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Possible dark energy stars in Rastall gravity

Asifa Ashraf[*,‖], M. R. Shahzad[†,**] and Wen-Xiu Ma[*,‡,§,¶,††]

[*]*School of Mathematical Sciences*
Zhejiang Normal University, Jinhua
Zhejiang 321004, P. R. China

[†]*Department of Mathematics*
Bahauddin Zakariya University
Vehari Campus, Vehari 61100, Pakistan

[‡]*Department of Mathematics*
King Abdulaziz University
Jeddah 21589, Saudi Arabia

[§]*Department of Mathematics and Statistics*
University of South Florida
Tampa, FL 33620-5700, USA

[¶]*School of Mathematical and Statistical Sciences*
North-West University, Mafikeng Campus
Private Bag X2046, Mmabatho 2735, South Africa
[‖]*asifamustafa3828@gmail.com*
[**]*dr.rizwanshahzad@bzu.edu.pk*
[††]*wma3@usf.edu*

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Observational evidence revealed that in the composition of the Universe, the dark energy is in heap proportion which is mainly responsible for the accelerated expansion of the Universe. The astrophysicists hypothesized that every astrophysical object is likely interact with the dark energy Sakti and Sulaksono [*Phys. Rev. D* **103** (2021) 084042]. Therefore, researchers are intended to study the astrophysical objects with dark energy interaction. In this paper, we proposed a dark energy stars (composed of with dark matter and ordinary matter) model in the Rastall theory of gravity. The Rastall field equations are obtained in the Tolman–Kuchowicz-type metric. To solve the field equations, the arbitrary constants involved in the field equations (due to Tolman–Kuchowicz ansatz) are obtained by matching the interior geometry with the exterior Schwarzschild geometry. The obtained solution is tested physically by computing some analytical results and plotting graphs of different physical parameters like pressure, density for ordinary matter, density for dark matter, mass function, compactness and surface redshift. Stability of the model is also analyzed by different tests like velocity

[††]Corresponding author.

of sound, adiabatic index and TOV equations. The influence of the Rastall parameter on the model has been studied by plotting graphs and computing numerical values of model's physical parameters for different values of the Rastall parameter. We infer that our propounded model fulfills all the necessary requirements for a physically plausible model.

Keywords: Compact stars; strange quark matter; stability; Rastall theory of gravity.

1. Introduction

General relativity (GR) has opened up a new research window with the discovery of the speedy expansion of the universe in early-time inflation and late-time cosmic evolution through a extensive data sets. This phenomenological fact was procured much attention in 1998 by the international collaborative research team by discovering type-Ia supernovae [1–9], which was later substantiated by observing of the cosmic microwave background radiation [10, 11], a large-scale model [12–17] and the Wilkinson Microwave Anisotropy Probe [18]. Indeed, there could be some hidden ingredients behind such an issue, including the theorized form of matter known as dark matter and the repulsive force called dark energy. To date, several well-consistent models of this standard theory have been made to address this phenomenon, they have not yet provided a substantial picture, and this situation sufficiently requires the alternative gravitational theories. In this perspective, many researchers have generalized the Einstein–Hilbert Lagrangian (EHL) corresponding to various alternative gravity theories [19–28]. Amongst generalized theories, Rastall gravity (RG) [27, 28] can be a viable alternative to dark energy without adding any matter supplement, and its formulation could be possible by improving the GR conservation law as $\nabla_\mu T^{\mu\nu} = \lambda R^\nu$, where λ and R represent the Rastall parameter and the function of scalar curvature, respectively.

The study of dark energy stars has become an interesting subject because the expansion of universe is tremendously speeding up from the day of creation to till. It has been noticed that the most probable ingredient that responsible to accelerate the universe evolution is dark energy. According to the recent survey of Plank mission team, the universe consists only 4.9% of baryonic matter, contribution of dark matter is 26.8% and dark energy accounts of 68.3%. Since the dark energy holds a strong negative pressure against the gravity that could qualitatively explains the expediting expansion of the universe and may interact any baryonic matter within any local astrophysical phenomenon in compact objects, white dwarfs, ultra-relativistic strange stars, quark made stars, pulsars, wormholes, and black holes. On the other hand, dark matter has owed enticing nature in universe because it cannot be viewed with telescope and does not eject light. However, the gravitational forces on the observable matter have proved its existence.

In the previous studies, ultra-compact star models composed of stellar relativistic matter have found immense interest in both the GR and alternative gravity theories. Baade and Zwicky [29] were the first authors to present the phenomenon of neutron stars after the discovery of the particle "neutron". Later, this notion was affirmed by the discovery of pulsars by the observational evidences [30]. Camenzind [31] reported that the strong gravitational magnetic fields spawned in the supermassive black hole rotation during the accretion process. Hossein *et al.* [32] have contemplated the solutions of Einstein field equations with cosmological constant to model the anisotropic dark energy stars. Raposo *et al.* [33] introduced the study on dynamical compact stars by encompassing the relativistic anisotropic fluids in the GR framework and shown that these anisotropic objects can aspect as ultra dense compact models as black holes. Astashenok *et al.* [34] described the consequences of neutron star models made up interiorly with quark matter in the perturbative $f(R)$ context by considering the realistic equations of state. Abbas *et al.* [35] have discussed the effects of cosmological constant on anisotropic compact star models by proposing the Krori–Barua ansatz as possible candidates of dark energy stars. Bhar [36] manifested various physical implications of isotropic compact star candidates made up of dark and baryonic matter and concluded that the dark energy can play a significant role to interact any astrophysical compact object. The same author and his collaborator [37] presented a new model of dark star candidate to observe the various physical traits as well as stability constraint under the perturbation approach. Very recently, several authors [38–47] have explored many notable consequences using Einstein field equations to study the relativistic evolution of compact objects in the background of baryonic matter coupled with various matter fields like dark energy and quark matter.

Several alternative gravitational theories have exhibited their notable interaction with structuring the ultra-relativistic stars and the phenomenon of accelerating expansion of the universe. Amongst, Rastall theory has one of the most competent candidates regarding its too simple Einstein field equations. In the recent past, this gravity theory has passed an adequate experimental tests into account of its astrophysical and cosmological backgrounds. Some theoretical physicists [48, 49] have revealed the most intriguing and unexpected features of Rastall scalar field theories by studying the different perturbative test models of the cosmological universe. From the literature survey, it is obvious that several groups in the research community have illustrated the keynote impacts of RG on the dynamics of the cosmological universe by describing the early time inflation and the present time cosmic expansion [48–59]. Moradpour *et al.* [60] proposed various viable findings of pressureless matter within the generalized version of RG to address the dark side of the present accelerating evolutionary universe and the primary inflationary phase of the cosmic expansion. In recent times, Saleem and Hassan [61] manifested different aspects of the inflationary model based on the Rastall background. The same author and his collaborators have determined the useful impacts of slow-roll warm inflation on irreversible thermodynamics and the holographic dark energy entropy profiles by allowing the cosmological fluids in the Rastall modifications [62, 63]. Contrary to cosmological evidence, the gravity theory must be consistent with astrophysical constraints and properly model the interior of stellar objects.

Manna *et al.* [64] considered some viable values of the parameter σ to compare the results of the Rastall framework with the standardized GR tests [64]. Heydarzade and Darabi [65] presented a comparative study of RG with an equivalent GR part for the Kiselev black hole. Spallucci and Smailagic [66] determined the physical attributes of RG for structuring the ultra-relativistic compact object, which is like a Gaussian BH. Several authors have divulged interesting characteristics of black holes in different setups of the Rastall corrections [67–72]. Moreover, Kumar and Ghosh [73] described the well-consistent features of the rotating black holes by generalizing the results developed in [65]. Kumar *et al.* [74] derived the analytical study for the shadow of a rotating black hole supported by gravitational lensing in the Rastall context. In some recent insights [75–78], the impact of RG on thermodynamical quantities has been analyzed with the counterpart GR. Nashed and his collaborators [79–90] investigated various stellar structures and black hole solutions in GR and modified theories of gravity.

Gosh *et al.* [91] studied a general model of a Gravastar in the RG which is consistent with the physical requirements and stability analysis has also been investigated. Recently, Afshar *et al.* [92] focused on the inflationary problem and the reheating in the RG and concluded that the Rastall model remain well consistent with the Plank 2018 data as well as the Plank 2018 data+BK18+BAO. They constraint the Rastall parameter to obtain a more compatible model with the early type Glaxies and thermodynamical predictions. In the background of Rastall gravity, strong energy condition (SEC) was discussed by Moradpour *et al.* [93] and later by some other authors [94–96]

Despite the aforementioned leading outcomes, there are some major arguments in favor and opposition of the RG, Visser [97] contended that the RG is the counterpart to the GR. Later, Darabi *et al.* [98] reasoned that RG is a distinctive alternative theory compared to GR. In particular, Visser alluded to the fact that the curvature part of the equations of motion is equivalent and claimed that the energy-momentum tensor characterizing the matter part is composed of an improved preserved quantity that is primarily isolated in two forms, each of which is not uniquely preserved. There is no matter with the first outcome (that the curvature part of the equations of motion are equivalent), but it is obvious that the improved stress-energy tensor does not coincide with a GR conservation law. Moreover, Visser did not provide any empirical approach to deciding which stress-energy tensor is a physically acceptable one. Hence, Darabi *et al.* [98] stated that the stress-energy tensor claimed by the Visser was not correct. To consolidate their assertion, they offered a counter example of $f(R)$ gravity by imposing the same approach as was imposed by the Visser for RG, and after specific derivations, authors have determined that the most viable $f(R)$ gravity would also be identical to the GR, which is incorrect, and so they commented that RG is distinctly non-identical to the GR. In addition, RG is further endorsed by observational results for the estimation of universe birth, the Hubble parameter, and the matter composition for old and present-type galaxies [99–101]. Also, this alternative gravitational version has foretold to be examined by several physical phenomena like massive compact objects, black holes, cosmological models, etc. [102–108]. Although each investigation admits that the parameter $\kappa\lambda$ always represents the re-order of the matter part of GR. Hence, from this point of view, it is still worth studying the RG from both theoretical and observational perspectives. In this debate, several novel implications of super dense cosmic structures have earned ample attention among the scientific groups [109–115]. The RG seems to be identical to the GR in the mergence of cosmological constant but physically presents the numerous cosmological phenomenona that cannot be properly allocated in the standard GR. Moradpour *et al.* [60] inquired that RG and standard GR are two different gravity theories. In an old study, these two gravity theories were non-identically reported in [116]. Recently, Hansraj *et al.* [117] considered their novel investigation whether the RG and GR are equivalent theories. They have studied the impacts of RG on the relativistic compact models by employing the static spherical symmetric system. In a recent work, Hansraj and Banerjee [118] more fortified their previous outcomes to counter the Visser's asserts. They determined the physical conditions for models of astrophysical compact stars in the RG and GR frameworks. Once again, the authors emphasized that the RG formulation seems viable with the stability constraints while equivalence GR disturbs from the necessary stability constraints. These novel arguments hold confirmations to the findings against Visser's assert. Moreover, many robust insights of the stellar relativistic models have affirmed to put in favor of the physical consistency of RG [119–128].

This paper is arranged as follows. Section 2 corresponds to the formulation of field equations along with solution in the modified RT. Section 3 comprises some physical tests to assess the physical requirements necessary for a physically realistic solution and comparative analysis. We summarize the consequences of our findings in Sec. 4.

2. Field Equations in the Rastall Background and Their Solution

In the background of Rastall's method, we improve the customary conservation law for EMT as a non-conservation as follows:

$$\nabla_\mu T^{\mu\nu} = \lambda R^\mu, \tag{1}$$

where λ is known as the Rastall coupling parameter. Rastall coupling parameter λ measures the deviation from GR. The following modified field equations are suggested by this revised conservation:

$$G_{\mu\nu} + \kappa\lambda g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{2}$$

where κ stands for gravitational constant. Clearly, we can see that the formation of above equation gives the following equation:

$$R(4\lambda\kappa - 1) = \kappa T, \tag{3}$$

The above equation implies that $\lambda\kappa = \frac{1}{4}$ is not possible because T (trace of EMT) would not be zero. If we use the limit suggested by the Newton and interpret the dimensionless parameter proposed by Rastall as $\gamma = \lambda\kappa$, where λ and κ are given by

$$\kappa = \frac{8\pi(1-4\gamma)}{(1-6\gamma)}, \tag{4}$$

$$\lambda = \frac{\gamma(1-6\gamma)}{8\pi(1-4\gamma)}, \tag{5}$$

which gives Rastall's parameter value λ i.e. if we set λ equals to zero then we will get γ equals to zero. From Eq. (11) when $\gamma = \frac{1}{6}$, gravitational constant κ diverges. Also from Eq. (12) when $\gamma = \frac{1}{4}$, the original Rastall parameter λ diverges, thus γ cannot be equal to $\frac{1}{6}$ and $\frac{1}{4}$ for the solution. Thus, the field equations suggested by Rastall with variable cosmological constant finally reduce to

$$G_{\mu\nu} + \gamma g_{\mu\nu} R = 8\pi T_{\mu\nu}\left(\frac{4\gamma-1}{6\gamma-1}\right). \tag{6}$$

Now, for modeling of dark energy stars in RG, we consider the line element in standard Schwarzschild form to describe a static spherically symmetric space-time as

$$ds^2 = e^{\nu}dt^2 - e^{\mu}dr^2 - r^2(d\theta^2 + \sin^2\theta d\Phi^2), \tag{7}$$

where metric potentials e^{ν} and e^{μ} are the functions of "r" and are defined as

$$\nu(r) = Br^2 + 2\ln D, \quad \mu(r) = \ln(1 + ar^2 + br^4), \tag{8}$$

where a, b, B and D are constants which are to be determined on the physical conditions. Let us assume that the energy-momentum tensor is made up of dark energy, which has density ρ^{de}, radial pressure p_r^{de}, and tangential pressure p_t^{de}, as well as normal matter, which has mass-energy density ρ and pressure p. The dark energy density is calculated using the variable cosmological constant and is given by $\rho^{de} = \frac{\Lambda}{8\pi}$ [129]. Consequently, the components of the EMT of two fluids can be written as [129]

$$T_0^0 = \rho^{\text{eff}} = \rho + \rho^{\text{de}}, \tag{9}$$

$$T_1^1 = -p^{\text{eff}} = -(p + p_r^{\text{eff}}), \tag{10}$$

$$T_2^2 = T_3^3 = -p^{\text{eff}} = -(p + p_t^{\text{de}}), \tag{11}$$

$$T_0^1 = T_1^0 = 0. \tag{12}$$

Now, the Rastall field equations (formulated as in Eq. (6)) for line element (Eq. (7)), with the above EMT (with G=1=c) can be written as

$$8\pi(\rho+\rho^{\rm de})\left(\frac{4\gamma-1}{6\gamma-1}\right)=e^{-\mu}\left[\frac{e^{\mu}-1}{r^2}+\frac{\mu'}{r}+\gamma\left\{\nu''-\mu'\nu'+\nu'^2+\frac{2}{r}\left(\mu'-\frac{e^{\mu}}{r^2}+\frac{1}{r}-\nu'\right)\right\}\right], \quad (13)$$

$$8\pi(p+p_r^{\rm de})\left(\frac{4\gamma-1}{6\gamma-1}\right)=e^{-\mu}\left[\frac{1-e^{\mu}}{r^2}+\frac{\nu'}{r}-\gamma\left(\nu''-\mu'\nu'+\nu'^2+\frac{2}{r}\left(\mu'-\frac{e^{\mu}}{r^2}+\frac{1}{r}-\nu'\right)\right)\right], \quad (14)$$

$$8\pi(p+p_t^{\rm de})\left(\frac{4\gamma-1}{6\gamma-1}\right)=e^{-\mu}\left[(\nu'-\mu')\left(\frac{\mu'}{4}+\frac{1}{2r}\right)+\frac{\nu''}{2}-\gamma\left(\nu''-\mu'\nu'+\nu'^2+\frac{2}{r}\left(\mu'-\frac{e^{\mu}}{r^2}+\frac{1}{r}-\nu'\right)\right)\right]. \quad (15)$$

Now by using metric (8) in Eqs. (13)–(15), the field equations are given as

$$8\pi(\rho+\rho^{\rm de})=\frac{1}{(4\gamma-1)r^3(ar^2+br^4+1)^2}\Big[(6\gamma-1)(-2\gamma(r(a^2r^3+ar(2br^4-2B^2r^5+3Br^3-3r+2)+r^2(-2B^2(br^6+r^2)+5bBr^4+br(br^4-5r+2)+B)-1)+1)+r^5(a^2+5b)+2abr^7+3ar^3+b^2r^9)\Big], \quad (16)$$

$$8\pi(p+p_r^{\rm de})=\frac{1}{(4\gamma-1)(ar^2+br^4+1)}\left[(6\gamma-1)\left(-2\gamma\left(\frac{2(r^2(aB+2b)+a+2B)}{ar^2+br^4+1}-\frac{a}{r}-br+2B^2r^2-5B-\frac{1}{r^3}+\frac{1}{r^2}\right)-a-br^2+2B\right)\right], \quad (17)$$

$$8\pi(p+p_t^{\rm de})=\frac{1}{(4\gamma-1)(ar^2+br^4+1)}\left[(6\gamma-1)\left(-2\left(\frac{2(r^2(aB+2b)+a+2B)}{ar^2+br^4+1}-\frac{a}{r}-br+2B^2r^2-5B-\frac{1}{r^3}+\frac{1}{r^2}\right)+\frac{(2ar^2+3br^4+1)(r^2(aB-2b)-a+bBr^4+B)}{(ar^2+br^4+1)^2}+B\right)\right]. \quad (18)$$

In order to solve Eqs. (16)–(18), we consider that the radial pressure relating to dark energy is proportional to the density relating to dark energy. Furthermore, the density relating to dark energy is proportional to the density of baryonic matter, i.e.

$$p_r^{\mathrm{de}} = -\rho^{\mathrm{de}}, \tag{19}$$

$$\rho^{\mathrm{de}} = \alpha\rho. \tag{20}$$

The value of α (which is a nonzero constant) can be determined by using the boundary conditions. In order to obtain the matter density (ρ) and pressure (p) for the normal baryonic matter, we solve Eqs. (16)–(18) with the help of Eqs. (19) and (20) as

$$\begin{aligned}\rho = &-\frac{1}{8\pi(\alpha+1)(4\gamma-1)(ar^2+br^4+1)^2}\Big[(6\gamma-1)(-a^2r^2\\&+2\gamma(r^4(2a(b-B^2)+bB)+r^2((a+B)(a-2B)+5b)+3a\\&+br^6(b-2B^2)-3B)-2abr^4-3a-b^2r^6-5br^2)\Big],\end{aligned} \tag{21}$$

$$\begin{aligned}p = &\frac{1}{8\pi(\alpha+1)(4\gamma-1)(ar^2+br^4+1)^2}\Big[(6\gamma-1)(a^2(2\gamma-1)r^2\\&+a(2\alpha+2b(2\gamma-1)r^4-4B^2\gamma r^4+2Br^2(\alpha-\gamma+1)+6\gamma-1)\\&-4B^2\gamma r^2(br^4+1)+2B(\alpha+br^4(\alpha+\gamma+1)-3\gamma+1)\\&+br^2(4\alpha+b(2\gamma-1)r^4+10\gamma-1))\Big].\end{aligned} \tag{22}$$

The matter density (ρ^{de}) radial pressure (p_r^{de}) and transverse pressure (p_t^{de}) of the dark energy are calculated as

$$\begin{aligned}\rho^{\mathrm{de}} = &-\frac{1}{8\pi(\alpha+1)(4\gamma-1)(ar^2+br^4+1)^2}\Big[\alpha(6\gamma-1)(-a^2r^2\\&+2\gamma(r^4(2a(b-B^2)+bB)+r^2((a+B)(a-2B)+5b)+3a\\&+br^6(b-2B^2)-3B)-2abr^4-3a-b^2r^6-5br^2)\Big],\end{aligned} \tag{23}$$

$$\begin{aligned}p_r^{\mathrm{de}} = &\frac{1}{8\pi(\alpha+1)(4\gamma-1)(ar^2+br^4+1)^2}\Big[\alpha(6\gamma-1)(-a^2r^2\\&+2\gamma(r^4(2a(b-B^2)+bB)+r^2((a+B)(a-2B)+5b)+3a\\&+br^6(b-2B^2)-3B)-2abr^4-3a-b^2r^6-5br^2)\Big],\end{aligned} \tag{24}$$

$$p_t^{\rm de} = \frac{1}{8\pi(\alpha+1)(4\gamma-1)(ar^2+br^4+1)^2}\Big[(6\gamma-1)(a^2r^2(2\alpha\gamma+1) + a(6\alpha\gamma-3\alpha+2br^4(2\alpha\gamma+1)+B^2r^4(-4\alpha\gamma+\alpha+1)-Br^2(2\alpha\gamma+\alpha+1)) + b^2r^6(2\alpha\gamma+1)+br^2(10\alpha\gamma-6\alpha+B^2r^4(-4\alpha\gamma+\alpha+1) + 2Br^2(\alpha(\gamma-1)-1)-1)+B(Br^2(-4\alpha\gamma+\alpha+1)-6\alpha\gamma))\Big]. \tag{25}$$

For our current model, the effective density, as well as the effective radial and transverse pressures are calculated as

$$\rho^{\rm eff} = 8\pi(\rho+\rho_{\rm de}) = \frac{1}{(4\gamma-1)r^3(ar^2+br^4+1)^2}[(6\gamma-1)(-2\gamma(r(a^2r^3+ar \times(2br^4-2B^2r^5+3Br^3-3r+2)+r^2(-2B^2(br^6+r^2) + 5bBr^4+br(br^4-5r+2)+B)-1)+1) + r^5(a^2+5b)+2abr^7+3ar^3+b^2r^9)], \tag{26}$$

$$p_r^{\rm eff} = 8\pi(p+p_r^{\rm de}) = \frac{1}{(4\gamma-1)(ar^2+br^4+1)}\Bigg[(6\gamma-1)\Bigg(-2\gamma \times\left(\frac{2(r^2(aB+2b)+a+2B)}{ar^2+br^4+1}-\frac{a}{r}-br+2B^2r^2-5B - \frac{1}{r^3}+\frac{1}{r^2}\right)-a-br^2+2B\Bigg)\Bigg], \tag{27}$$

$$p_t^{\rm eff} = 8\pi(p+p_t^{\rm de}) = \frac{1}{(4\gamma-1)(ar^2+br^4+1)}\Bigg[(6\gamma-1)\Bigg(-2\gamma \times\left(\frac{2(r^2(aB+2b)+a+2B)}{ar^2+br^4+1}-\frac{a}{r}-br+2B^2r^2-5B - \frac{1}{r^3}+\frac{1}{r^2}\right)+\frac{(2ar^2+3br^4+1)(r^2(aB-2b) -a+bBr^4+B)}{(ar^2+br^4+1)^2}+B\Bigg)\Bigg]. \tag{28}$$

3. Matching Conditions

We calculate the values of four unknown constants a, b, B and D by matching our interior space-time with the exterior Schwarzschild line element at the boundary

$r = R$. The exterior space-time is given as

$$ds^2 = F(r)dt^2 - F(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{29}$$

where $F(r) = (1 - \frac{2M}{r})$, M is the mass of compact object. The metric coefficients g_{tt}, g_{rr} and $\frac{\partial}{\partial r}(g_{tt})$ all are continuous at the boundary $r = R$, yielding the following equations:

$$1 - \frac{2M}{R} = e^{\mathrm{BR}^2}D, \tag{30}$$

$$\left(1 - \frac{2M}{R}\right)^{-1} = 1 + aR^2 + bR^4, \tag{31}$$

$$\frac{2M}{R^2} = 2\mathrm{BR}e^{\mathrm{BR}^2}D^2. \tag{32}$$

By solving Eqs. (30)–(32), we can get

$$a = \frac{1}{R^2}\left[\left(1 - \frac{2M}{R}\right)^{-1} - 1 - bR^4\right], \tag{33}$$

$$B = \frac{M}{R^3}\left(1 - \frac{2M}{R}\right)^{-1}, \tag{34}$$

$$D = e^{-\frac{\mathrm{BR}^2}{2}}\sqrt{1 - \frac{2M}{R}}. \tag{35}$$

We can obtain the value of α by putting $p(r = R) = 0$ as

$$\begin{aligned}\alpha = {} & \frac{1}{2(aBR^2 + a + bBR^4 + 2bR^2 + B)}[-2a^2\gamma R^2 + a^2R^2 - 4ab\gamma R^4 + 2abR^4 \\ & + 4aB^2\gamma R^4 + 2aB\gamma R^2 - 2aBR^2 - 6a\gamma + a - 2b^2\gamma R^6 + b^2R^6 + 4bB^2\gamma R^6 \\ & - 2bB\gamma R^4 - 2bBR^4 - 10b\gamma R^2 + bR^2 + 4B^2\gamma R^2 + 6B\gamma - 2B].\end{aligned} \tag{36}$$

So, all of the constants in the metric potentials have been obtained.

4. Some Physical Aspects of the Model

In this part, we'll perform a detailed analysis of the stellar configuration by computing analytical results for a few key physical variables. In this case, we'll additionally offer a graphical representation of these physical variables so that we may observe their behavior more closely. We will compute the numerical values of the physical parameters and display them in tabular form to compare our results to the GR results and observational data.

4.1. *Nature of metric potential*

Both metric potentials have no singularities inside the radius of the star. Furthermore, $e^{\nu(0)} = D^2$ is a nonzero constant and $e^{-\mu(0)} = 1$ for our proposed stellar model. At the center of the star, the derivative of the metric potentials is zero. Within the star's interior, they are also positive and steady.

4.2. *Pressure and density*

The density and pressure profiles are shown in Fig. 1. They are all monotonic decreasing functions of radius "r", as seen in the Figs. 1 and 2. At the center of the star, the star's density (ρ) is maximum. At the star's boundary $r = R$, the star's pressure (p) disappears. Density, on the other hand, is positive at the boundary.

4.3. *Equation of state parameter*

The pressure density ratio which explains the concept of the EoS parameter denoted by w is calculated as

$$w = \frac{1}{2(\alpha+1)(r^2(aB+2b)+a+bBr^4+B)}[a^2r^2 - 2\gamma(r^4(2a(b-B^2)+bB) + r^2((a+B)(a-2B)+5b)+3a+br^6(b-2B^2)-3B) + 2abr^4+3a+b^2r^6+5br^2]-1 \quad (37)$$

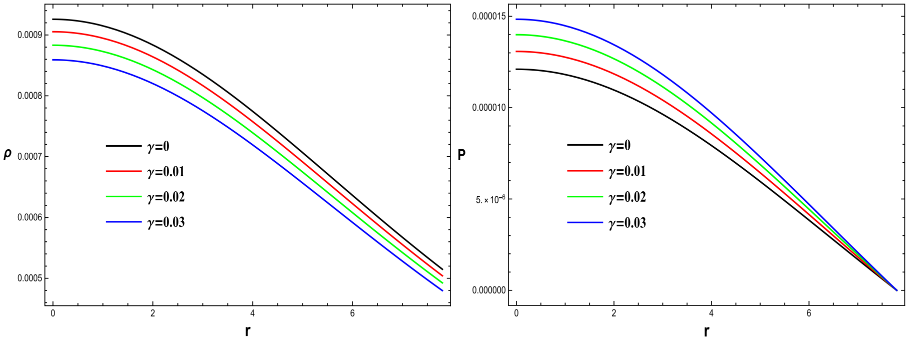

Fig. 1. Behavior of density and pressure for $4U1538-52$.

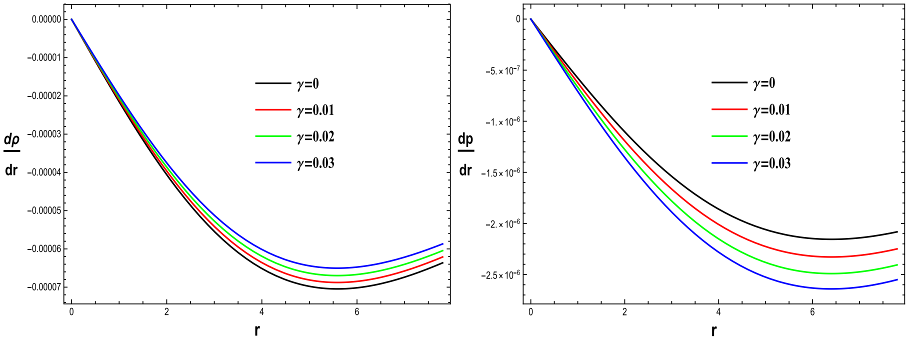

Fig. 2. Gradients of density and pressure(for ordinary matter) for $4U1538-52$.

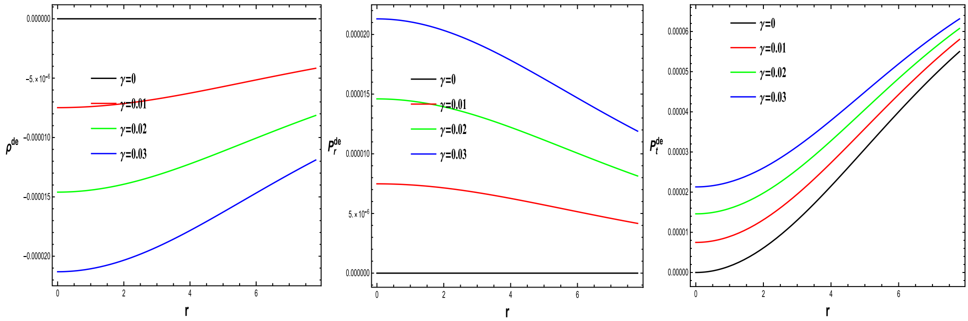

Fig. 3. Demeanor of density and pressure components of dark energy for $4U1538-52$.

W is an important component to consider while identifying the type of star formation.

4.4. *Pressure and density due to dark energy*

Dark energy density and radial pressure are monotonically decreasing as a function of "r", while transverse pressure is monotonically increasing. In our model, both density and transverse pressure are positive, but radial pressure negatively corresponds to dark energy (see Fig. 3).

4.5. *Maximality criteria*

In our model, the density and pressure gradient owing to normal baryonic matter are calculated as follows:

$$\begin{aligned}\frac{d\rho}{dr} &= \frac{1}{4\pi(\alpha+1)(4\gamma-1)(ar^2+br^4+1)^3}[(6\gamma-1)r(a^3(2\gamma-1)r^2 \\ &\quad + a^2(3b(2\gamma-1)r^4 - 2\gamma(Br^2-5)-5) + a(3b^2(2\gamma-1)r^6 \\ &\quad - br^2(\gamma(4B^2r^4+6Br^2-26)+13) + 2B\gamma(2Br^2-5)) + b^3(2\gamma-1)r^8 \\ &\quad - 4b^2r^4(\gamma(Br^2-3)(Br^2+2)+3) + b(5-2\gamma(14Br^2+5)) + 4B^2\gamma)], \qquad (38)\end{aligned}$$

$$\begin{aligned}\frac{dp}{dr} &= -\frac{1}{4\pi(\alpha+1)(4\gamma-1)(ar^2+br^4+1)^3}[(6\gamma-1)r(a^3(2\gamma-1)r^2 \\ &\quad + a^2(4\alpha+3b(2\gamma-1)r^4+2Br^2(\alpha-\gamma+1)+10\gamma-1) + a(-4B^2\gamma r^2(br^4-1) \\ &\quad + 2B(\alpha+3br^4(\alpha-\gamma+1)-5\gamma+1) + br^2(12\alpha+3b(2\gamma-1)r^4+26\gamma-1)) \\ &\quad + b^3(2\gamma-1)r^8 + 4b^2r^4(3(\alpha+2\gamma)+Br^2(\alpha-B\gamma r^2+\gamma+1)) \\ &\quad + b(-4\alpha+4Br^2(\alpha-7\gamma+1)-10\gamma+1)+4B^2\gamma)], \qquad (39)\end{aligned}$$

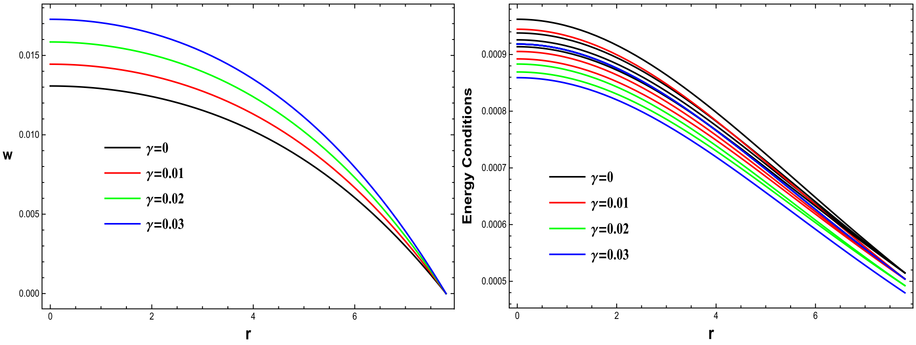

Fig. 4. Conduct of the equation of state parameter w for $4U1538-52$.

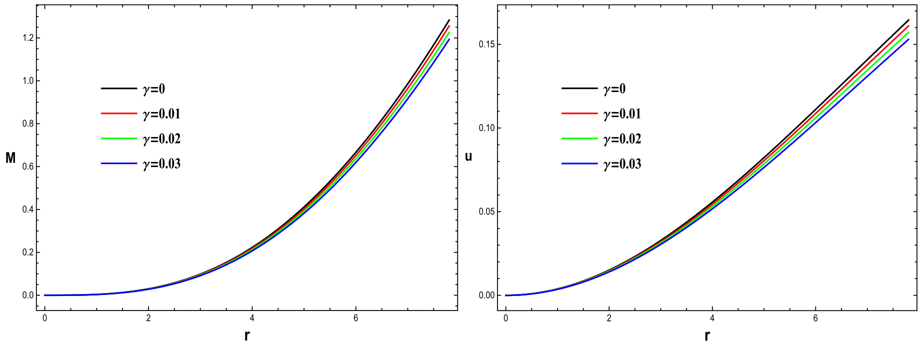

Fig. 5. Conduct of the mass function and compactness for $4U1538-52$.

5. Energy Condition

In this section, we shall discuss whether our present model satisfies the energy conditions. The four energy conditions are given by

$$\text{NEC}: \rho+p\geq 0, \quad \text{WEC}: \rho+p\geq 0, \quad \rho\geq 0, \tag{40}$$

$$\text{SEC}: \rho+p\geq 0, \quad \rho+3p+\frac{2\gamma}{4\gamma-1}(\rho-p)\geq 0 \quad \text{DEC}: \rho-p\geq 0, \quad \rho\geq 0. \tag{41}$$

Table 1. Numerical values of physical parameters for $SMCX-4$.

χ	α	Mass From our Model	ρ_0 gm/cm^3	ρ_s gm/cm^3	p_0 dyne/cm^2	Compactness u	Z_s Max Value
0	-3.033148^{-17}	1.290000	1.5223556^{15}	6.932844^{14}	1.870918^{34}	0.216222	0.327380
0.01	-0.006866	1.261385	1.486550^{15}	6.698632^{14}	$2.200013^{3}4$	0.211425	0.316302
0.02	-0.013731	1.230489	1.448120^{15}	6.454792^{14}	2.513699^{34}	0.206247	0.304649
0.03	-0.020597	1.197007	1.406719^{15}	6.2002469^{14}	2.808909^{34}	0.200635	0.292362

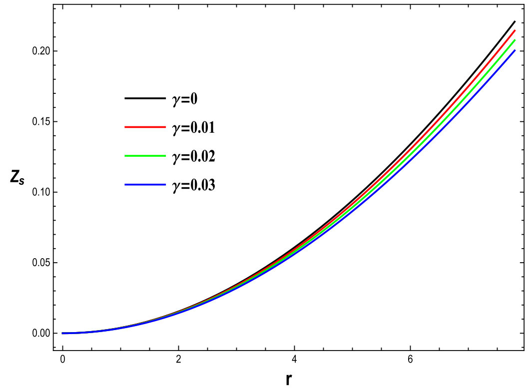

Fig. 6. Graphical trend of the surface red shift for $4U1538-52$.

Table 2. Numerical values of physical parameters for $HerX-1$.

χ	α	Mass From our Model	ρ_0 gm/cm^3	ρ_s gm/cm^3	p_0 dyne/cm^2	Compactness u	Z_s Max Value
0	-2.997724^{-17}	0.850000	8.746255^{14}	7.430858^{14}	1.745529^{34}	0.147500	0.190983
0.01	-0.008824	0.832078	8.559630^{14}	7.194009^{14}	1.748859^{34}	0.144390	0.185763
0.02	-0.017649	0.812623	8.357289^{14}	6.946054^{14}	1.747889^{34}	0.141014	0.180174
0.03	-0.026473	0.791425	8.137099^{14}	6.685747^{14}	1.741899^{34}	0.137336	0.174174

All these energy conditions are satisfied for our presented model as shown in Fig. 4

6. Mass Radius Relation and Surface Redshift

We check the compactness of our present model by a dimensionless parameter u. We want to observe the demeanor of mass function of our present model which can be obtained from the solution of the following differential equation with the condition $m(0) = 0$:

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2, \tag{42}$$

Mass function is calculated as

$$m = -\frac{m_1}{m_2}, \tag{43}$$

Table 3. Numerical values of physical parameters for Vela$X-1$.

χ	α	Mass From our Model	ρ_0 gm/cm^3	ρ_s gm/cm^3	p_0 dyne/cm^2	Compactness u	Z_s Max Value
0	-7.474638^{-17}	1.770000	2.105304^{15}	5.948803^{14}	2.180401^{34}	0.274816	0.490102
0.01	-0.004025	1.727900	2.049791^{15}	5.731441^{14}	3.183702^{34}	0.268279	0.468935
0.02	-0.008049	1.682795	1.990910^{15}	5.506883^{14}	4.142860^{34}	0.261276	0.447228
0.03	-0.012075	1.634282	1.928222^{15}	5.274297^{14}	5.050176^{34}	0.253744	0.424923

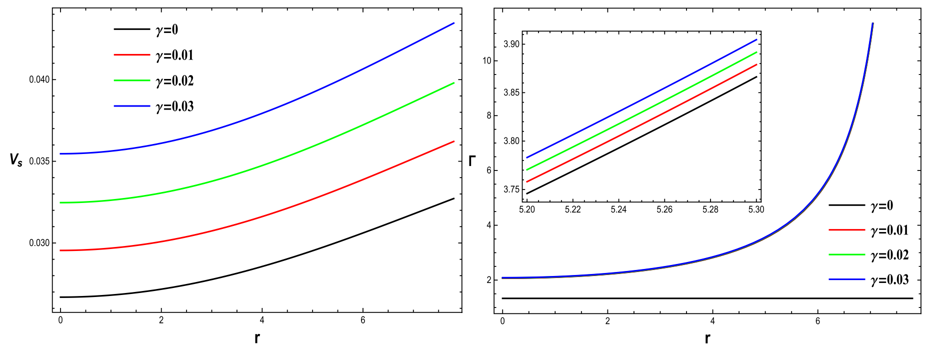

Fig. 7. Sound speed representation for $4U1538-52$.

$$
\begin{aligned}
m_1 = {} & (6\gamma-1)(\sqrt{b}r\sqrt{a^2-4b}\sqrt{a-\sqrt{a^2-4b}}\sqrt{\sqrt{a^2-4b}+a}(r^2(2\gamma(a(b-2B^2)\\
& + br^2(b-2B^2)-bB)-b(a+br^2))-4B^2\gamma)+2\sqrt{2}B^2\gamma(ar^2+br^4+1)\\
& \times\left((a(\sqrt{a^2-4b}+a)-2b)\sqrt{a-\sqrt{a^2-4b}}\tan^{-1}\left(\frac{\sqrt{2}\sqrt{b}r}{\sqrt{\sqrt{a^2-4b}+a}}\right)\right.\\
& \left.+\left(a(\sqrt{a^2-4b}-a)+2b)\sqrt{\sqrt{a^2-4b}+a}\tan^{-1}\left(\frac{\sqrt{2}\sqrt{b}r}{\sqrt{a-\sqrt{a^2-4b}}}\right)\right)\right),
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
m_2 = {} & 1.475(2(\alpha+1)b^{3/2}(4\gamma-1)\sqrt{a-\sqrt{a^2-4b}}\\
& \times\sqrt{\sqrt{a^2-4b}+a}\sqrt{a^2-4b}(ar^2+br^4+1)).
\end{aligned}
\tag{45}
$$

The compactness factor (U) is obtained as

$$u=\frac{m}{r}, \tag{46}$$

the graphical evolution of the mass function and compactness can be seen in Fig. 5.

The surface redshift (Z_s) for our model is obtained as

$$Z_s=\frac{1}{\sqrt{1-2u}}-1. \tag{47}$$

The graphical behaviour of surface redshift is shown in Fig. 6.

7. Equilibrium and Model Stability Through Different Tests

7.1. *Velocity of sound*

In this section, we calculate the velocity of sound by using the following relation:

$$V_s=\sqrt{\frac{dp}{d\rho}}=\sqrt{\frac{p'}{\rho'}}, \tag{48}$$

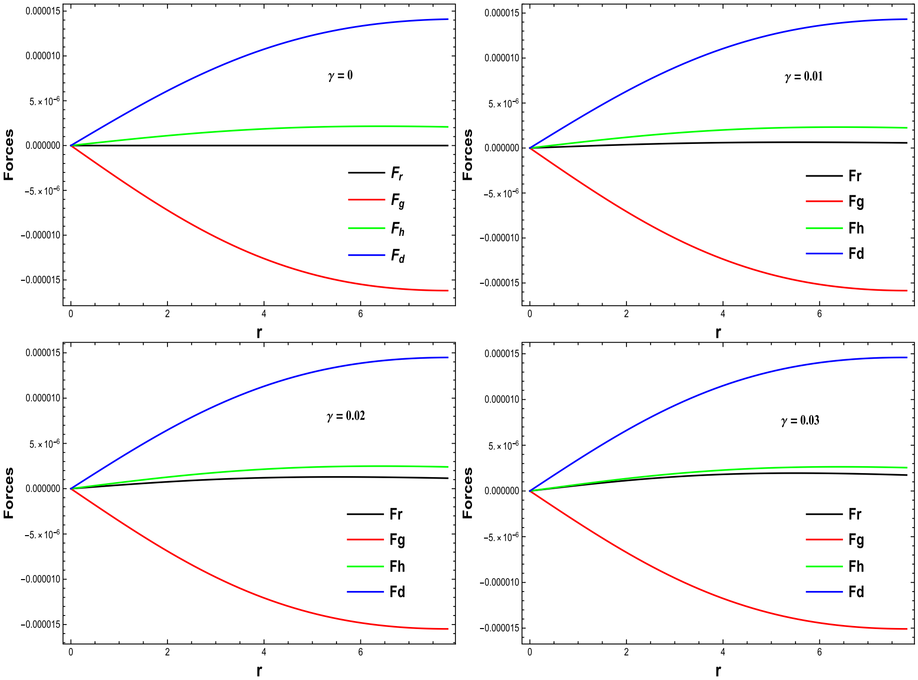

Fig. 8. Stability via TOV equation for different values of γ for $4U1538-52$.

We can obtain the square of sound velocity from our present model as follows:

$$V_s^2 = \frac{v_1}{v_2} - 1,$$

where

$$\begin{aligned} v_1 &= -2(\alpha+1)(a^2(Br^2+2) + a(3bBr^4 + 6br^2 + B) \\ &\quad + 2b(B(br^6 + r^2) + 3br^4 - 1)), \\ v_2 &= a^3(2\gamma-1)r^2 + a^2(3b(2\gamma-1)r^4 - 2B\gamma r^2 + 10\gamma - 5) + 2a\gamma(br^6(3b - 2B^2) \\ &\quad + r^2(13b + 2B^2) - 3bBr^4 - 5B) - abr^2(3br^4 + 13) + b^3(2\gamma-1)r^8 \\ &\quad + 4b^2r^4(\gamma(-B^2r^4 + Br^2 + 6) - 3) + b(5 - 2\gamma(14Br^2 + 5)) + 4B^2\gamma - 1. \end{aligned} \tag{49}$$

The graphical conduct of the squared sound speed is shown in Fig. 7, which remain between o and 1, which is an important condition for realistic stellar object.

7.2. *Relativistic adiabatic index*

The ratio of two specific heat represents the stiffness of the equation of state for a given density known as the adiabatic index Γ. This adiabatic index is used to

Table 4. Numerical values of physical parameters for $LMCX-4$.

χ	α	Mass From our Model	ρ_0 gm/cm^3	ρ_s gm/cm^3	p_0 dyne/cm^2	Compactness u	Z_s Max Value
0	-2.396785^{-16}	1.040000	1.315496^{15}	7.793286^{14}	1.664353^{34}	0.184819	0.259520
0.01	-0.007814	1.017450	1.285799^{15}	7.537199^{14}	1.835873^{34}	0.180811	0.251589
0.02	-0.015627	0.993044	1.253789^{15}	7.269877^{14}	1.998040^{34}	0.176475	0.243171
0.03	-0.023441	0.966529	1.219158^{15}	6.990061^{14}	2.148968^{34}	0.171763	0.234216

Table 5. Numerical values of physical parameters for $CenX-3$.

χ	α	Mass From our Model	ρ_0 gm/cm^3	ρ_s gm/cm^3	p_0 dyne/cm^2	Compactness u	Z_s Max Value
0	-9.139039^{-17}	1.490000	1.672773^{15}	6.343094^{14}	2.044864^{34}	0.238886	0.383789
0.01	-0.005988	1.456242	1.631984^{15}	6.123397^{14}	2.536763^{34}	0.233474	0.369666
0.02	-0.011977	1.419878	1.588370^{15}	5.895221^{14}	3.006565^{34}	0.227643	0.354928
0.03	-0.017965	1.380559	1.541558^{15}	5.657613^{14}	3.450015^{34}	0.221339	0.339514

Table 6. Numerical values of physical parameters for $4U1538-52$.

χ	α	Mass From our Model	ρ_0 gm/cm^3	ρ_s gm/cm^3	p_0 dyne/cm^2	Compactness u	Z_s Max Value
0	3.935359^{-17}	0.870000	1.249253^{15}	8.824449^{14}	1.469866^{34}	0.164519	0.220819
0.01	-0.008266	0.851323	1.221572^{15}	8.538371^{14}	1.588217^{34}	0.160987	0.214442
0.02	-0.016532	0.831087	1.191677^{15}	8.239356^{14}	1.699460^{34}	0.157161	0.207646
0.03	-0.024797	0.809080	1.159271^{15}	7.925959^{14}	1.802165^{34}	0.152999	0.200383

examine at the star structure's dynamical stability. Inside a dynamically stable fluid distribution, the adiabatic index for a stable Newtonian sphere should be greater than $\frac{4}{3}$. The adiabatic index is defined as

$$\Gamma = \left(\frac{\rho + p}{p}\right)\frac{dp}{d\rho}, \tag{50}$$

which remain in the stability range as shown in Fig. 7.

7.3. *TOV equation*

In this part, we check the equilibrium for our present model in the existence of dark energy. For this purpose, we use the Tolman–Oppenheimer–Volkoff (TOV) equation, which is defined as

$$\left(\frac{\gamma}{4\gamma - 1}\right)\frac{d}{dr}(\rho - 3p) - \frac{\nu'}{2}(\rho + p) - \frac{2}{r}(p_r^{de} - p_t^{de}) - \frac{d}{dr}(p_r^{\text{eff}}) = 0. \tag{51}$$

There are three forces namely, Rastall force (due to Rastall theory), gravitational force (F_g), hydrostatic force (F_h and force due to dark energy (F_d). In term of these forces, the above equation can be written as

$$F_r + F_g + F_h + F_d = 0, \tag{52}$$

It is noted that dark energy forces and hydrostatic forces are positive, while the gravitational force is negative. So, the effect of these forces vanishes and the equilibrium state of our model is obtained as shown in Fig. 8.

8. Discussion and Concluding Remarks

Rastall theory of gravity has grabbed the much interest of the research community during the last few decades due to its fascinating implications in the astrophysical and cosmological background.

In this paper, we have studied dark energy stars made up of dark matter and ordinary matter with the help of a specific metric potential (Tolman–Kuchowicz) in the Rastall theory of gravity. To solve the field equations, we have obtained the expressions for the constants involved in the field equation due to the specific metric potential by matching the interior space-time geometry with the corresponding exterior space-time geometry by applying some physical assumptions. The effects of the Rastall parameter are observed on the different physical parameters of the model, graphically and numerically. We observed that all the necessary physical requirements are satisfied. The salient features of the model are as follows.

The behavior of density is shown in Fig. 1, which indicates that density is maximum at the core and decreases steadily as radius increases. The gradient of density is zero at the core and negative in the remaining region of the star (see Fig. 2), which also indicate the maximality of density at the center. Thus, an important physical requirement for a plausible model has been achieved.

The physically acceptable model also demands that the pressure is maximum at the center and vanishes at the boundary. For our model, the traits of pressure are illustrated in Fig. 1 indicating that pressure is maximum at the core and gradually decreases along the radius and vanishes at the surface.

On the other hand, the density and pressure due to dark energy are shown in Figs. 3. The dark energy density remains negative throughout the region of the star which is favorable condition for dark energy star. The radial pressure due to dark energy is decreasing towards the surface whereas transverse pressure increases along the radius. The equation of state parameter w is plotted in Fig. 4, which indicates that w remains less than 1 (which is also a positive indicator for realistic model)and vanishes at the surface. For a physically reasonable solution, all the energy conditions listed in Eqs. 42 and 43 should be satisfied. Figure 4 indicates that all the energy conditions are positive throughout the stellar interior, i.e. energy conditions given in Eqs. 42 and 43 are satisfied.

The behavior of mass function is depicted in Fig. 5, which is zero at the core and increases gradually toward the surface. We also calculated the numerical values of mass of different representative compact stars for different values of the Rastall parameter and tabulated these values (Tables 2–6). We observed that our calculated values of mass function remain in the observational limit. The conduct of the compactness can be seen in Fig. 5 to observe the Buchdahl limt, which is $u = \frac{M}{R} < \frac{4}{9}$. Plot of u indicate that our model is also consonant with the Bhuchdahl limit. The value of the surface redshift is also in the specified limit. We checked the stability of the obtained model by different tests like velocity of sound, TOV equation and adiabatic index. In all the cases, we infer that our model fulfills all the stability criterion for different chosen values of the Rastall parameter.

In summary, we conclude that our proposed model of dark energy star is well behaved and compatible with the necessary physical requirements of realistic stellar model.

Acknowledgments

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