^2+v_l^2)\right]\\ &\quad-\alpha_{j,t}-\frac{1}{2}\beta_{j,xx},\,1\le j\le n,\\ Q_j^*&=2v_j\sum_{l=1}^{n}(v_l\alpha_l-u_l\beta_l)+\alpha_j\\ &\quad\left[\sum_{l=1}^{n}(\,u_l^2+v_l^2)\right]-\beta_{j,t}+\frac{1}{2}\alpha_{j,xx},\,1\le j\le n. \end{aligned} \tag{66}$$

If α_j is replaced by u_j and β_j by v_j in (66), we obtain systems (4)-(5). The conservation vector $C=(C^1,C^2,C^3,\ldots)$ has the following form

$$\begin{aligned} C^n&=\xi^n L+W^\alpha\left[\frac{\partial\,L}{\partial\,u_n^\alpha}-D_j\left(\frac{\partial\,L}{\partial\,u_{nj}^\alpha}\right)\right.\\ &\quad\left.+D_jD_k\left(\frac{\partial\,L}{\partial\,u_{njk}^\alpha}-\cdots\right.\right]\\ &\quad+D_j(W^\alpha)\left[\frac{\partial\,L}{\partial\,u_{nj}^\alpha}-D_k\left(\frac{\partial\,L}{\partial\,u_{njk}^\alpha}\right)+\cdots\right]\\ &\quad+D_jD_k(W^\alpha)\left[\frac{\partial\,L}{\partial\,u_{njk}^\alpha}-\cdots\right], \end{aligned} \tag{67}$$

where $W^\alpha=\eta^\alpha-\xi^j u_j^\alpha$ $(\alpha=1,2,\ldots,m)$.

According to (67), the conservation vectors of (63) are writen as

$$\begin{aligned} C^x&=\xi L+\sum_{j=1}^{n}\left[W^{u_j}\left(-D_x\frac{\partial L}{\partial u_{j,xx}}\right)\right]\\ &\quad+\sum_{j=1}^{n}\left[D_x(W^{u_j})\frac{\partial L}{\partial u_{j,xx}}\right]\\ &\quad+\sum_{j=1}^{n}\left[W^{v_j}\left(-D_x\frac{\partial L}{\partial v_{j,xx}}\right)\right] \end{aligned}$$

$$+\sum_{j=1}^{n}\left[D_x(W^{v_j})\frac{\partial L}{\partial v_{j,xx}}\right],$$

$$C^t=\tau L+\sum_{j=1}^{n}(W^{u_j}\frac{\partial L}{\partial u_{j,t}})+\sum_{j=1}^{n}\left(W^{v_j}\frac{\partial L}{\partial v_{j,t}}\right). \tag{68}$$

In the following, we will use Lie point symmetry (14) to construct conservation laws of systems (4)-(5).

Case 1: For the generator $V_1=t\frac{\partial}{\partial x}+\sum_{j=1}^{n}(u_jx\frac{\partial}{\partial v_j}-v_jx\frac{\partial}{\partial u_j})$, we have

$$\begin{aligned}W^{u_j}&=-v_jx-u_{j,x}t, \quad 1\le j\le n,\\ W^{v_j}&=u_jx-v_{j,x}t, \quad 1\le j\le n.\end{aligned} \tag{69}$$

Substituting (69) into (68), we obtain the conserved vectors

$$\begin{aligned}C_1^x&=\frac{1}{2}\left[\sum_{j=1}^{n}(u_j^2+v_j^2)\right]+t\left[\sum_{j=1}^{n}(u_ju_{j,t}+v_jv_{j,t})\right],\\ C_1^t&=-t\left[\sum_{j=1}^{n}(u_ju_{j,x}+v_jv_{j,x})\right].\end{aligned} \tag{70}$$

The vectors C_1^x and C_1^t satisfy

$$D_x(C_1^x)\ +\ D_t(C_1^t)\ =\ 0. \tag{71}$$

Then substituting $u_j=\frac{p_j+p_j^*}{2}$ and $v_j=\frac{p_j-p_j^*}{2i}$ into (70), we get the conservation laws of (2) as follows

$$\begin{aligned}T_1^x&=\frac{1}{2}\{\sum_{j=1}^{n}(p_jp_j^*)+t\left[\sum_{j=1}^{n}(p_j^*p_{j,t}+p_jp_{j,t}^*)\right]\},\\ T_1^t&=-\frac{1}{2}t\left[\sum_{j=1}^{n}(p_j^*p_{j,x}+p_jp_{j,x}^*)\right].\end{aligned} \tag{72}$$

Case 2: For the generator $V_2=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}-\sum_{j=1}^{n}(u_j\frac{\partial}{\partial u_j}+v_j\frac{\partial}{\partial v_j})$, we have

$$\begin{aligned}W^{u_j}&=-u_j-xu_{j,x}-2tu_{j,t}, \quad 1\le j\le n,\\ W^{v_j}&=-v_j-xv_{j,x}-2tv_{j,t}, \quad 1\le j\le n.\end{aligned} \tag{73}$$

For this case, (68) yields the conserved vectors

$$\begin{aligned}C_2^x&=x\left[\sum_{j=1}^{n}(u_ju_{j,t}+v_jv_{j,t})\right]+\frac{3}{2}\left[\sum_{j=1}^{n}(v_ju_{j,x}-v_{j,x}u_j)\right]\\ &\quad+t\left[\sum_{j=1}^{n}(u_{j,x}v_{j,t}-u_{j,t}v_{j,x}+v_ju_{j,tx}-u_jv_{j,tx})\right],\\ C_2^t&=t\left[\sum_{j=1}^{n}(u_jv_{j,xx}-v_ju_{j,xx})\right]\\ &\quad-x\left[\sum_{j=1}^{n}(u_ju_{j,x}+v_jv_{j,x})\right]-\left[\sum_{j=1}^{n}(u_j^2+v_j^2)\right].\end{aligned} \tag{74}$$

The vectors C_2^x and C_2^t satisfy

$$D_x(C_2^x)\ +\ D_t(C_2^t)\ =\ 0. \tag{75}$$

Then substituting $u_j=\frac{p_j+p_j^*}{2}$ and $v_j=\frac{p_j-p_j^*}{2i}$ into (41), we get the conservation laws of (2) as follows

$$\begin{aligned}T_2^x&=\frac{x}{2}\left[\sum_{j=1}^{n}(p_j^*p_{j,t}+p_jp_{j,t}^*)\right]\\ &\quad+\frac{3}{4i}\left[\sum_{j=1}^{n}\left(p_jp_{j,x}^*-p_j^*p_{j,x}\right)\right]\\ &\quad+\frac{t}{2i}\left[\sum_{j=1}^{n}\left(p_{j,x}^*p_{j,t}-p_{j,x}p_{j,t}^*\right.\right.\\ &\quad\left.\left.+p_jp_{j,tx}^*-p_j^*p_{j,tx}\right)\right],\\ T_2^t&=-\frac{t}{2i}\left[\sum_{j=1}^{n}(p_j^*p_{j,xx}-p_jp_{j,xx}^*)\right]\\ &\quad-\frac{x}{2}\left[\sum_{j=1}^{n}(p_j^*p_{j,x}+p_jp_{j,x}^*)\right]\\ &\quad-\left[\sum_{j=1}^{n}(p_j^*p_j)\right].\end{aligned} \tag{76}$$

Case 3: For the generator $V_3=\frac{\partial}{\partial x}$, we have

$$\begin{aligned}W^{u_j}&=-u_{j,x}, \quad 1\le j\le n,\\ W^{v_j}&=-v_{j,x}, \quad 1\le j\le n,\end{aligned} \tag{77}$$

and conserved vectors are written as:

$$\begin{aligned}C_3^x&=\sum_{j=1}^{n}(u_ju_{j,t}+v_jv_{j,t}),\\ C_3^t&=-\left[\sum_{j=1}^{n}(u_ju_{j,x}+v_jv_{j,x})\right].\end{aligned} \tag{78}$$

The vectors C_3^x and C_3^t satisfy the following equation:

$$D_x(C_3^x)\ +\ D_t(C_3^t)\ =\ 0. \tag{79}$$

Consequently, the conservation laws of (2) are writen as

$$T_3^x = \frac{1}{2}\left[\sum_{j=1}^{n}\left(p_j^* p_{j,t} + p_j p_{j,t}^*\right)\right],$$
$$T_3^t = -\frac{1}{2}\left[\sum_{j=1}^{n}\left(p_j^* p_{j,x} + p_j p_{j,x}^*\right)\right]. \tag{80}$$

Case 4: For the generator $V_4 = \frac{\partial}{\partial t}$, we have

$$W^{u_j} = -u_{j,t}, \quad 1 \le j \le n,$$
$$W^{v_j} = -v_{j,t}, \quad 1 \le j \le n, \tag{81}$$

and the conserved vectors:

$$C_4^x = \frac{1}{2}\left[\sum_{j=1}^{n}\left(-v_{j,x}u_{j,t} + v_j u_{j,tx} + v_{j,t}u_{j,x} - u_j v_{j,tx}\right)\right],$$
$$C_4^t = \frac{1}{2}\left[\sum_{j=1}^{n}\left(u_j v_{j,xx} - v_j u_{j,xx}\right)\right]. \tag{82}$$

The vectors C_4^x and C_4^t in (82) satisfy the following equation:

$$D_x(C_4^x) + D_t(C_4^t) = 0. \tag{83}$$

Substitution of $u_j = \frac{p_j + p_j^*}{2}$ and $v_j = \frac{p_j - p_j^*}{2i}$ into (82) leads to the conservation laws of (2) as follows

$$T_4^x = \frac{1}{4i}\left[\sum_{j=1}^{n}\left(p_{j,x}^* p_{j,t} - p_{j,x} p_{j,t}^* - p_j^* p_{j,tx} + p_j p_{j,tx}^*\right)\right],$$
$$T_4^t = \frac{1}{4i}\left[\sum_{j=1}^{n}\left(p_j^* p_{j,xx} - p_j p_{j,xx}^*\right)\right]. \tag{84}$$

Case 5: The generator $V_{j+4} = u_j \frac{\partial}{\partial v_j} - v_j \frac{\partial}{\partial u_j}$ yields the trivial conservation laws

$$D_x(C_{j+4}^x) + D_t(C_{j+4}^t) = 0, \quad 1 \le j \le n, \tag{85}$$

where

$$C_{j+4}^x = 0, \quad 1 \le j \le n,$$
$$C_{j+4}^t = 0, \quad 1 \le j \le n. \tag{86}$$

The conservation laws of $V_{1,lk}$ and $V_{2,lk}$ are obtained likewise.

5 Conclusion

This paper aims to investigate symmetry reductions and conservation laws of MNLS equations. As far as we know, there are few studies on the symmetry reduction and conservation laws of nonlinear Schrödinger equations with arbitrary components, except for the two or three-component nonlinear Schrödinger equations. Here, we present the general formula of infinitesimal symmetries of MNLS equations, which is essential for analyzing the specific types of MNLS equations in practice. Using the symmetry invariance property, the system (2) was reduced systematically and some new exact solutions were constructed successfully. More interestingly, the conservation laws of the MNLS equations were presented with a detailed derivation by utilizing Ibragimov's method. The resulting conservation laws can be used to find exact solutions and to expand numerical methods.

The MNLS equations can have various applications in different fields, including Bose-Einstein condensates in atomic physics, nonlinear optics, and superfluidity in condensed matter physics, among others. We hope that this study might be important for researchers specializing in applied mathematics, optical fibers, nonlinear transmission network, etc. In future work, we will focus on the derivative NLS equation with arbitrary components derived by adding nonlinear terms or high-order linear dispersion terms to the standard NLS equations.

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Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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