Optical and Quantum Electronics (2023) 55:1084
https://doi.org/10.1007/s11082-023-05370-y

On the dynamics of the generalized unstable nonlinear Schrödinger equation in dispersive media

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Received: 21 July 2023 / Accepted: 30 August 2023

Abstract

The nonlinear Schrödinger equation (NLSE) is one of the most important physical model in optical fiber theory for explaining changes in optical soliton growth. Due to the large range of opportunities for ultrafast signal routing systems and brief pulses of light in communications, optical soliton transmission in nonlinear fibers is currently a topic of significant interest for researchers. In this article, we study the dynamics of the generalised unstable NLSE which specified the time evolution of perturbations in barely steady or unstable media. A set of efficient analytical approaches are implemented to develop various interesting travelling wave structures such as bright, dark, singular, optical, bell shaped and periodic solitons solutions. Additionally, the stability of the obtained results is validated. Finally, the modulation instability of the governing model has also been investigated to present change effects that occurs as a result of the irregular refractive indices in an optical fiber, leading to distinct behaviour patterns. The methods as reported are applicable in clarity, effectiveness, and simplicity, establishing their applicability to various sets of dynamic and static nonlinear equations concerning evolutionary phenomena in computational physics, as well as other practical domains and numerous areas of research.

Keywords The nonlinear Schrödinger equation · Stability analysis · Modulation instability · Optical solitons

1 Introduction

Some floating wave solutions were computed using compositional programmes including Mathematica and Matlab (Hassan 2020). Several effective methodologies have been developed for the study of dynamical strictures, including the $\exp(-\psi(\phi))$ method (Lakestani and Manafian 2018), the unified method (Osman et al. 2018), the Jacobi elliptic expansion method (Tarla et al. 2022), the generalized exponential rational function method (Ghanbari and Gómez-Aguilar 2019), the Sardar sub-equation method (Ibrahim et al. 2023), the modified Khater method (Rezazadeh et al. 2019), the F-expansion method (Ali Akbar and Ali 2017), the He's semi-inverse method (Kohl et al. 2020), the Riccati equation method (Mohammed et al. 2022), the Sine-Gordon expansion method (Kundu et al. 2021),

Extended author information available on the last page of the article homogeneous balance method (Yang et al. 2020), the Darboux transformation method (Shi et al. 2023), the trial equation method (Gepreel et al. 2019), the auxiliary equation method (Rezazadeh et al. 2019), the Sine-Cosine method (Yao et al. 2022), the the Hirota bilinear method (Hoque and Alshammari 2020), and the homogeneous balance method (Iqbal et al. 2021). The nonlinear Schrödinger equation (NLSE) which have the anomalous dispersal phase can be written as

$$iv_t + \frac{1}{2}v_{xx} + \tau_2 v|v|^2 = 0, \qquad i = \sqrt{-1}, \tag{1}$$

where $v(x, t)$ is a complex function that demonstrates the profile of the soliton. The form of normal dispersal phase is described as

$$iv_t - \frac{1}{2}v_{xx} + \tau_2 v|v|^2 = 0. \tag{2}$$

The NLSE plays a vital role and importance in the area of optical fibers for the transmission of signals (Souleymanou et al. 2023). The governing model is a special form of NLSE which comes along the displacement of used variables. Further the integrable generalize form of NLSE can be write as

$$iv_t + v_{xx} + 2v|v|^2 - 2v = 0. \tag{3}$$

In Iqbal et al. (2022), implemented the the exponential rational function method while the extended simple equation method is utilized, while the trial equation method is used for obtaining the wave solution (Abbagari et al. 2021; Houwe et al. 2023). In this work, we utilize the more modified form of Eq. (3) name as the generalized nonlinear Schrödinger equation (Akinyemi et al. 2022) is considered by employing a variety of efficient analytical techniques namely the $(\frac{G'}{G^2})$ expansion method (Arshed and Sadia 2018), the modified extended *tan* hyperbolic function method (Alam and Jiang 2021) and the polynomial expansion method (Huang 2006) to construct various new soliton solutions for the generalized unstable NLSE, which reads

$$iv_t + \tau_1 v_{xx} + \tau_2 v|v|^2 + \tau_3 v = 0, \qquad i = \sqrt{-1}, \tag{4}$$

where τ_1, τ_2 and τ_3 are constants. The methods used in this work have great significant importance to study the dynamics of various complex models describing many physical phenomenon. Such techniques provide an easy and strong mathematical approach to deal with nonlinear dynamical structures more efficiently that occur in engineering disciplines, theoretical physics, and contemporary fields (Akinyemi et al. 2023). The solutions obtain are exclusive and novel; according to our knowledge the techniques applied in this paper has not implemented in past which conclude that the results are unique.

The framework of this article can be organized as follows: In order to compute the pulse solution of the generalized unstable nonlinear Schrödinger equation, three distinct techniques are employed. These techniques include the $(\frac{G'}{G^2})$ expansion method, the modified extended *tan* hyperbolic function method, and the polynomial expansion method. The proposed methods and their applications are discussed in Sects. 2 and 3 respectively. Furthermore, Sects. 3 and 4 address the topics of stability and modulation instability (MI), respectively. In Sect. 5, a concise depiction of the graphical representation of the solutions is presented. Finally, in Sect. 6, the paper culminates with a summary of the findings and conclusions.

2 Description of the methods

Consider the NPDE in the form

$$G\left(\upsilon, \upsilon_t, \upsilon_x, \upsilon_y, \upsilon_{tt}, \upsilon_{xx}, \upsilon_{xt}, \cdots\right) = 0, \tag{5}$$

where υ is a function of x and t. By taking the transformation of the form

$$\upsilon(x, t) = \upsilon(\zeta), \quad \zeta = \alpha x - \rho t, \tag{6}$$

where α and ρ denote constants. An ODE is obtained by using (6) in (5)

$$p\left(\upsilon, \upsilon', \upsilon'', \upsilon''', \cdots\right) = 0, \tag{7}$$

where $'$ denotes derivative of υ.

2.1 The $\left(\frac{G'}{G^2}\right)$ expansion method

The following is a concise overview of the key steps involved in the process:

Step 1: The solution to Eq. (7) can be represented as

$$\upsilon(\zeta) = c_0 + \sum_{i=1}^{n}\left[c_i\left(\frac{G'(\zeta)}{G(\zeta)^2}\right)^{i} + \sigma_i\left(\frac{G'(\zeta)}{G(\zeta)^2}\right)^{-i}\right], \tag{8}$$

where n represents a real variable and c_0, c_i and σ_i are the unknowns, also $\psi(\zeta)$ meets

$$\psi'(\zeta) = p + q\psi(\zeta)^2, \tag{9}$$

where p, q are real constants and

$$\frac{G'(\zeta)}{G(\zeta)^2} = \psi(\zeta).$$

Case 1: If $pq > 0$, accordingly, we acquire solution as

$$\psi(\zeta) = \frac{\sqrt{\frac{p}{q}}\left(A_2 \sin\left(\zeta\sqrt{pq}\right) + A_1 \cos\left(\zeta\sqrt{pq}\right)\right)}{A_2 \cos\left(\zeta\sqrt{pq}\right) - A_1 \sin\left(\zeta\sqrt{\mathrm{pq}}\right)}.$$

Case 2: If $pq < 0$, the solution can be obtained as

$$\psi(\zeta) = -\frac{\sqrt{|pq|}\left(A_1 \sinh\left(2\zeta\sqrt{|pq|}\right) + A_1 \cosh\left(2\zeta\sqrt{|pq|}\right) + A_2\right)}{q\left(A_1 \sinh\left(2\zeta\sqrt{|pq|}\right) + A_1 \cosh\left(2\zeta\sqrt{|pq|}\right) - A_2\right)}.$$

Case 3: For $q \neq 0$ and $p = 0$, the solution is obtained as

$$\psi(\zeta) = -\frac{A_1}{q\left(A_1\zeta + A_2\right)}.$$

Step 2: By substituting Eqs. (8) and (9) into Eq. (7) and equating the resulting expression to zero, we obtain multiple mathematical equations by collecting the common value of $\psi(\zeta)$.
Step 3: Therefore to properly attain Eq. (4)'s accurate wave solutions, it is necessary to evaluate the entire set of equations in the final stage by combining the quantities derived from the responses of Eqs. (9) and (8).

2.2 The modified extended *tan* hyperbolic function technique

The following is a concise overview of the key steps involved in the process:

Step 1: The solution to Eq. (7) can be represented as

$$\upsilon(\zeta) = a_0 + \sum_{i=1}^{n} a_i\psi^i(\zeta) + \sum_{i=1}^{n} \frac{d_i}{\psi^i(\zeta)}, \tag{10}$$

where n represents a real variable and a_0, a_i, d_i are the unknowns, also $\psi(\delta)$ satisfies

$$\psi'(\zeta) = b + \psi(\zeta)^2, \tag{11}$$

where b is a real constant.

Case 1: If $b < 0$, accordingly, we acquire solution as

$$\psi(\zeta) = -\sqrt{-b}\tanh\left(\sqrt{-b}\zeta\right),$$
$$\psi(\zeta) = -\sqrt{-b}\coth\left(\sqrt{-b}\zeta\right).$$

Case 2: If $b > 0$, the solution is obtained as

$$\psi(\zeta) = \sqrt{b}\tan\left(\sqrt{b}\zeta\right),$$
$$\psi(\zeta) = -\sqrt{b}\cot\left(\sqrt{b}\zeta\right).$$

Case 3: For $b = 0$, we acquire solution as

$$\psi(\zeta) = -\frac{1}{\zeta}.$$

Step 2: By incorporating Eqs. (10) and (11) into Eq. (7) and equating the resulting expression to zero, we obtain a set of multiple mathematical equations that share the common value of $\psi(\zeta)$.
Step 3: In order to obtain accurate wave solutions of Eq. (4), one needs to evaluate the entire set of equations in the final stage by combining the quantities found with the Eq. (10) and to the Eq. (11) response.

2.3 The polynomial expansion method

The following is a concise overview of the key steps involved in the process:

Step 1: The solution to Eq. (7) can be represented as

$$v(\zeta) = \sum_{i=1}^{n} a_i \psi(\zeta)^{-i} + \sum_{i=1}^{n} d_i \psi(\zeta)^{i} + d_0, \tag{12}$$

where n expresses a real variable and a_i, d_i are the unknowns, also $\psi(\zeta)$ meets

$$\psi'(\zeta) = \beta\psi(\zeta) + \gamma + \psi(\zeta)^2, \tag{13}$$

such that the real constants are labeled as β and γ. The equation defined above provides various types of solutions based on the following conditions.

Case 1: If $\beta = 0, \ \ \gamma = 0$, accordingly, we acquire solution as

$$\psi(\zeta) = -\frac{1}{\zeta}.$$

Case 2: For $\beta \neq 0, \ \ \gamma = 0$, we acquire solution as

$$\psi(\zeta) = -\frac{\beta}{A_0 e^{-\beta\zeta} - 1}.$$

where A_0 is an integration constant.

Case 3: If $\beta = 0, \ \ \gamma \neq 0, \ \ \gamma > 0$, accordingly, we acquire solution as

$$\psi(\zeta) = \sqrt{\gamma} \tan\left(\sqrt{\gamma}\zeta\right),$$
$$\psi(\zeta) = -\sqrt{-\gamma} \cot\left(\sqrt{\gamma}\zeta\right).$$

Case 4: If $\beta = 0, \ \ \gamma \neq 0, \ \ \gamma < 0$, consequently, the solution is obtained as

$$\psi(\zeta) = -\sqrt{-\gamma} \tanh\left(\sqrt{-\gamma}\zeta\right),$$
$$\psi(\zeta) = \sqrt{-\gamma} \coth\left(\sqrt{-\gamma}\zeta\right).$$

Case 5: If $\gamma \neq 0, \ \ \ \beta \neq 0$, then solution is acquired as

$$\psi(\zeta) = \frac{\delta_1 - A_1\delta_2 e^{\zeta(\delta_1 - \delta_2)}}{1 - A_1 e^{\zeta(\delta_1 - \delta_2)}},$$

where A_1 is an integration constant and δ_1 and δ_2 are the roots of the equation $\delta^2 + \beta\delta + \gamma$, i.e.,

$$\delta_1 = \frac{-p + \sqrt{p^2 - 4q}}{2}, \qquad \delta_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}$$

Step 2: Placing Eqs. (12) and (13) to (7) and if, we placed all of them to zero, we acquire multiple mathematical equations by gathering the same value of $\psi(\zeta)$.

Step 3: Therefore to properly attain Eq. (4) accurate wave solutions, one evaluates the entire set of equations in the final stage by combining the quantities found with the Eq. (12) and to the Eq. (13) response.

3 Traveling wave solutions

The primary objective is to obtain accurate traveling pulse solutions for Eq. (4) using a variety of approaches. We assessed the transformation accordingly

$$\upsilon(x,t) = \upsilon(\zeta)e^{i\phi}, \quad \phi = \rho_2 t + \alpha_2 x, \quad \zeta = \alpha_1 x - \rho_1 t, \tag{14}$$

An ODE is obtained by using given transformation Eq. (14) into the Eq. (4), which gives both real and imaginaryparts

$$(\alpha_1^2\tau_1)\upsilon''(\zeta) + (\alpha_2^2\left(-\tau_1\right) - \rho_1 + \tau_3)\upsilon(\zeta) + \tau_2\upsilon^3(\zeta) = 0, \tag{15}$$

and

$$(2\tau_1\alpha_1\alpha_2 + \rho_2)\upsilon'(\zeta) = 0. \tag{16}$$

3.1 Applications of the $\left(\frac{G'}{G^2}\right)$ expansion method

By employing a homogeneous balancing approach, we achieve the balancing number $n = 1$. It is observed that Eq. (15) possesses a solution in the form

$$\upsilon(\zeta) = c_0 + c_1\psi(\zeta) + \frac{\sigma_1}{\psi(\zeta)}, \tag{17}$$

by adding Eq. (17) with Eq. (9) into Eq. (15), we obtain the following outcomes for the equations in mathematical forms:

$$\begin{aligned}
2\alpha_1^2 c_1 q^2\tau_1 + c_1^3\tau_2 &= 0,\\
2\alpha_1^2 p^2\sigma_1\tau_1 + \sigma_1^3\tau_2 &= 0,\\
-\alpha_2^2 c_0\tau_1 - c_0\rho_1 + 6c_1c_0\sigma_1\tau_2 + c_0^3\tau_2 + c_0\tau_3 &= 0,\\
-\alpha_2^2\sigma_1\tau_1 + 3c_0^2\sigma_1\tau_2 + 3c_1\sigma_1^2\tau_2 + 2\alpha_1^2 pq\sigma_1\tau_1 - \rho_1\sigma_1 + \sigma_1\tau_3 &= 0,\\
-\alpha_2^2 c_1\tau_1 + 2\alpha_1^2 c_1 pq\tau_1 - c_1\rho_1 + 3c_1^2\sigma_1\tau_2 + 3c_1c_0^2\tau_2 + c_1\tau_3 &= 0.
\end{aligned}$$

By examining these frameworks, one might come to this conclusion:

Family 1:

$$c_0 = 0,\ c_1 = -\frac{i\sqrt{2}\alpha_1 q\sqrt{\rho_1 - \tau_3}}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}},\ \sigma_1 = 0,\ \tau_1 = \frac{\rho_1 - \tau_3}{2\alpha_1^2 pq - \alpha_2^2}. \tag{18}$$

Therefore, by substituting the following outcomes in Eq. (17), one get

Case 1: If $b < 0$, in consequence, its solution is

$$v_1(x,t) = \left(-\frac{i\sqrt{2}\alpha_1 q\sqrt{\frac{p}{q}}\sqrt{\rho_1-\tau_3}\left(A_2 \sin\left(\sqrt{pq}(M)\right) + A_1 \cos\left(\sqrt{pq}(M)\right)\right)}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}\left(A_2 \cos\left(\sqrt{pq}(M)\right) - A_1 \sin\left(\sqrt{\mathrm{pq}}(M)\right)\right)} \right) \times e^{i\phi}. \quad (19)$$

Case 2: If $b > 0$, as a result, its solution is given by

$$v_2(x,t) = \left(\frac{i\sqrt{2}\alpha_1\sqrt{\rho_1-\tau_3}\sqrt{|pq|}\left(A_1 \sinh\left(2\sqrt{|pq|}(M)\right) + A_1 \cosh\left(2\sqrt{|pq|}(M)\right) + A_2\right)}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}\left(A_1 \sinh\left(2\sqrt{|pq|}(M)\right) + A_1 \cosh\left(2\sqrt{|pq|}(M)\right) - A_2\right)} \right) \times e^{i\phi} \quad (20)$$

Case 3: If $b = 0$, consequently, the solution is expressed as

$$v_3(x,t) = \left(\frac{i\sqrt{2}\alpha_1 A_1\sqrt{\rho_1-\tau_3}}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}\left(A_1(M) + A_2\right)} \right) \times e^{i\phi}, \quad (21)$$

where $M = \alpha_1 x - \rho_1 t$.

Family 2:

$$c_0 = 0,\ c_1 = 0,\ \sigma_1 = -\frac{i\sqrt{2}\alpha_1 p\sqrt{\rho_1-\tau_3}}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}},\ \tau_1 = \frac{\rho_1-\tau_3}{2\alpha_1^2 pq - \alpha_2^2}. \quad (22)$$

Therefore, by substituting the following outcomes in Eq. (17), one get

Case 1: If $b < 0$, consequently, the solution is given by

$$v_4(x,t) = \left(-\frac{i\sqrt{2}\alpha_1 p\sqrt{\rho_1-\tau_3}\left(A_2 \cos\left(\sqrt{pq}(M)\right) - A_1 \sin\left(\sqrt{\mathrm{pq}}(M)\right)\right)}{\sqrt{\frac{p}{q}}\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}\left(A_2 \sin\left(\sqrt{pq}(M)\right) + A_1 \cos\left(\sqrt{pq}(M)\right)\right)} \right) \times e^{i\phi}, \quad (23)$$

Case 2: If $b > 0$, as a consequence, its solution is

$$v_5(x,t) = \left(\frac{i\sqrt{2}\alpha_1 pq\sqrt{\rho_1-\tau_3}\left(A_1 \sinh\left(2\sqrt{|pq|}(M)\right) + A_1 \cosh\left(2\sqrt{|pq|}(M)\right) - A_2\right)}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}\sqrt{|pq|}\left(A_1 \sinh\left(2\sqrt{|pq|}(M)\right) + A_1 \cosh\left(2\sqrt{|pq|}(M)\right) + A_2\right)} \right) \times e^{i\phi} \quad (24)$$

Case 3: If $b = 0$, consequently, the solution is given by

$$v_6(x,t) = \left(\frac{i\sqrt{2}\alpha_1 pq\sqrt{\rho_1-\tau_3}\left(A_1(M) + A_2\right)}{A_1\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}} \right) \times e^{i\phi}, \quad (25)$$

where $M = \alpha_1 x - \rho_1 t$.

Family 3:

$$c_0 = 0,\ c_1 = \frac{i\sqrt{2}\alpha_1 q\sqrt{\rho_1 - \tau_3}}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}},\ \sigma_1 = 0,\ \tau_1 = \frac{\rho_1 - \tau_3}{2\alpha_1^2 pq - \alpha_2^2}. \tag{26}$$

Therefore, by substituting the following outcomes in Eq. (17), one get

Case 1: If $b < 0$, in consequence, its solution is

$$v_7(x,t) = \left(\frac{i\sqrt{2}\alpha_1 q\sqrt{\frac{p}{q}}\sqrt{\rho_1 - \tau_3}\left(A_2 \sin\left(\sqrt{pq}(M)\right) + A_1 \cos\left(\sqrt{pq}(M)\right)\right)}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}\left(A_2 \cos\left(\sqrt{pq}(M)\right) - A_1 \sin\left(\sqrt{\mathrm{pq}}(M)\right)\right)} \right) \times e^{i\phi}. \tag{27}$$

Case 2: If $b > 0$, as a result, its solution is given by

$$v_8(x,t) = \left(-\frac{i\sqrt{2}\alpha_1\sqrt{\rho_1 - \tau_3}\sqrt{|pq|}\left(A_1 \sinh\left(2\sqrt{|pq|}(M)\right) + A_1 \cosh\left(2\sqrt{|pq|}(M)\right) + A_2\right)}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}\left(A_1 \sinh\left(2\sqrt{|pq|}(M)\right) + A_1 \cosh\left(2\sqrt{|pq|}(M)\right) - A_2\right)} \right) \times e^{i\phi}, \tag{28}$$

Case 3: If $b = 0$, consequently, its solution is expressed as

$$v_9(x,t) = \left(-\frac{i\sqrt{2}\alpha_1 A_1\sqrt{\rho_1 - \tau_3}}{\sqrt{2\alpha_1^2 pq\tau_2 - \alpha_2^2\tau_2}\left(A_1(M) + A_2\right)} \right) \times e^{i\phi}. \tag{29}$$

where $M = \alpha_1 x - \rho_1 t$.

3.2 Application of the tanh-function method

By employing a homogeneous balancing approach, we obtain the balancing number $n = 1$. It is observed that Eq. (15) exhibits a solution in the form

$$v(\zeta) = a_0 + a_1\psi(\zeta) + \frac{d_1}{\psi(\zeta)}, \tag{30}$$

By adding Eq. (30) with Eq. (11) into Eq. (15), the resulting mathematical equations are as follows:

$$\begin{aligned}
\alpha_1^2 b^2 d_1\tau_1 + d_1^3\tau_2 &= 0,\\
2\alpha_1^2 a_1\tau_1 + a_1^3\tau_2 &= 0,\\
3a_0^2 d_1\tau_2 + 3a_1 d_1^2\tau_2 + 2\alpha_1^2 b d_1\tau_1 - \alpha_2^2 d_1\tau_1 - d_1\rho_1 + d_1\tau_3 &= 0,\\
-\alpha_2^2 a_0\tau_1 + 6a_1 a_0 d_1\tau_2 - a_0\rho_1 + a_0^3\tau_2 + a_0\tau_3 &= 0,\\
-a_1\alpha_2^2\tau_1 + 2a_1\alpha_1^2 b\tau_1 + 3a_1^2 d_1\tau_2 - a_1\rho_1 + 3a_1 a_0^2\tau_2 + a_1\tau_3 &= 0.
\end{aligned}$$

Upon examining these frameworks, one could arrive at the following conclusions:

Family 1:

$$a_0 = 0,\ a_1 = -\frac{i\sqrt{2}\alpha_1\sqrt{\tau_1}}{\sqrt{\tau_2}},\ d_1 = -\frac{i\sqrt{2}\alpha_1 b\sqrt{\tau_1}}{\sqrt{\tau_2}},\ \rho_1 = -\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3. \tag{31}$$

Therefore, by substituting the following outcomes in Eq. (30), one get

Case 1: If $b < 0$, in consequence, its solution is

$$v_{10}(x,t) = \left(\frac{i\sqrt{2}\alpha_1\sqrt{-b}\sqrt{\tau_1}\tanh\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} + \right.$$
$$\left. + \frac{i\sqrt{2}\alpha_1 b\sqrt{\tau_1}\coth\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{-b}\sqrt{\tau_2}}\right) \times e^{\pi\phi}, \tag{32}$$

$$v_{11}(x,t) = \left(\frac{i\sqrt{2}\alpha_1 b\sqrt{\tau_1}\tanh\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{-b}\sqrt{\tau_2}}\right.$$
$$\left. + \frac{i\sqrt{2}\alpha_1\sqrt{-b}\sqrt{\tau_1}\coth\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}}\right) \times e^{\pi\phi}. \tag{33}$$

Case 2: If $b > 0$, as a result, its solution is

$$v_{12}(x,t) = \left(-\frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\tan\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}}\right.$$
$$\left. - \frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\cot\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}}\right) \times e^{\pi\phi}, \tag{34}$$

$$v_{13}(x,t) = \left(\frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\tan\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}}\right.$$
$$\left. + \frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\cot\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}}\right) \times e^{\pi\phi}. \tag{35}$$

Case 3: If $b = 0$, consequently, its solution is

$$\upsilon_{14}(x,t) = \left(\frac{i\sqrt{2}\alpha_1\sqrt{\tau_1}}{\sqrt{\tau_2}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right)\right)} - \frac{i\sqrt{2}\alpha_1 b\sqrt{\tau_1}\left(t\left(-\alpha_2^2\tau_1 - 4\alpha_1^2 b\tau_1 + \tau_3\right) - \alpha_1 x\right)}{\sqrt{\tau_2}} \right) \times e^{\pi\phi}. \tag{36}$$

Family 2:

$$a_0 = 0,\ a_1 = \frac{i\sqrt{2}\alpha_1\sqrt{\tau_1}}{\sqrt{\tau_2}},\ d_1 = 0,\ \rho_1 = -\alpha_2^2\tau_1 + 2\alpha_1^2 b\tau_1 + \tau_3.$$

Case 1: If $b < 0$, in consequence, its solution is

$$\upsilon_{15}(x,t) = \left(-\frac{i\sqrt{2}\alpha_1\sqrt{-b}\sqrt{\tau_1}\tanh\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 2\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} \right) \times e^{\pi\phi}, \tag{37}$$

$$\upsilon_{16}(x,t) = \left(-\frac{i\sqrt{2}\alpha_1\sqrt{-b}\sqrt{\tau_1}\coth\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 2\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} \right) \times e^{\pi\phi}. \tag{38}$$

Case 2: If $b > 0$, in consequence, its solution is

$$\upsilon_{17}(x,t) = \left(\frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\tan\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 2\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} \right) \times e^{\pi\phi}, \tag{39}$$

$$\upsilon_{18}(x,t) = \left(-\frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\cot\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 2\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} \right) \times e^{\pi\phi}. \tag{40}$$

Case 3: If $b = 0$, in consequence, its solution is

$$\upsilon_{19}(x,t) = \left(-\frac{i\sqrt{2}\alpha_1\sqrt{\tau_1}}{\sqrt{\tau_2}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 2\alpha_1^2 b\tau_1 + \tau_3\right)\right)} \right) \times e^{\pi\phi}. \tag{41}$$

Family 3:

$$a_0 = 0,\ a_1 = \frac{i\sqrt{2}\alpha_1\sqrt{\tau_1}}{\sqrt{\tau_2}},\ d_1 = -\frac{i\sqrt{2}\alpha_1 b\sqrt{\tau_1}}{\sqrt{\tau_2}},\ \rho_1 = -\alpha_2^2\tau_1 + 8\alpha_1^2 b\tau_1 + \tau_3.$$

Case 1: If $b < 0$, in consequence, its solution is

$$v_{20}(x,t) = \left(\frac{i\sqrt{2}\alpha_1 b\sqrt{\tau_1}\coth\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 8\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{-b}\sqrt{\tau_2}} - \frac{i\sqrt{2}\alpha_1\sqrt{-b}\sqrt{\tau_1}\tanh\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 8\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} \right) \times e^{\pi\phi}, \tag{42}$$

$$v_{21}(x,t) = \left(\frac{i\sqrt{2}\alpha_1 b\sqrt{\tau_1}\tanh\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 8\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{-b}\sqrt{\tau_2}} - \frac{i\sqrt{2}\alpha_1\sqrt{-b}\sqrt{\tau_1}\coth\left(\sqrt{-b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 8\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} \right) \times e^{\pi\phi}. \tag{43}$$

Case 2: If $b > 0$, in consequence, its solution is

$$v_{22}(x,t) = \left(\frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\tan\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 8\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} - \frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\cot\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 8\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} \right) \times e^{\pi\phi}, \tag{44}$$

$$v_{23}(x,t) = \left(\frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\tan\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 8\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} - \frac{i\sqrt{2}\alpha_1\sqrt{b}\sqrt{\tau_1}\cot\left(\sqrt{b}\left(\alpha_1 x - t\left(-\alpha_2^2\tau_1 + 8\alpha_1^2 b\tau_1 + \tau_3\right)\right)\right)}{\sqrt{\tau_2}} \right) \times e^{\pi\phi}. \tag{45}$$

Case 3: If $b = 0$, in consequence, its solution is

$$v_{24}(x,t)=\left(-\frac{i\sqrt{2}\alpha_1 b\sqrt{\tau_1}\left(t\left(-\alpha_2^2\tau_1+8\alpha_1^2 b\tau_1+\tau_3\right)-\alpha_1 x\right)}{\sqrt{\tau_2}}\right.$$
$$\left.-\frac{i\sqrt{2}\alpha_1\sqrt{\tau_1}}{\sqrt{\tau_2}\left(\alpha_1 x-t\left(-\alpha_2^2\tau_1+8\alpha_1^2 b\tau_1+\tau_3\right)\right)}\right)\times e^{\pi\phi}. \tag{46}$$

3.3 Application to the polynomial expansion method

By employing a homogeneous balancing method, we are able to attain a balancing number of n = 1. It is observed that the Eq. (15) possesses a solution in the form

$$v(\zeta)=d_0+d_1\psi(\zeta)+\frac{a_1}{\psi(\zeta)}. \tag{47}$$

Through the addition of Eqs. (47) and (13), and their subsequent inclusion in Eq. (15), we obtain

$$\begin{aligned}
22\alpha_1^2 a_1\gamma^2\tau_1+a_1^3\tau_2 &= 0,\\
3\alpha_1^2 a_1\beta\gamma\tau_1+3a_1^2 d_0\tau_2 &= 0,\\
3\alpha_1^2\beta d_1\tau_1+3d_0 d_1^2\tau_2 &= 0,\\
3a_1 d_1^2\tau_2+\alpha_1^2\beta^2 d_1\tau_1+2\alpha_1^2\gamma d_1\tau_1-\alpha_2^2 d_1\tau_1-d_1\rho_1+3d_1 d_0^2\tau_2+d_1\tau_3 &= 0,\\
\alpha_1^2 a_1\beta^2\tau_1+2\alpha_1^2 a_1\gamma\tau_1-\alpha_2^2 a_1\tau_1+3a_1^2 d_1\tau_2+3a_1 d_0^2\tau_2-a_1\rho_1+a_1\tau_3 &= 0,\\
a_1\alpha_1^2\beta\tau_1+6a_1 d_1 d_0\tau_2+\alpha_1^2\beta\gamma d_1\tau_1-\alpha_2^2 d_0\tau_1-d_0\rho_1+d_0^3\tau_2+d_0\tau_3 &= 0.
\end{aligned}$$

By examining these frameworks, one might come to these conclusions:

Family 1:

$$d_0=-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}},\ a_1=-\frac{2i\alpha_1\gamma\sqrt{\rho_1-\tau_3}}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}},$$
$$d_1=0,\ \tau_1=-\frac{2\left(\rho_1-\tau_3\right)}{\alpha_1^2\beta^2-4\alpha_1^2\gamma+2\alpha_2^2}.$$

Therefore, by substituting the following outcomes in Eq. (47), one get

Case 1: If $\beta=0,\ \gamma=0$ in consequence, its solution is

$$\upsilon_{25}(x,t)=\left(-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}-\frac{2i\alpha_1\gamma\sqrt{\rho_1-\tau_3}\left(\rho_1 t-\alpha_1 x\right)}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}}\right)\times e^{\pi\phi}. \tag{48}$$

Case 2: If $\beta \neq 0,\ \ \gamma = 0$ consequently, its solution is

$$\upsilon_{26}(x,t)=\left(\frac{2i\alpha_1\gamma\sqrt{\rho_1-\tau_3}\left(A_0 e^{-\beta(\alpha_1 x-\rho_1 t)}-1\right)}{\beta\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}}-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}\right)\times e^{\pi\phi}. \tag{49}$$

Case 3: If $\beta = 0,\ \ \gamma \neq 0,\ \ \gamma > 0$ in consequence, its solution is

$$\upsilon_{27}(x,t)=\left(-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}-\frac{2i\alpha_1\sqrt{\gamma}\sqrt{\rho_1-\tau_3}\cot\left(\sqrt{\gamma}\left(\alpha_1 x-\rho_1 t\right)\right)}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}}\right)\times e^{\pi\phi}, \tag{50}$$

$$\upsilon_{28}(x,t)=\left(\frac{2i\alpha_1\gamma\sqrt{\rho_1-\tau_3}\tan\left(\sqrt{\gamma}\left(\alpha_1 x-\rho_1 t\right)\right)}{\sqrt{-\gamma}\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}}-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}\right)\times e^{\pi\phi}, \tag{51}$$

Case 4: If $\beta = 0,\ \ \gamma \neq 0,\ \ \gamma < 0$ then the solution obtained is

$$v_{29}(x,t)=\left(\frac{2i\alpha_1\gamma\sqrt{\rho_1-\tau_3}\coth\left(\sqrt{-\gamma}\left(\alpha_1x-\rho_1t\right)\right)}{\sqrt{-\gamma}\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}}-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}\right)\times e^{\pi\phi}, \tag{52}$$

$$v_{30}(x,t)=\left(-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}-\frac{2i\alpha_1\gamma\sqrt{\rho_1-\tau_3}\tanh\left(\sqrt{-\gamma}\left(\alpha_1x-\rho_1t\right)\right)}{\sqrt{-\gamma}\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}}\right)\times e^{\pi\phi}. \tag{53}$$

Case 5: If $\gamma\neq 0,\quad \beta\neq 0$ in consequence, its solution is

$$v_{31}(x,t)=\left(-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}-\frac{2i\alpha_1\gamma\sqrt{\rho_1-\tau_3}\left(1-A_1e^{\left(\kappa_1-\kappa_2\right)\left(\alpha_1x-\rho_1t\right)}\right)}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}\left(\kappa_1-A_1\kappa_2e^{\left(\kappa_1-\kappa_2\right)\left(\alpha_1x-\rho_1t\right)}\right)}\right)\times e^{\pi\phi}. \tag{54}$$

Family 2:

$$d_0=\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}},\ d_1=\frac{2i\alpha_1\sqrt{\rho_1-\tau_3}}{\sqrt{\tau_2\left(-\left(\alpha_1^2\beta^2-4\alpha_1^2\gamma+2\alpha_2^2\right)\right)}},$$

$$a_1=0,\ \tau_1=-\frac{2\left(\rho_1-\tau_3\right)}{\alpha_1^2\beta^2-4\alpha_1^2\gamma+2\alpha_2^2}.$$

Case 1: If $\beta=0,\quad \gamma=0$ in consequence, its solution is

$$v_{32}(x,t)=\left(\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}-\frac{2i\alpha_1\sqrt{\rho_1-\tau_3}}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}\left(\alpha_1x-\rho_1t\right)}\right)\times e^{\pi\phi}. \tag{55}$$

Case 2: If $\beta \neq 0$, $\gamma = 0$ in consequence, its solution is

$$\begin{aligned} \upsilon_{33}(x,t) = & \left(\frac{i\alpha_1 \beta \sqrt{\rho_1 - \tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2 + 4\alpha_1^2\gamma\tau_2 - 2\alpha_2^2\tau_2}} \right. \\ & \left. - \frac{2i\alpha_1 \beta \sqrt{\rho_1 - \tau_3}}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right) + 4\alpha_1^2\gamma - 2\alpha_2^2\right)}\left(A_0 e^{-\beta(\alpha_1 x - \rho_1 t)} - 1\right)} \right) \times e^{\pi\phi}. \end{aligned} \tag{56}$$

Case 3: If $\beta = 0$, $\gamma \neq 0$, $\gamma > 0$ in consequence, its solution is

$$\begin{aligned} \upsilon_{34}(x,t) = & \left(\frac{i\alpha_1 \beta \sqrt{\rho_1 - \tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2 + 4\alpha_1^2\gamma\tau_2 - 2\alpha_2^2\tau_2}} \right. \\ & \left. + \frac{2i\alpha_1 \sqrt{\gamma}\sqrt{\rho_1 - \tau_3}\tan\left(\sqrt{\gamma}\left(\alpha_1 x - \rho_1 t\right)\right)}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right) + 4\alpha_1^2\gamma - 2\alpha_2^2\right)}} \right) \times e^{\pi\phi}, \end{aligned} \tag{57}$$

$$\begin{aligned} \upsilon_{35}(x,t) = & \left(\frac{i\alpha_1 \beta \sqrt{\rho_1 - \tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2 + 4\alpha_1^2\gamma\tau_2 - 2\alpha_2^2\tau_2}} \right. \\ & \left. + \frac{2i\alpha_1 \sqrt{-\gamma}\sqrt{\rho_1 - \tau_3}\cot\left(\sqrt{\gamma}\left(\alpha_1 x - \rho_1 t\right)\right)}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right) + 4\alpha_1^2\gamma - 2\alpha_2^2\right)}} \right) \times e^{\pi\phi}. \end{aligned} \tag{58}$$

Case 4: If $\beta = 0$, $\gamma \neq 0$, $\gamma < 0$ in consequence, its solution is

$$\begin{aligned} \upsilon_{36}(x,t) = & \left(\frac{i\alpha_1 \beta \sqrt{\rho_1 - \tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2 + 4\alpha_1^2\gamma\tau_2 - 2\alpha_2^2\tau_2}} \right. \\ & \left. + \frac{2i\alpha_1 \sqrt{-\gamma}\sqrt{\rho_1 - \tau_3}\tanh\left(\sqrt{-\gamma}\left(\alpha_1 x - \rho_1 t\right)\right)}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right) + 4\alpha_1^2\gamma - 2\alpha_2^2\right)}} \right) \times e^{\pi\phi}, \end{aligned} \tag{59}$$

$$v_{37}(x,t)=\left(\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2(-\beta^2)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}+\frac{2i\alpha_1\sqrt{-\gamma}\sqrt{\rho_1-\tau_3}\coth\left(\sqrt{-\gamma}(\alpha_1x-\rho_1t)\right)}{\sqrt{\tau_2\left(\alpha_1^2(-\beta^2)+4\alpha_1^2\gamma-2\alpha_2^2\right)}}\right)\times e^{\pi\phi}. \tag{60}$$

Case 5: If $\gamma\neq0,\quad\beta\neq0$ in consequence, its solution is

$$v_{38}(x,t)=\left(\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2(-\beta^2)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}+\frac{2i\alpha_1\sqrt{\rho_1-\tau_3}\left(\kappa_1-A_1\kappa_2e^{(\kappa_1-\kappa_2)(\alpha_1x-\rho_1t)}\right)}{\sqrt{\tau_2\left(\alpha_1^2(-\beta^2)+4\alpha_1^2\gamma-2\alpha_2^2\right)}\left(1-A_1e^{(\kappa_1-\kappa_2)(\alpha_1x-\rho_1t)}\right)}\right)\times e^{\pi\phi}. \tag{61}$$

Family 3:

$$d_0=-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2(-\beta^2)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}},\ d_1=-\frac{2i\alpha_1\sqrt{\rho_1-\tau_3}}{\sqrt{\tau_2\left(-\left(\alpha_1^2\beta^2-4\alpha_1^2\gamma+2\alpha_2^2\right)\right)}}$$
$$a_1=0,\ \tau_1=-\frac{2(\rho_1-\tau_3)}{\alpha_1^2\beta^2-4\alpha_1^2\gamma+2\alpha_2^2}.$$

Case 1: If $\beta=0,\ \ \gamma=0$ in consequence, its solution is

$$v_{39}(x,t)=\left(\frac{2i\alpha_1\sqrt{\rho_1-\tau_3}}{\sqrt{\tau_2\left(\alpha_1^2(-\beta^2)+4\alpha_1^2\gamma-2\alpha_2^2\right)}(\alpha_1x-\rho_1t)}-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2(-\beta^2)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}}\right)\times e^{\pi\phi}. \tag{62}$$

Case 2: If $\beta\neq0,\ \ \gamma=0$ in consequence, its solution is

$$v_{40}(x,t) = \left(\frac{2i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}\left(A_0e^{-\beta\left(\alpha_1x-\rho_1t\right)}-1\right)} - \frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}} \right) \times e^{\pi\phi}. \tag{63}$$

Case 3: If $\beta = 0, \;\; \gamma \neq 0, \;\; \gamma > 0$ in consequence, its solution is

$$v_{41}(x,t) = \left(-\frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}} - \frac{2i\alpha_1\sqrt{\gamma}\sqrt{\rho_1-\tau_3}\tan\left(\sqrt{\gamma}\left(\alpha_1x-\rho_1t\right)\right)}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}} \right) \times e^{\pi\phi}, \tag{64}$$

$$v_{42}(x,t) = \left(\frac{2i\alpha_1\sqrt{-\gamma}\sqrt{\rho_1-\tau_3}\cot\left(\sqrt{\gamma}\left(\alpha_1x-\rho_1t\right)\right)}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}} - \frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}} \right) \times e^{\pi\phi}. \tag{65}$$

Case 4: If $\beta = 0, \;\; \gamma \neq 0, \;\; \gamma < 0$ in consequence, its solution is

$$v_{43}(x,t) = \left(\frac{2i\alpha_1\sqrt{-\gamma}\sqrt{\rho_1-\tau_3}\tanh\left(\sqrt{-\gamma}\left(\alpha_1x-\rho_1t\right)\right)}{\sqrt{\tau_2\left(\alpha_1^2\left(-\beta^2\right)+4\alpha_1^2\gamma-2\alpha_2^2\right)}} - \frac{i\alpha_1\beta\sqrt{\rho_1-\tau_3}}{\sqrt{\alpha_1^2\left(-\beta^2\right)\tau_2+4\alpha_1^2\gamma\tau_2-2\alpha_2^2\tau_2}} \right) \times e^{\pi\phi}, \tag{66}$$

$$\upsilon_{44}(x,t) = \left(-\frac{i\alpha_1 \beta \sqrt{\rho_1 - \tau_3}}{\sqrt{\alpha_1^2(-\beta^2)\tau_2 + 4\alpha_1^2\gamma\tau_2 - 2\alpha_2^2\tau_2}} - \frac{2i\alpha_1 \sqrt{-\gamma}\sqrt{\rho_1 - \tau_3} \coth\left(\sqrt{-\gamma}(\alpha_1 x - \rho_1 t)\right)}{\sqrt{\tau_2\left(\alpha_1^2(-\beta^2) + 4\alpha_1^2\gamma - 2\alpha_2^2\right)}} \right) \times e^{\pi\phi}. \tag{67}$$

Case 5: If $\gamma \neq 0, \quad \beta \neq 0$ in consequence, its solution is

$$\upsilon_{45}(x,t) = \left(-\frac{i\alpha_1 \beta \sqrt{\rho_1 - \tau_3}}{\sqrt{\alpha_1^2(-\beta^2)\tau_2 + 4\alpha_1^2\gamma\tau_2 - 2\alpha_2^2\tau_2}} - \frac{2i\alpha_1 \sqrt{\rho_1 - \tau_3}\left(\kappa_1 - A_1\kappa_2 e^{(\kappa_1-\kappa_2)(\alpha_1 x - \rho_1 t)}\right)}{\sqrt{\tau_2\left(\alpha_1^2(-\beta^2) + 4\alpha_1^2\gamma - 2\alpha_2^2\right)}\left(1 - A_1 e^{(\kappa_1-\kappa_2)(\alpha_1 x - \rho_1 t)}\right)} \right) \times e^{\pi\phi}. \tag{68}$$

4 Stability analysis

Here, we define the Hamiltonian and the momentum for Eq. (4). In this kind of system, the technique is written as

$$G = \frac{1}{2} \int_{-\infty}^{\infty} \upsilon^2 \, dx, \tag{69}$$

The momentum of the system is denoted by G and the electric potential is represented by υ. An essential component for stabilised solitary pulses is

$$\frac{\partial G}{\partial \alpha_1} > 0, \tag{70}$$

where α_1 is the frequency. By incorporating the solution of the moving pulse described by Eq. (39) into Eq. (69), we derive the following outcome

$$G = \frac{1}{2} \int_{-10}^{10} \left(\frac{i\sqrt{2}\alpha_1 \sqrt{b}\sqrt{\tau_1} \tan\left(\sqrt{b}\left(\alpha_2^2\tau_1 - 2\alpha_1^2 b\tau_1 - \tau_3 + \alpha_1 x\right)\right)}{\sqrt{\tau_2}} \right)^2 dx, \tag{71}$$

shortly after being simplified, the result becomes

$$G = -\frac{\alpha_1 \sqrt{b}\tau_1\left(-20\alpha_1\sqrt{b} + \tan\left(\sqrt{b}D\right) + \tan\left(\sqrt{b}C\right)\right)}{\tau_2}. \tag{72}$$

We ultimately arrive at the result utilising the solitude waves stabilisation condition given in Eq. (70)

$$\frac{\sqrt{b}\tau_1\left(20\alpha_1\sqrt{b}-\tan\left(\sqrt{b}D\right)-\tan\left(\sqrt{b}C\right)+\alpha_1\left(-\sqrt{b}\right)f\right)}{\tau_2}>0, \tag{73}$$

where $f=2\left(2\alpha_1 b\tau_1+5\right)\sec^2\left(\sqrt{b}D\right)+\left(10-4\alpha_1 b\tau_1\right)\sec^2\left(\sqrt{b}C\right)-20$, $C=-\alpha_2^2\tau_1+10\alpha_1+2\alpha_1^2 b\tau_1+\tau_3$ and $D=\alpha_2^2\tau_1+10\alpha_1-2\alpha_1^2 b\tau_1-\tau_3$.

There exists an exceedingly evident scenario in which it becomes apparent, through Eq. (73), that Eq. (4) represents a stable model.

5 Modulation instability

The modulation instability (MI) is a change effects that occurs as a result of the irregular refractive indices in an optical fiber, leading to distinct behaviour patterns. When a high-power optical beam propagates through an optical fiber exhibiting irregular variations, it induces optical amplification in two wide spectral regions symmetrically positioned at both ends of the beam's wavelength spectrum. The sustainability of a solutions could be lowered by nonlinear elements and nonlinear mechanisms dispersion (Cheung et al. 2023; Yin et al. 2023). A persistent planar waveform can generate modulation instability, that leads to intensity and frequencies self-modulation, within a complex dispersion system. Consequently, this leads to an exponential amplification of tiny disturbances across a flattened surface. A review of modulation instability regions, that is applied in numerous domains, provides a foundation for controlling various patterns that don't involve linear occurrences (Biondini and Mantzavinos 2016; Li et al. 2022).

The moving waveform modulation instability can be observed in Eq. (4) within this section. Let's say that the disrupted solution offered by

$$\upsilon(x,t)=\nu\mathcal{L}(x,t)+T, \tag{74}$$

where ν and $\mathcal{L}$ be the coefficient of perturbation and perturbation term, T represents the steady flow constant of Eq. (4). By substituting Eq. (74) into Eq. (4) and linearizing the resulting expression, one can obtain

$$i\mathcal{L}_t+\tau_1\mathcal{L}_{xx}+3\tau_2T^2\mathcal{L}+\tau_3\mathcal{L}=0, \tag{75}$$

The solution of Eq. (75) can be assumed like as

$$\mathcal{L}(x,t)=\delta_1 e^{i\left(\alpha_1 x-\rho_1 t\right)}+\delta_2 e^{-i\left(\alpha_1 x-\rho_1 t\right)}. \tag{76}$$

Here δ_1 and δ_2 are constants. Upon substituting Eq. (76) into Eq. (75), we obtain the normal wave number

$$-\alpha_1^2\delta_1\tau_1+\delta_1\rho_1+3\delta_1\tau_2T^2+\delta_1\tau_3=0=0,$$
$$-\alpha_1^2\delta_1\tau_1-\delta_1\rho_1+3\delta_1\tau_2T^2+\delta_1\tau_3=0=0,$$

after simplification, we have

$$\rho_1 = \pm\left(\alpha_1^2\tau_1 - 3\tau_2 T^2 - \tau_3\right). \tag{77}$$

The equation presented above, Eq. (77), demonstrates the characteristics of steady stability. The solution is deemed stable when the real value of ρ_1 is satisfied, while an unstable behavior is exhibited when the imaginary value of ρ_1 is considered. Furthermore, the perturbation grows exponentially in such cases. Hence, we express the modulation instability gain in the following manner

$$h(\alpha_1) = 2Im(\alpha_1^2\tau_1 - 3\tau_2 T^2 - \tau_3).$$

We can write it as the incidence power influence on the rate of MI, it also have effects on other factor like group velocity and tune-off implications, for details see Fig. 1.

6 Discussion and results

In this section of the paper, we utilize Wolfram Mathematica to present graphical illustrations of the generalized unstable nonlinear Schrödinger equation. By utilizing the parameter values intrinsic to the model, we generate a wide range of graphical representations that capture distinct behaviors of solitons. These representations include but are not limited to bell-shaped solitons, periodic solitons, and dark solitons. This comprehensive set of graphical depictions enables us to explore the diverse dynamics and characteristics exhibited by solitons under different parameter configurations.

If we consider Fig. 2, it represents the graph depicting the solution $|v_9(x,t)|$ of Eq. (4) based on the appropriate parameter values $A_1 = 0.8,\ \alpha_1 = 0.25,\ \rho_1 = 0.71,\ \tau_3 = 0.62,\ A_2 = 0.16,\ \beta = 0.43,\ \alpha_2 = 0.4,\ \tau_2 = 0.5,\ p = 0.42,$ and $q = 0.71$. This graph vividly depicts the characteristic bell-shaped behavior exhibited by the desired solution. The smooth curve rises symmetrically, reaching a peak before gradually decaying on both sides. This distinctive bell-shaped pattern is a hallmark of the solution under investigation, revealing its unique nature and behavior within the given context. Similar to the Fig. 3 showcases the presence of bell-shaped soliton solution with parameters $\alpha_1 = 0.8,\ \tau_1 = 0.25,\ \tau_2 = 0.71,\ b = 1.2,\ \tau_3 = 0.16,$ and $\alpha_2 = 0.31$. Also Fig. 4 showcases the presence of a bell-shaped soliton solution. However, it is important to note that Fig. 4 corresponds to a different set of specified parameter values $\gamma = 0.8,\ \alpha_1 = 0.25$, $\rho_1 = 0.71,\ \gamma = -4.2,\ \tau_3 = 0.16,\ \beta = 0.43,\ \alpha_2 = 0.4,$ and $\tau_2 = 1.5$. The fact that both figures display a similar bell-shaped soliton pattern despite varying parameter configurations suggests the robustness and consistency of this particular solution. The bell-shaped soliton solution is a characteristic feature observed in a wide range of parameter combinations within the system. This consistency can be attributed to the inherent nonlinear dynamics of the system, where certain parameter values lead to the formation and stability of bell-shaped solitons. Despite the parameter variations between Figs. 3 and 4, the underlying dynamics of the system give rise to similar soliton behavior, reinforcing the robustness of this phenomenon.

Figure 5 showcases the periodic behavior of solitons, elucidating the dynamic characteristics resulting from the specific parameter values employed for $\alpha_1 = 0.8,\ \tau_1 = 0.25$, $\tau_2 = 0.71,\ b = 1.2,\ \tau_3 = 0.16,\ \alpha_2 = 0.31$. The solitons exhibit repetitive patterns over time, displaying regular oscillations and maintaining their shape. Similarly, Fig. 6 also demonstrates the periodic behavior of solitons with parameter $\alpha_1 = 0.8,\ \tau_1 = 0.25$, $\tau_2 = 0.71,\ b = 1.42,\ \tau_3 = 0.16,$ and $\alpha_2 = 0.31$. It reveals analogous characteristics to

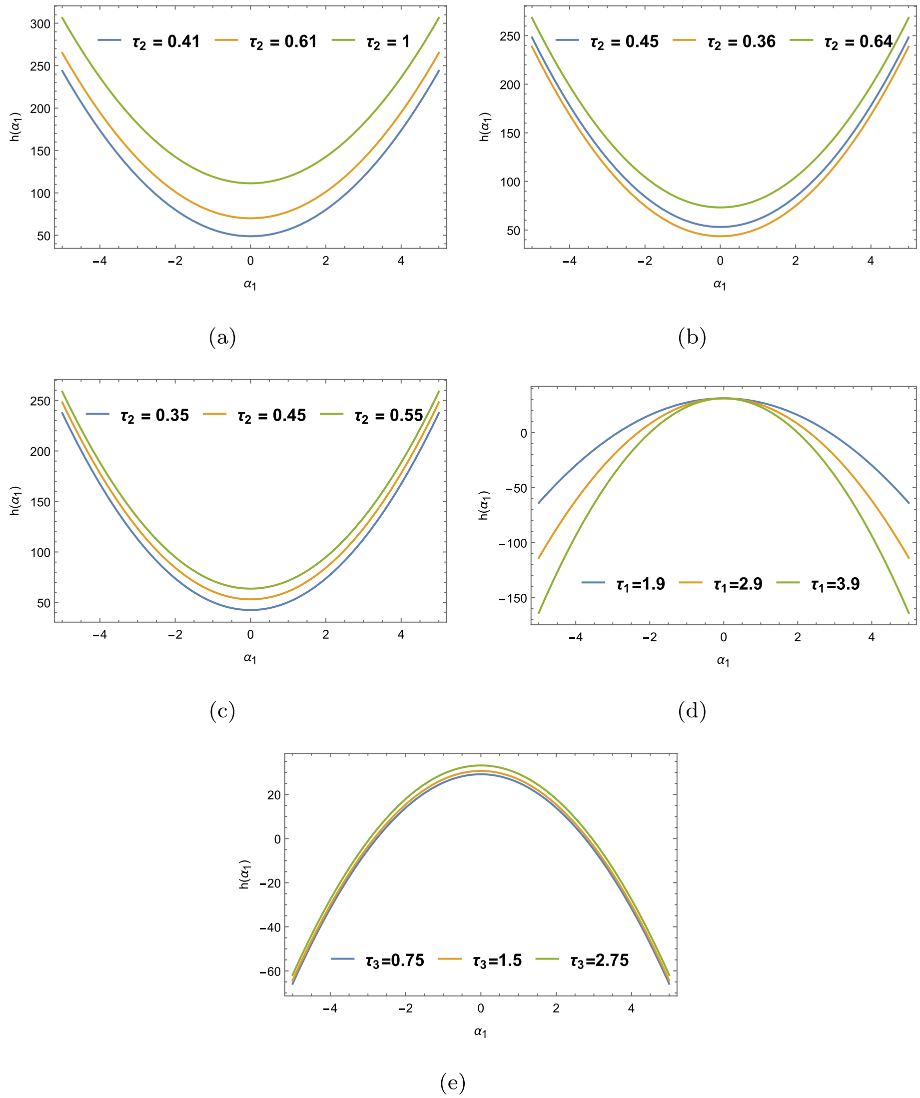
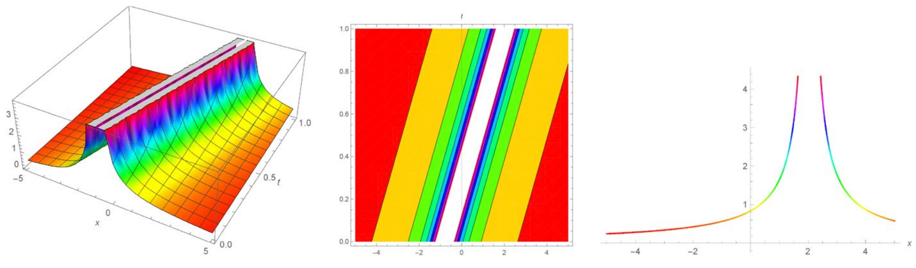


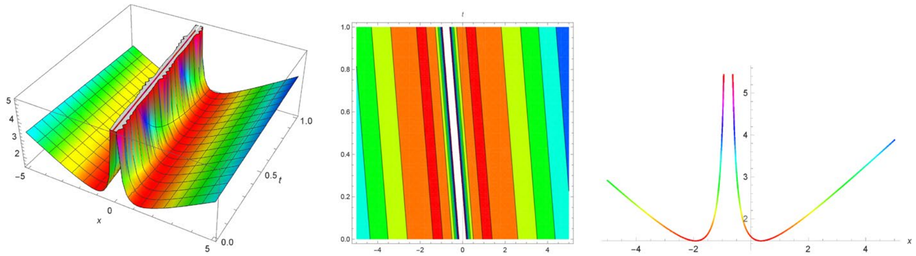
Fig. 1 Modulation instability gaining behaviour for τ_1, τ_2, τ_3

Fig. 5, portraying solitons with consistent periodic oscillations with parameter $\gamma = 0.8$, $\alpha_1 = 0.25$, $\rho_1 = 0.71$, $\gamma = -4.2$, $\tau_3 = 0.16$, $\beta = 0.43$, $\alpha_2 = 0.4$, and $\tau_2 = 1.5$. The resemblance in periodic behavior between these two figures suggests a similarity in the dynamic properties of the solitons. Concluding this discussion, Figs. 7, 8 and 9 demonstrate the manifestation of dark soliton solutions for the specified parameter values. For Fig. 7, the parameter are $\gamma = 0.8$, $\alpha_1 = 0.25$, $\rho_1 = 0.71$, $\gamma = -4.2$, $\tau_3 = 0.16$, $\beta = 0.43$, $\alpha_2 = 0.4$, and $\tau_2 = 1.5$. And for Fig. 8, parameters are $\gamma = -0.8$, $\alpha_1 = -0.25$, $\rho_1 = -0.71$, $\tau_3 = -0.16$, $\beta = 0.43$, $\alpha_2 = 0.4$, and $\tau_2 = 1.5$. Also for Fig. 9, the parameter are $\alpha_1 = 0.25$, $\rho_1 = 0.71$, $\gamma = -3.2$, $\tau_3 = 0.16$, $\beta = 0.43$, $\alpha_2 = 0.4$, and $\tau_2 = 1.5$. The dark



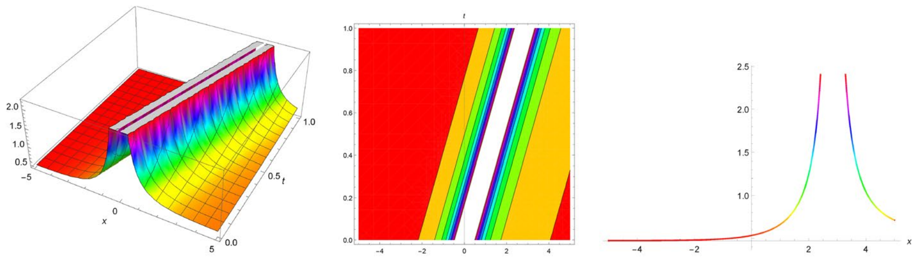
(a) (b) (c)

Fig. 2 Graphical pictures for the solution $|\upsilon_9(x,t)|$ of (4) for $A_1 = 0.8,\ \alpha_1 = 0.25,\ \rho_1 = 0.71,\ \tau_3 = 0.62$, $A_2 = 0.16$, $\beta = 0.43$, $\alpha_2 = 0.4$, $\tau_2 = 0.5$, $p = 0.42$, $q = 0.71$



(a) (b) (c)

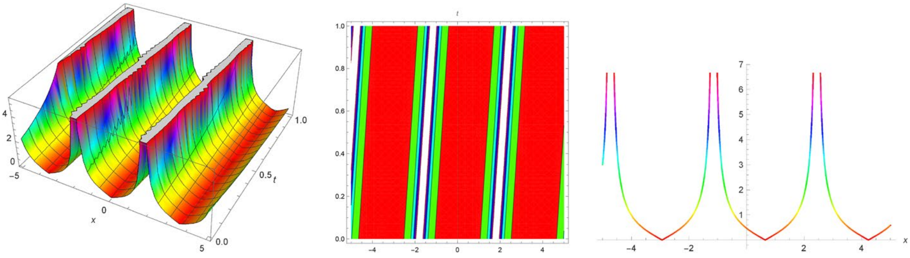
Fig. 3 Graphical pictures for the solution $|\upsilon_{12}(x,t)|$ of (4) for $\alpha_1 = 0.8,\ \tau_1 = 0.25,\ \tau_2 = 0.71,\ b = 1.2$, $\tau_3 = 0.16$, $\alpha_2 = 0.31$



(a) (b) (c)

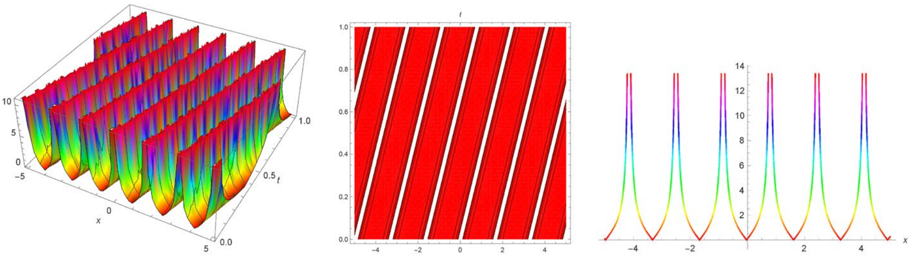
Fig. 4 Graphical pictures for the solution $|\upsilon_{37}(x,t)|$ of (4) for $\gamma = 0.8,\ \alpha_1 = 0.25,\ \rho_1 = 0.71,\ \gamma = -4.2$, $\tau_3 = 0.16$, $\beta = 0.43$, $\alpha_2 = 0.4$, $\tau_2 = 1.5$

soliton appears as a localized, dark region that stands out from the surrounding environment. In the curve, the depth of the depression or dip and the width of the soliton depend on the specific parameters of the system. The manifestation of dark soliton solutions within the analyzed system implies the ability of the system to support stable, localized wave structures that maintain their shape and characteristics over time.



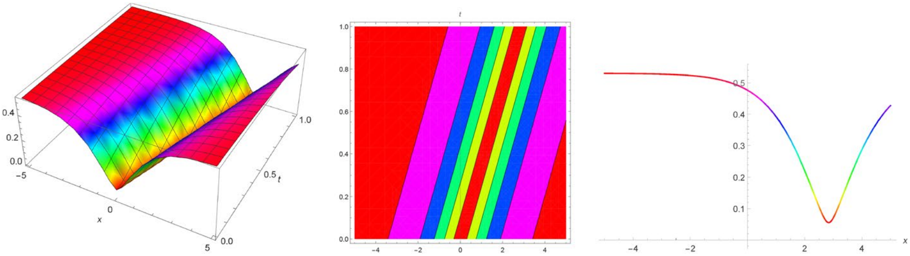
(a) (b) (c)

Fig. 5 Graphical pictures for the solution $|\upsilon_{15}(x,t)|$ of (4) for $\alpha_1 = 0.8,\ \tau_1 = 0.25,\ \tau_2 = 0.71,\ b = 1.2$, $\tau_3 = 0.16, \alpha_2 = 0.31$



(a) (b) (c)

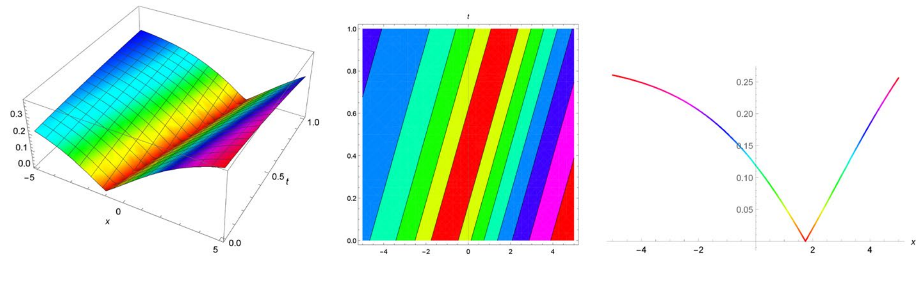
Fig. 6 Graphical pictures for the solution $|\upsilon_{19}(x,t)|$ of (4) for $\alpha_1 = 0.8,\ \tau_1 = 0.25,\ \tau_2 = 0.71,\ b = 1.42$, $\tau_3 = 0.16, \alpha_2 = 0.31$



(a) (b) (c)

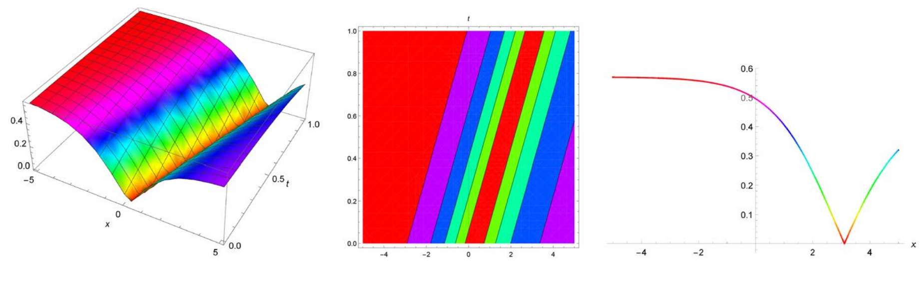
Fig. 7 Graphical pictures for the solution $|\upsilon_{28}(x,t)|$ of (4) for $\gamma = 0.8,\ \alpha_1 = 0.25,\ \rho_1 = 0.71,\ \gamma = -4.2$, $\tau_3 = 0.16, \beta = 0.43, \alpha_2 = 0.4, \tau_2 = 1.5$

In order to enhance our comprehension of the computed structure of these solutions, we present both 2D and 3D visualizations that depict the generated solutions for different combinations of variables. These representations offer a comprehensive view of the soliton structures, enabling a more detailed analysis and interpretation of their



(a) (b) (c)

Fig. 8 Graphical pictures for the solution $|\upsilon_{30}(x,t)|$ of (4) for $\gamma = -0.8$, $\alpha_1 = -0.25$, $\rho_1 = -0.71$, $\tau_3 = -0.16$, $\beta = 0.43$, $\alpha_2 = 0.4$, $\tau_2 = 1.5$



(a) (b) (c)

Fig. 9 Graphical pictures for the solution $|\upsilon_{43}(x,t)|$ of (4) for $\alpha_1 = 0.25$, $\rho_1 = 0.71$, $\gamma = -3.2$, $\tau_3 = 0.16$, $\beta = 0.43$, $\alpha_2 = 0.4$, $\tau_2 = 1.5$

properties. By examining the solutions from multiple perspectives, we aim to gain a deeper insight into their intricate characteristics and behavior.

7 Conclusion

The study of the generalised unstable nonlinear Schrödinger equation describing pulse propagation in optical fibers is the topic of this article. Optical solitons have received a lot of attention in the field of nonlinear optics due to their potential applications in telecommunications nd signal processing systems. The governing model displays a series of analytical solutions graphically represented by contour plots, 3D visuals, and 2D graphics that result in a variety of soliton solutions such as bright, dark, optical, periodic, and single bell-shaped solitons. In order to better comprehend the nature of the established results, we also considered the stability analysis and modulation instabilities connected with them. The ability to obtain solutions is important in the domains of physics, mathematics, and, in particular, fibre optics. The methodologies adopted in this research can be applied to a wide range of nonlinear dynamical models encountered in numerous scientific and engineering areas.

Acknowledgements The authors would like to extend their sincere appreciations to: The financial support provided by the Hubei University of Automotive Technology, P. R. China in the form of a start-up research grant (BK202212). The Higher Education of Russian Federation, Ural Federal University Program of Development within the Priority-2030 Program, Project: 4.38 (FEUZ 2023-0014).

Author Contributions FB: Identification of the research problem, software, review and editing. KUT: Methodology, conceptualization, resources and planning. MA: Scientific computing, visualizations and validation. W-XM: Supervision, project administration. SMRK: Formal analysis and investigation, writing original draft.

Funding Not available.

Declarations

Conflict of interest The authors declare no conflict of interest.

Consent for publication All the authors have agreed and given their consent for the publication of this research paper.

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