International Journal of Geometric Methods in Modern Physics
Vol. 21, No. 9 (2024) 2450161 (19 pages)

DOI: 10.1142/S0219887824501615

Structural properties of compact stars in extended Teleparallel gravity

Asifa Ashraf[*,**], Faisal Javed[†,††], Wen-Xiu Ma[*,‡,§,¶,‡‡]
and G. Mustafa[†,‖,§§]

[*]*School of Mathematical Sciences, Zhejiang Normal University*
Jinhua, Zhejiang 321004, P. R. China

[†]*Department of Physics, Zhejiang Normal University*
Jinhua, Zhejiang 321004, P. R. China

[‡]*Department of Mathematics, King Abdulaziz University*
Jeddah 21589, Saudi Arabia

[§]*Department of Mathematics and Statistics*
University of South Florida, Tampa, FL 33620-5700, USA

[¶]*School of Mathematical and Statistical Sciences*
North-West University, Mafikeng Campus, Private Bag X2046
Mmabatho 2735, South Africa

[‖]*New Uzbekistan University, Mustaqillik Avenue 54*
100007 Tashkent, Uzbekistan
[**]*asifamustafa3828@gmail.com*
[††]*faisaljaved.math@gmail.com*
[‡‡]*wma3@usf.edu*
[§§]*gmustafa3828@gmail.com*

Received 15 January 2024
Accepted 12 February 2024
Published 22 April 2024

In this study, the structures of stars are examined using the Karmarkar condition (KC) to assess the metric components. The study also takes into account the anisotropic source of the matter distribution in the context of Modified Teleparallel Rastall Gravity (MTRG). Various values of the model parameter η are tested by assuming different metric coefficients for the embedding spacetime. To calculate the involved model parameters, the Schwarzschild black hole solution is employed as an exterior solution. The stability and admissibility of the solutions obtained are demonstrated by analyzing some core properties of stellar objects, using observed data from established stellar models such as "PSRJ1416-2230". It is worth noting that the results obtained in this study are physically feasible and acceptable.

Keywords: Teleparallel gravity; Rastall gravity; stellar structures; modified gravity.

PACS Nos.: 04.50.Kd, 95.35.+d, 95.36.+x

[‡‡]Corresponding author.

1. Introduction

After Einstein proposed the original version of General Relativity (GR) [1], the Teleparallel Theory (TT) was introduced years later as a modified theory of gravity. However, the relationship between GR and TT was re-examined based on the hypotheses presented by [2–7]. According to GR, the gravitational effect of a gravitating source is best explained by curvature. It is generally believed that spacetime can have both torsion and curvature, similar to a Cartan space. Terms such as the Riemann tensor can be derived from the torsion of spacetime. Therefore, a theory that describes gravity based on the curvature of spacetime can be considered a theory that only involves torsion without the involvement of the Riemann tensor [8].

Several modifications to GR have been proposed in the light of the current expansion model and the existence of dark energy as an exotic form of energy. Merging theories of gravity that include the terms R^2, $R^{\alpha\beta}R_{\alpha\beta}$, and $R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$ in their Lagrangian action is applicable at low energy scales. Among these theories, the $f(R)$ gravity received widespread attention [9–16] and is in great accord with cosmological and astrophysical observational data [17]. However, the field equations (FEs) in $f(R)$ gravity are fourth-order differential equations, making them more complicated to solve than the second-order GR field equations [18]. Torsion-based theory, often known as $f(T)$ gravity, is a generalization of TT that eliminates curvature and the Riemann tensor caused by the torsion-free factor. Several other successful modified theories of gravity have been proposed in the literature, including $f(G)$ gravity [19], $f(R)$ gravity [20], $f(R,T)$ gravity [21–25], $f(T,B)$ gravity [26,27], and $f(T,\tau)$ gravity [28–32]. Interferometric detection of gravitational waves has been discussed in [33]. Recent years have seen a comprehensive and valuable body of research on compact stars [19–32]. This paper employs the framework of Modified Teleparallel Rastall Gravity (MTRG) [34] and the well-known Karmarkar condition (KC) [35]. By embedding spacetime into higher dimensions via the KC technique, we may identify one of spacetime's gravitational components while the other is known. Different modified theories of gravity have been discussed by Capozziello and his other coauthors [8,36–43] for different aspects of compact objects and cosmology.

Ruderman [44] has discussed the effects of anisotropy on stars, stating that at high densities such as 10^{15} g/cm^3, nuclear interactions become relativistic, resulting in anisotropic features. Following this, Bowers and Liang [45] examined the properties of static spherically symmetric distributions of relativistic anisotropic matter, which are generally dense. In recent years, several studies have explored the physics of anisotropic pressures. Dev and Gleiser [46,47] demonstrated through their research that pressure variations affect the mass, structure, and pressure regimes of objects. Additionally, a number of authors [48–54] have discussed analytical solutions in their recent works. The consequences of local anisotropy in self-gravitating systems, where anisotropy results from the relaxation of isotropic conditions, i.e. $p_r \neq p_t$, were initially identified by Herrera and Santos [55]. Crucially, the generic method described in [50,56,57] may be used to derive all feasible solutions for static isotropic, anisotropic, and charged anisotropic systems of Einstein's field equations based on the spherically symmetric spacetime.

During the last phases of regular star formation, compact stars provide an ideal platform for studying the arrangement of highly dense matter under maximum conditions. Pulsars and other revolving stars with strong magnetic fields, which are thought to be compact objects with great densities, are among the major discoveries made by astronomy [58]. It is thought that the substance of these objects is made up of subatomic particles like baryons, leptons, mesons, etc., including weird quark matter, although a thorough explanation is not yet available. Therefore, the key challenge in astrophysics is to identify the geometry and configuration of matter distribution within the internal structure of these compact stars. The literature [59, 61] contains a number of excellent studies on the examination of compact stars.

The paper is structured as follows: Section 2 provides an in-depth explanation of the basic structure of MTRG and its field equations. We introduce the generalized field equations utilizing embedding spacetime in Sec. 3. Section 4 examines the determination of unknown parameters using matching conditions. In Sec. 5, we finally analyze the outcomes of our solutions and make inferences about what conduct is necessary and appropriate. We conclude in Sec. 6.

2. MTRG: Basic Formulation

Understanding the structure of stars is necessary for studying them since spherically symmetric spacetime is used as a common paradigm. This isotropic spacetime can be represented theoretically by a number of models, including the Schwarzschild metric. The behavior and evolution of stars can be better understood by researchers by examining the spherically symmetric spacetimes, which is important for many areas of astronomy. This spacetime's expression form is as follows:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dt^2 + r^2\sin^2\theta d\phi^2 + d\theta^2. \tag{1}$$

It is feasible to depict the metric tensor $g_{\mu\nu}$ defined on a manifold using the tetrad fields e^i_μ and the Minkowski metric $\eta_{ij} = \mathrm{diag}(-1,1,1,1)$, which is defined as

$$g_{\mu\nu} = \eta_{ij}e^i_\mu e^j_\nu, \tag{2}$$

where $i,j = 0,1,2,3,\ldots$ indicate the tangent space indices, and $\mu,\nu = 0,1,2,3,\ldots$ indicate the spacetime indices. The theoretical concept's Weitzenböck relationship is described as follows:

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_i\partial_\nu e^i_\mu = -e^i_\mu\partial_\nu e^\alpha_i. \tag{3}$$

The Teleparallel Theory makes use of a particular kind of link mentioned above, which has zero curvature but nonzero torsion. The torsion tensor, which is defined by the connections, is represented by the following notation:

$$T^{\sigma}_{\mu\nu} \equiv \Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\mu\nu} = e^{\sigma}_{i}\left(\partial_{\mu}e^{i}_{\nu} - \partial_{\nu}e^{i}_{\mu}\right). \tag{4}$$

The following relation connects the Levi-Civita connection, $\bar{\Gamma}^{\sigma}_{\mu\nu}$, to the Weitzenböck connection:

$$\Gamma^{\sigma}_{\mu v} = \Gamma^{\sigma}_{\mu\nu} - K^{\sigma}_{\mu v}, \tag{5}$$

where $K_{\mu\nu}$ defines the contortion tensor, which is further defined as

$$K^{\sigma}_{\mu\nu} = \frac{1}{2}(T^{\sigma}_{\mu}\sigma_{\nu} + T^{\sigma}_{\nu}\sigma_{\mu} - T^{\sigma}_{\mu\nu}). \tag{6}$$

The torsion scalar is read as

$$T = S^{\sigma\mu\nu}T_{\sigma\mu\nu}; \tag{7}$$

further, the superpotential $S^{\sigma\mu\nu}$ is expressed as

$$S^{\sigma\mu\nu} = -S^{\sigma\nu\mu} = \frac{1}{2}(K^{\mu\nu\sigma} - g^{\sigma\nu}T^{\alpha\mu\mu} + g^{\sigma\mu}T^{a\nu}). \tag{8}$$

The action for the current extended framework is defined as

$$S = \int eL_{m}d^{4}x + \frac{1}{4\kappa}\int ef(T)d^{4}x. \tag{9}$$

The function $f(T)$ is based on the torsion scalar, and the determinant of the tetrad field ${e^{a}}_{F}$ is indicated as e. In addition, the matter Lagrangian is represented by L_{m}. By varying the action with respect to the tetrad field, we can obtain the matching FE,

$$S^{\mu\nu}_{i}f_{\mathrm{TT}}\partial_{\mu}T + e^{-1}\partial_{\mu}(eS^{\mu\nu}_{i})f_{T} - {T^{\alpha}}_{\mu i}S^{\nu\mu}_{\sigma}f_{T} - \frac{1}{4}{e_{i}}^{\nu}f = -\kappa\theta^{\nu}_{\mu}, \tag{10}$$

where Θ^{v}_{μ} refers to the typical energy–momentum tensor of the ideal fluid and it is demonstrated that

$$\left(S^{\mu v}_{i}f_{\mathrm{TT}}\partial_{\mu}T + e^{-1}\partial_{\mu}(e{S_{i}}^{\mu\nu})f_{T} - {T^{\sigma}}_{\mu i}S^{\nu\mu}_{\sigma}f_{T} - \frac{1}{4}e^{v}_{i}f\right)_{;v} = 0. \tag{11}$$

The covariant derivative is expressed as

$$V^{\mu}_{;\nu} = \partial_{\nu}V^{\mu} + (\Gamma^{\mu}_{\lambda\nu} - K^{\mu}_{\lambda\nu})V^{\lambda}, \tag{12}$$

for any vector V^{μ} in spacetime. The energy–momentum tensor's covariant derivative likewise vanishes, as

$$\Theta^{\nu}{}_{\mu;\nu} = 0. \tag{13}$$

Both Einstein's theory and our Modified Teleparallel Gravity Theory have the same energy–momentum conservation equation, which is $\nabla_{\mu}T^{\mu\nu} = 0$. Rastall [62], on the other hand, presented a novel equation that contradicts the Einstein theory's conservation equation,

$$T^{\mu}_{\nu\ \mu} = \lambda R_{,\nu}. \tag{14}$$

This equation leads to the modified FEs and involves an interesting matter–geometry relationship. We extend Rastall's idea by including the same presumption into our revised Teleparallel Gravity Theory. We create a relationship where the divergence of the energy–momentum tensor ${\Theta^{\nu}}_{\mu}$ is proportionate to the divergence of the torsion scalar by connecting matter and geometry via the scalar torsion of geometry,

$$ {\Theta^{\nu}}_{\mu;\nu} = \lambda h(T)_{,\mu}, \tag{15} $$

where $h(T)$ is an analytical function of torsion and λ is a real constant. Next, the following is a restatement of the field equation (11):

$$ S_u^{\mu\nu} f\gamma\tau\partial_\mu T + e^{-1}\partial_\mu(S_i^{\mu\nu})f_T - T_\mu^\sigma S_\sigma^{\nu\mu} f_T - \frac{1}{4}e_i^\nu f - \delta_\mu^\nu \kappa\lambda h(T) = -\kappa\Theta_\mu^\nu. \tag{16} $$

In this case, the gravitational constant in Rastall theory, κ, is written as

$$ \kappa = \frac{4\gamma - 1}{6\gamma - 1}\kappa_G, \tag{17} $$

where $\gamma = \lambda\kappa$ and κ_G denotes the Einstein coupling constant $\kappa_C = 4\pi G$. The energy–momentum tensor Θ_μ is assumed to describe a nonisotropic fluid given as

$$ \Theta_\mu^v = (p + p_t)u_\mu u^v - p_t b_\mu^v + (p_r - p_t)v_\mu v^v, \tag{18} $$

where the vector u_μ represents the four-velocities in the time-like direction, whereas v_μ is a unit space-like vector in the radial direction. The relationship between these vectors is established by $u_0u^0 = -v_1v^1 = 1$. e_μ^i represents the tetrad fields, which are a crucial component of the Teleparallel method in General Relativity. Using the off-diagonal tetrad matrix provided in [63–65], one can calculate the FEs; further, it is expressed as

$$ {e^a}_\mu = \begin{pmatrix} e^{\nu/2} & 0 & 0 & 0 \\ 0 & e^{\lambda/2}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & e^{\lambda/2}\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & e^{\lambda/2}\cos\theta & -r\sin\theta & 0 \end{pmatrix}, \tag{19} $$

where $g_{tt} = e^{\nu(r)}$ and $g_{rr} = e^{\lambda(r)}$. The determinant of the tetrad field can be defined as: $e = \det({e^a}_\mu) = r^2\sin\theta e^{(\nu+\lambda)/2}$. The torsion scalar from Eqs. (4), (7), and (8) is realized as

$$ T(r) = \frac{2e^{-\lambda}}{r^2}(e^{\frac{\lambda}{2}} - 1)(e^{\frac{\lambda}{2}} - 1 - r\nu'). \tag{20} $$

Now, by replacing Eqs. (18)–(20) into the field equation (1), one can get the following expressions:

$$ \kappa\rho(r) = \frac{e^{-\lambda/2}}{r}(1 - e^{-\lambda/2})f_T' - \left(\frac{T}{4} - \frac{1}{2r^2}\right)f_T + \frac{e^{-\lambda}}{2r^2}(r\lambda' - 1)f_T + \frac{f}{4} + \gamma h(T), \tag{21} $$

$$\kappa p_r(r) = \left[\frac{T}{4} - \frac{1}{2r^2} + \frac{e^{-\lambda}}{2r^2}(1 + r\nu')\right] f_T - \frac{f}{4} - \gamma h(T), \tag{22}$$

$$\kappa p_t(r) = \frac{e^{-\lambda}}{2}\left(\frac{\nu'}{2} + \frac{1}{r} - \frac{e^{\lambda/2}}{r}\right) f'_T - \frac{f}{4} - \gamma h(T)$$
$$+ f_T\left\{\frac{T}{4} + \frac{e^{-\lambda}}{2r}\left[\left(\frac{1}{2} + \frac{r\nu'}{4}\right)(\nu' - \lambda') + \frac{r\nu''}{2}\right]\right\}. \tag{23}$$

The Rastall term $\gamma h(T)$ and the coefficient κ have a significant impact on component behavior and magnitude, which may lead to modifications in the energy conditions. For compact objects, different assumptions about the accessible $f(T)$ functions must be considered in order to obtain the solutions.

3. Embedding Spacetime

To complete the system, we choose two unknowns $e^{\nu(r)}$ and $e^{\lambda(r)}$ using the well-known Karmarkar condition, with an embedding approach stated as

$$R_{1414}R_{2323} = R_{1212}R_{3434} + R_{1224}R_{1334}, \tag{24}$$

where $R_{2323} \neq 0$. The spacetime that satisfies this condition is known as the embedding spacetime, and it can be represented by the differential equation (25) derived by substituting the Ricci scalars into Eq. (24),

$$\frac{\lambda'(r)\nu'(r)}{1 - e^{\lambda(r)}} = -2(\nu''(r) + {\nu'}^2(r)) + {\nu'}^2(r) + \lambda'(r)\nu'(r), \tag{25}$$

where $e^{\lambda(r)} \neq 1$. On solving the above equation for $e^{\nu(r)}$, we get

$$e^{\nu(r)} = \left(A + B\int \sqrt{e^{\lambda(r)} - 1}\, dr\right)^2, \tag{26}$$

where A and B are constants of integration. By utilizing Eq. (26), we can determine $e^{\nu(r)}$, which represents the g_{tt} component of the spacetime, when we have the g_{rr} component. For simplicity, we assume that $e^{\lambda(r)}$ is already known as

$$e^{\lambda(r)} = 1 + ar^2 e^{\eta \sin^{-1}(br^2 + c)}, \tag{27}$$

where $a,\ b$, and c are real constants and $\eta > 0$. By using Eq. (27) to solve Eq. (26), we get $e^{v(r)}$ in the simplified form as

$$e^{\nu(r)} = \left(A + \frac{B(\eta\sqrt{1 - (br^2 + c)^2} + 2br^2 + 2c)\sqrt{ar^2 e^{\eta \sin^{-1}(br^2 + c)}}}{b(\eta^2 + 4)r}\right)^2. \tag{28}$$

Here, we make a compatible choice of the functions $f(T)$ and $h(T)$ as given below:

$$f(T) = \beta T^n, \tag{29}$$

$$h(T) = \psi \log(\phi T^{\chi}). \tag{30}$$

By replacing Eqs. (27)–(30) into Eqs. (21)–(23), we get the expressions for ρ, p_r, and p_t as

$$\begin{aligned}\rho &= \frac{(6\gamma-1)}{4\pi(4\gamma-1)r^2}\left[-\frac{\beta 2^{n-2}n(1-g_4(r))(g_5(r))^{n-1}}{g_1^2(r)}\right.\\&\quad+\frac{\beta 2^{n-2}n(2g_1(r)+g_2(r)-1)(g_5(r))^{n-1}}{g_1^2(r)}+\beta 2^{n-2}r^2(g_3(r))^n\\&\quad\left.-\frac{\beta 2^{n-2}(1-n)nr\left(1-\frac{1}{g_1(r)}\right)(g_5(r))^{n-2}}{g_1(r)}+\gamma r^2\psi\log(2^{\chi}\phi(g_5(r))^{\chi})\right],\end{aligned} \tag{31}$$

$$p_r = \frac{(6\gamma-1)\begin{pmatrix}-\frac{\beta 2^{n-1}n(g_1(r)+g_2(r)-1)(g_5(r))^{n-1}}{r^2(g_1^2(r))}-\beta 2^{n-2}(g_3(r))^n\\-\gamma\psi\log(2^{\chi}\phi(g_5(r))^{\chi}\end{pmatrix}}{4\pi(4\gamma-1)}, \tag{32}$$

$$\begin{aligned}p_t &= \frac{(6\gamma-1)}{4\pi(4\gamma-1)}\left[\frac{\beta g_7 2^{n-2}nr(g_5(r))^{n-1}}{r^2(g_1^2(r))}\right.\\&\quad-\beta 2^{n-2}\left(\frac{(g_1(r)-1)(g_1(r)-g_2(r)-1)}{r^2(g_1^2(r))}\right)^n\\&\quad\left.-\frac{\beta 2^{n-4}(1-n)n(-2g_1(r)+g_2(r)+2)(g_5(r))^{n-2}}{r(g_1^2(r))}-\gamma\psi\log(2^{\chi}\phi(g_5(r))^{\chi})\right],\end{aligned} \tag{33}$$

where

$$g_1(r) = \sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}+1},$$

$$g_2(r) = \frac{2abB(\eta^2+4)r^3e^{\eta\sin^{-1}(br^2+c)}}{Ab(\eta^2+4)\sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}}+aBr(2br^2+2c+\eta g_6(r))e^{\eta\sin^{-1}(br^2+c)}},$$

$$g_3(r) = \frac{(g_1(r)-1)\begin{pmatrix}-\frac{2abB(\eta^2+4)r^3e^{\eta\sin^{-1}(br^2+c)}}{\begin{matrix}Ab(\eta^2+4)\sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}}\\+aBr(2br^2+2c+\eta g_6(r))e^{\eta\sin^{-1}(br^2+c)}\end{matrix}}+g_1(r)-1\end{pmatrix}}{r^2(g_1^2(r))},$$

$$g_4(r) = \frac{2are^{\eta\sin^{-1}(br^2+c)}+\frac{2ab\eta r^3e^{\eta\sin^{-1}(br^2+c)}}{\sqrt{1-(br^2+c)^2}}}{g_1^2(r)},$$

$$g_5(r) = \frac{(g_1(r)-1)(g_1(r)-g_2(r)-1)}{r^2(g_1^2(r))},$$

$$g_6(r) = \sqrt{-b^2r^4-2bcr^2-c^2+1},$$

$$g_7(r) = \frac{1}{4}r\left[\frac{1}{g_8(r)}[4bB(\eta^2+4)(ar^2e^{\eta\sin^{-1}(br^2+c)})^{3/2}[B(b^2\eta r^4 - 2bg_6(r)r^2 - c^2\eta + 2cg_6(r) + \eta)\sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}} + Ab(\eta^2+4)r(b\eta r^2 + g_6(r))]] + (g_2(r)+2)(g_2(r)-g_4(r))\right] + (g_1(r)-1)(g_1(r)-g_2(r)-1),$$

$$g_8(r) = r\sqrt{1-(br^2+c)^2}(Ab(\eta^2+4)\sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}} + aBr(2br^2+2c+\eta g_6(r))e^{\eta\sin^{-1}(br^2+c)})^2.$$

4. Evaluations of Unknowns by Comparison of Interior and Exterior Spacetimes

To properly characterize the model and physical properties such as mass and radius of anisotropic fluid spheres, it's crucial to ensure a seamless alignment between the interior spacetime $\mathcal{M}^{\text{int}}$ and the exterior geometry $\mathcal{M}^{\text{ext}}$. However, achieving this in modified theories of gravity can be quite challenging. When studying compact stars in GR, the exterior geometry can vary depending on the case — it could be the Schwarzschild vacuum solution, the Reissner–Nordström model, the Kerr–Newman model, or others. For this research, we are examining anisotropic compact objects without charge. To achieve this alignment, we've matched the inner geometry described in Eq. (1) with the Schwarzschild outer spacetime,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{34}$$

Matching (1) with Eq. (34) at the boundary, i.e. $r = R$, we get the system of equations as

$$\left(\frac{B(\eta\sqrt{1-(br^2+c)^2} + 2br^2 + 2c)\sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}}}{b(\eta^2+4)r} + A\right)^2 = 1 - \frac{2M}{R}, \tag{35}$$

$$ar^2e^{\eta\sin^{-1}(br^2+c)} + 1 = \frac{1}{1-\frac{2M}{R}}, \tag{36}$$

$$2\left[\frac{B\left(4br - \frac{2b\eta r(br^2+c)}{\sqrt{1-(br^2+c)^2}}\right)\sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}}}{b(\eta^2+4)r} - \frac{B(\eta\sqrt{1-(br^2+c)^2} + 2br^2 + 2c)}{b(\eta^2+4)r^2}\sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}}\right.$$

Table 1. Values of constants by taking the mass and radius of PSRJ1416-2230 ($M_0 = 1.97$ and $R = 9949$) by fixing $n = 1,\ \beta = 5,\ \xi = 0.3,\ \gamma = 1,\ \phi = -2,\ \chi = 5,\ \psi = -2 \times 10^{-8},\ b = 0.1,$ and $c = 0.1$.

η	a	A	B	$\frac{p_{rc}}{\rho_c}$ $(r = 0)$
1.5	0.00570024	0.0801307	0.0316263	<1
1.6	0.00570024	0.0801307	0.0316263	<1
1.7	0.0055872	0.0705327	0.0316263	<1
1.8	0.00553153	0.0693179	0.0316263	<1
1.9	0.0054764	0.0700784	0.0316263	<1
2.0	0.0725344	0.00542183	0.0316263	<1
2.1	0.0053678	0.0764324	0.0316263	<1
2.2	0.00531431	0.0815453	0.0316263	<1
2.3	0.00526136	0.0876715	0.0316263	<1

$$+\frac{B(\eta\sqrt{1-(br^2+c)^2}+2br^2+2c)}{2b(\eta^2+4)r\sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}}}$$
$$\left.\times\left(2are^{\eta\sin^{-1}(br^2+c)}+\frac{2ab\eta r^3e^{\eta\sin^{-1}(br^2+c)}}{\sqrt{1-(br^2+c)^2}}\right)\right]$$
$$\times\left[\frac{B(\eta\sqrt{1-(br^2+c)^2}+2br^2+2c)}{b(\eta^2+4)r}\sqrt{ar^2e^{\eta\sin^{-1}(br^2+c)}}+A\right]=\frac{2M}{R^2}. \tag{37}$$

By solving the above system of equations (35)–(37), we get the following constants:

$$a=-\frac{2Me^{-\eta\sin^{-1}(bR^2+c)}}{R^2(2M-R)}, \tag{38}$$

$$A=\sqrt{1-\frac{2M}{R}}-\frac{\sqrt{2}B\sqrt{\frac{M}{R-2M}}(\eta\sqrt{-(bR^2+c-1)(bR^2+c+1)}+2bR^2+2c)}{b\,(\eta^2+4)\,R}, \tag{39}$$

$$B=\frac{M}{R^2\sqrt{2-\frac{4M}{R}}\sqrt{-\frac{M}{2M-R}}}. \tag{40}$$

We enlist the numerical values of these constants in Table 1.

5. Discussion About the Physical Features

In this paper, we are interested in exploring the essential physical properties of stellar objects. We present a summary of our concluding remarks and highlights as follows:

- In order for spacetime of gravitation to function properly, both gravitational components must exhibit consistent and seamless behavior, such that $e^{a(r)} > 0$

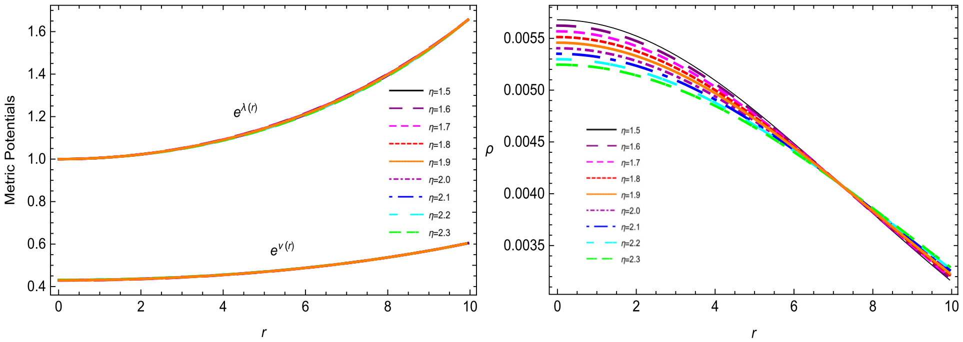

Fig. 1. (Left) Metric potential [$e^{\nu(r)}$, g_{tt} component, in increasing curves, and $e^{\lambda(r)}$, inverse of g_{rr} component, in decreasing curves] and (right) energy density $\rho\,(\mathrm{km}^{-2})$ versus the radial coordinate $r\,(\mathrm{km})$. Information about constant parameters is given in Table 1.

and $e^{b(r)} > 0$. This investigation demonstrates that the metric functions satisfy the required behavior for the stable configuration and their graphical behaviors can be checked from Fig. 1.

- For the stable and consistent stellar configuration, the energy profile is a crucial attribute. It attains its maximum values at the center, i.e. $r \to 0$, and then it declines steadily and smoothly toward the boundary, i.e. $r \to R$, and remains positive throughout the stellar configuration. The second graph of Fig. 1 provides the graphical interpretation of the energy density under the current scenario.
- Both the pressure components p_r and p_t are necessary factors determining the physical existence of compact stars. Both the profiles have maximum values at the center of the stars and gradually decrease toward the boundary, eventually reaching a minimum value. It is noteworthy that p_t is greater than p_r, at the boundary. Figure 2 demonstrates that our calculated solutions have good agreement with the necessary requirements for pressure profiles.

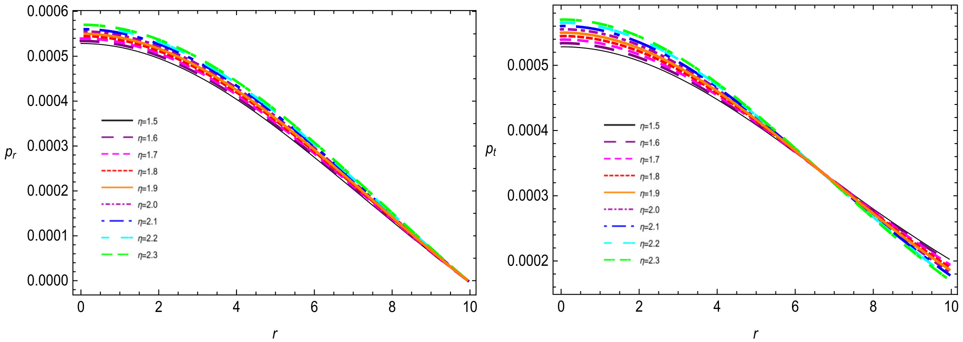

Fig. 2. (Left) Radial pressure $p_r\,(\mathrm{km}^{-2})$ and (right) tangential pressure $p_t\,(\mathrm{km}^{-2})$ versus the radial coordinate $r\,(\mathrm{km})$. Information about constant parameters is given in Table 1.

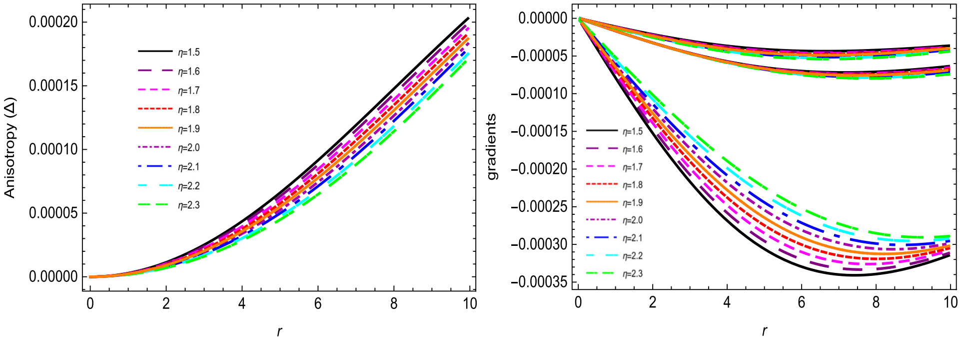

Fig. 3. (Left) Anisotropy $\Delta\,(\mathrm{km}^{-2})$ and (right) gradients versus the radial coordinate $r\,(\mathrm{km})$. Information about constant parameters is given in Table 1.

- Anisotropy is one of the most important features that indicate the stability of a stellar configuration. The positive nature of anisotropy confirms the important condition like $p_t > p_r$. Furthermore, as we move toward the center of the stellar body where p_t and p_r are equal, Δ approaches zero, resulting in the disappearance of anisotropy. The first graph in Fig. 3 confirms the smooth and steady behavior of anisotropy.
- In order to check the maximality of the involved physical quantities like energy density and pressure components at $r = 0$, gradients of these described physical quantities should be negative, i.e. $\frac{\rho}{dr}|_{r=0} = \frac{p_r}{dr}|_{r=0} = \frac{p_t}{dr}|_{r=0} = 0$. It is interesting to notice that the calculated gradients reveal a negative behavior, which can be confirmed in Fig. 3.
- The equation of state (EOS), which includes both radial and tangential components, is a significant factor in checking the nature of the matter within the stellar configurations. In the case of anisotropic matter, the EOS lies between the values of 0 and 1, i.e. $0 < w_r, w_t < 1$. Both the EOS parameters are defined mathematically as

$$w_r = \frac{p_r}{\rho}, \quad w_t = \frac{p_t}{\rho}, \tag{41}$$

where p_r and p_t represent the radial and tangential pressures, respectively, and ρ is the density of the matter. Figure 4 illustrates the graphical behaviors of both EOS parameters. It is noticed that both EOS parameters are lying within the described range.
- To study compact bodies and their distribution of matter, one of the key properties is energy conditions that must be satisfied throughout the configuration. Some important energy conditions, known as strong energy conditions (SEC), null energy conditions (NEC), weak energy conditions (WEC), and dominant energy conditions (DEC), are defined as

$$\mathrm{SEC}: \rho + p_\gamma \geq 0, \quad \rho + p_r + 2p_t \geq 0, \tag{42}$$

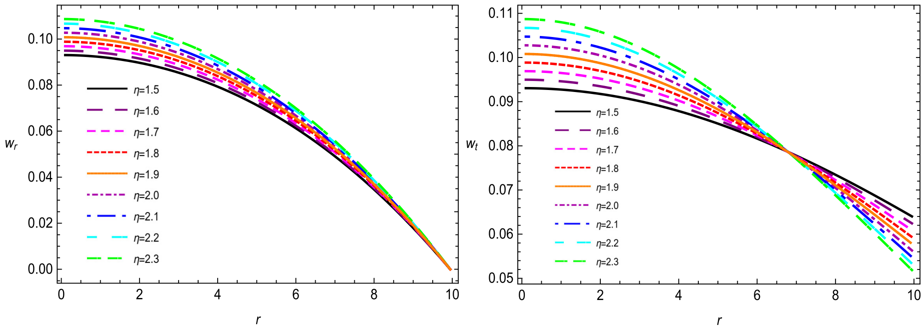

Fig. 4. (Left) EOS radial component and (right) tangential component versus the radial coordinate r (km). Information about constant parameters is given in Table 1.

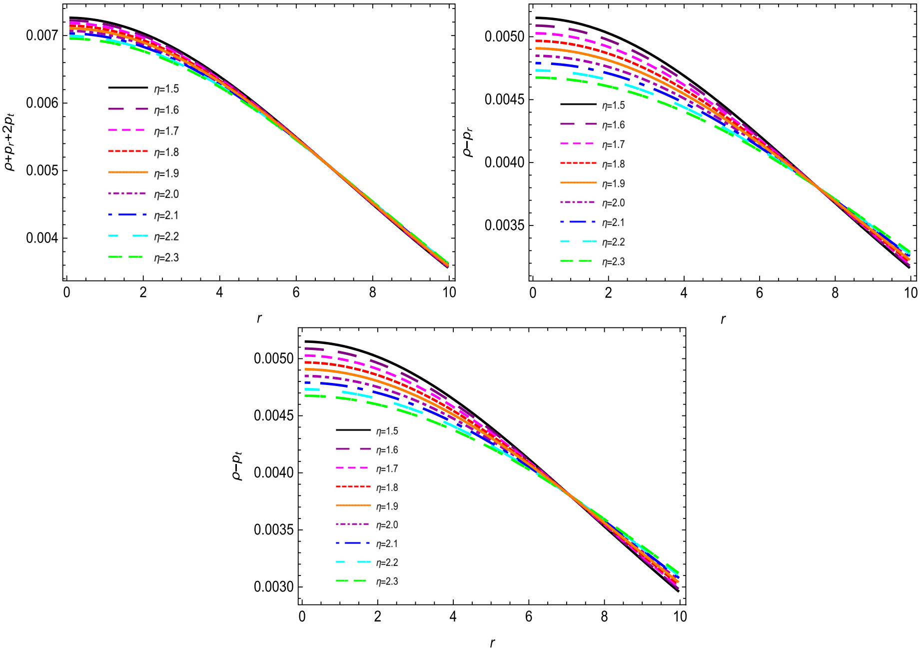

Fig. 5. Energy conditions (km^{-2}) versus the radial coordinate r (km). Information about constant parameters is given in Table 1.

$$\text{WEC}: \rho \geq 0, \quad \rho + p_\gamma \geq 0, \tag{43}$$

$$\text{NEC}: \rho + p_\gamma \geq 0, \tag{44}$$

$$\text{DEC}: \rho > |p_\gamma|. \tag{45}$$

Figure 5 provides the validity region of energy conditions. The positive behavior of energy conditions confirms the presence of ordinary matter, which supports for a stable and consistent stellar configuration.

- In order to check the stability of the described solutions, we use the Tolman–Oppenheimer–Volkoff (TOV) equation [66,67]. The TOV equation for MTRG is expressed as

$$p_r' + \frac{\nu'(\rho + p_r)}{2} - \frac{2(p_t - p_r)}{r} - \frac{\gamma}{4\gamma - 1}(\rho' - p_r' - p_t') = 0, \tag{46}$$

$$F_g + F_h + F_a + F_r = 0, \tag{47}$$

where

$$F_g = -\frac{\nu'(\rho + p_r)}{2}, \quad F_h = -p_r', \quad F_a = \frac{2(p_t - p_r)}{r}, \quad F_r = \frac{\gamma}{4\gamma - 1}(\rho' - p_r' - p_t'). \tag{48}$$

Another crucial property that demonstrates the stability of compact objects in a spherically symmetric spacetime is the adiabatic index. The study of adiabatic index becomes an essential ingredient in understanding the stability of these objects. According to the literature [45,68,69], the adiabatic index must have a limit of $\Gamma > \frac{4}{3}$ to achieve stability within the interior of a compact star. The mathematical expression for adiabatic index Γ is given as

$$\Gamma = \frac{p_r + \rho}{p_r} v_r^2. \tag{49}$$

In this study, we present a graphical representation of the adiabatic index in the second graph of Fig. 6. We can easily verify that our stellar system obeys the stability range of the adiabatic index.

- Another criterion for demonstrating the stability of the system is the sound speed parameters, i.e. v_{sr}^2 and v_{st}^2. Both the speed of sound parameters are defined as

$$v_r^2 = \frac{dp_r}{d\rho}, \quad v_t^2 = \frac{dp_t}{d\rho}. \tag{50}$$

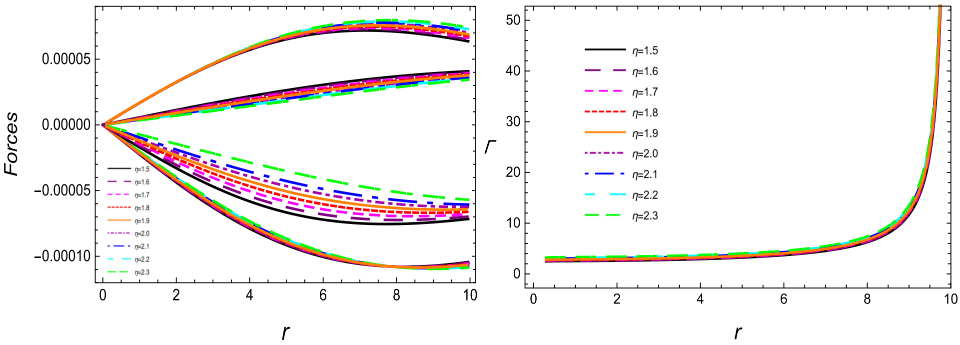

Fig. 6. (Left) TOV forces (km^{-2}) and (right) adiabatic index versus the radial coordinate r (km). Information about constant parameters is given in Table 1.

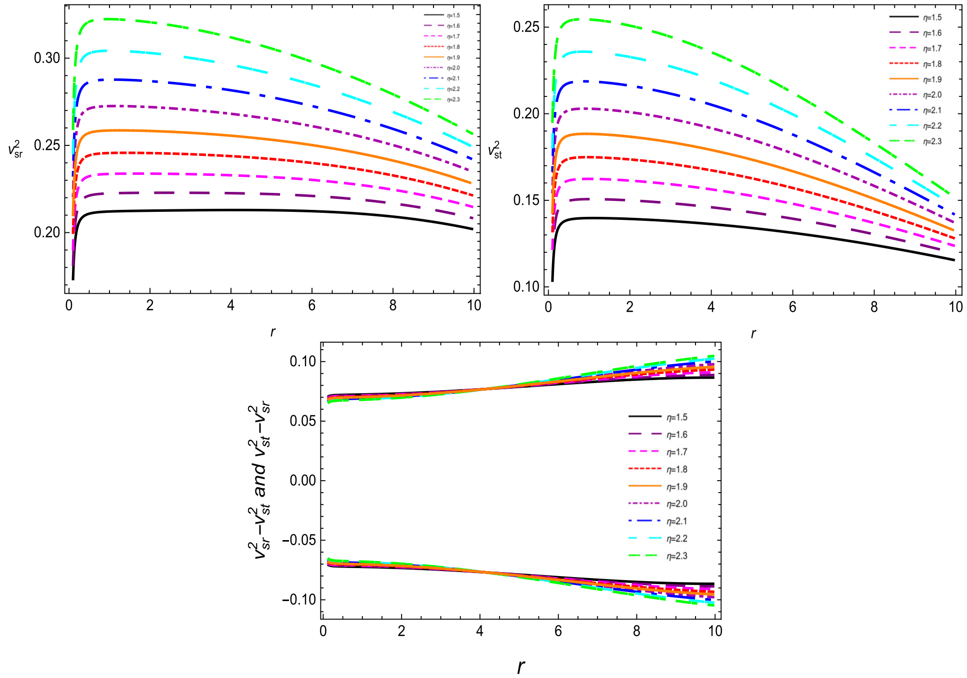

Fig. 7. Sound speeds and Abreu limit versus the radial coordinate r (km). Information about constant parameters is given in Table 1.

Abreu *et al.* [70] recommended a condition for the stable region, i.e. $v_r^2 > v_t^2$. Figure 7 in this paper confirms that our results fall within the stable region, ensuring the stability of our solutions for the study of compact stars.

- The compactification level of the stellar system is illustrated by the ratio $\frac{m(R)}{R}$, where the mass function $m(r)$ is given by

$$m(r) = \frac{r}{2}(1 - e^{-b(r)}). \tag{51}$$

The compactness u defined in Eq. (52) is contributed by Eq. (51), and is used to elaborate the redshift function z_s as

$$u = \frac{m(R)}{R}, \tag{52}$$

$$z_s = e^{-\frac{a(r)}{2}} - 1. \tag{53}$$

The maximum limit for compactification was set by the author as $u(r) = \frac{m(r)}{R} < \frac{4}{9}$. This limit was further generalized for anisotropic cases by [70]. Buchdahl [71] suggested a peak value for the redshift parameter, given by $z_s \leq 4.77$. The mass function in the first graph of Fig. 8 shows a smooth and regular trend that perfectly matches the original masses of the stars studied. The second and third

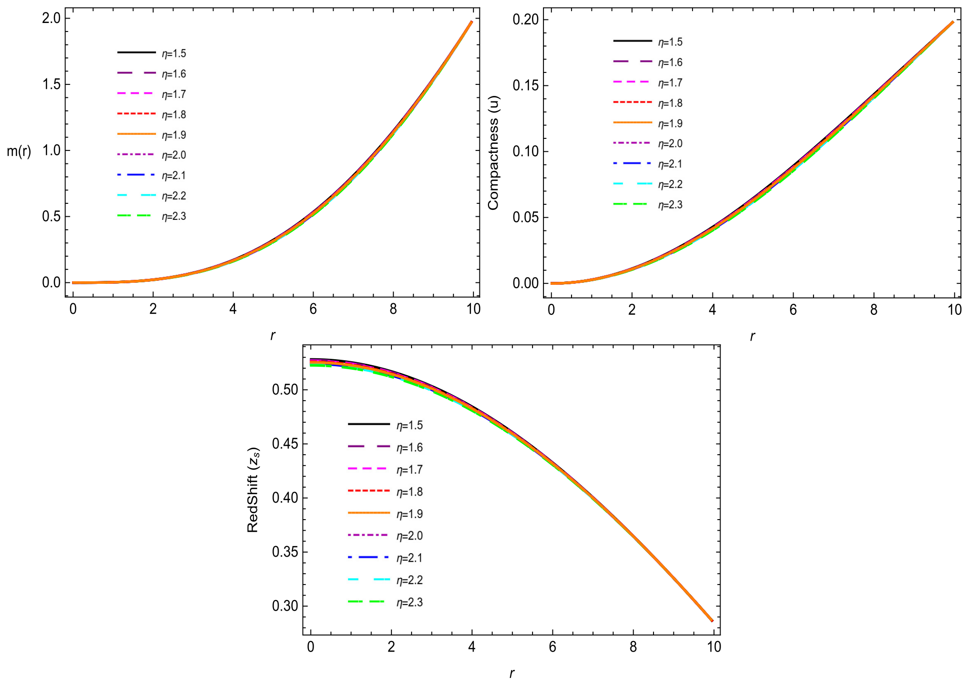

Fig. 8. Mass function, compactification, and redshift versus the radial coordinate r (km). Information about constant parameters is given in Table 1.

graphs in Fig. 8 clearly demonstrate that the plotted values of compactness and redshift parameters are regular and fully adhere to the criteria defined above.

6. Conclusion

In this study, we used an embedding approach to investigate the physical properties of spherically symmetric compact stars in MTRG. To summarize our findings, the stellar system, subject to the effects of MTRG, is physically possible since it adheres to the correct gravitational constituents of spacetime, namely $e^{a(r)}$ and $e^{b(r)}$, as indicated in the first graph of Fig. 1. The second graph of Fig. 1 shows the correct and smooth energy density and pressure profiles, which prove the physical existence of stellar bodies. Figure 2 illustrates the physical behavior of pressure profiles. The system's anisotropic character, equilibrium, and stability are proved by the counteracting action of anisotropic repulsive forces and attractive gradient behavior, as illustrated in Fig. 3. Energy conditions ensure a realistic distribution of matter, as shown in Fig. 4. Our solutions contain actual anisotropic matter, as shown by the EOS in Fig. 5. The TOV equation, visually depicted in Fig. 6, confirms the system's equilibrium state. Furthermore, our stellar system's stability is corroborated by the exact behavior of the adiabatic index, as demonstrated in Fig. 6. Our system's stability is proved by the smooth and correct behavior of sound speeds within

Abreu and Andreasson's stated range, as illustrated in Fig. 7. Finally, we provide the accurate and smooth presentation of the mass function, redshift parameter, and compactification, as shown in Fig. 8. In conclusion, our computed results are physically sound and fully support the study of stellar models, which may have implications for future research.

Acknowledgments

The work was supported in part by NSFC under the Grants 12271488, 11975145, and 11972291, the Ministry of Science and Technology of China (G2021016032L), and the Natural Science Foundation for Colleges and Universities in Jiangsu Province (17 KJB 110020).

Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability Statement

This paper has no associated data, or the data will not be deposited. (There is no observational data related to this paper. The necessary calculations and graphic discussion can be made available on request.)

ORCID

Asifa Ashraf https://orcid.org/0000-0001-5522-5491
Faisal Javed https://orcid.org/0000-0001-6970-1305
Wen-Xiu Ma https://orcid.org/0000-0001-5309-1493
G. Mustafa https://orcid.org/0000-0003-1409-2009

References

[1] A. Unzicker and T. Case, Translation of Einstein's attempt of a unified field theory with teleparallelism, arXiv: physics/0503046v1 [physics.hist-ph].
[2] J. W. Maluf, Hamiltonian formulation of the teleparallel description of general relativity, *J. Math. Phys.* **35** (1994) 335–343.
[3] C. Moller, On the crisis in the theory of gravitation and a possible solution, *Mat.-Fys. Skr. K. Dan. Vidensk. Selsk.* **39**(13) (1978) 1–31.
[4] C. Pellegrini and J. Plebanski, Tetrad fields and gravitational fields, *Mat.-Fys. Skr. K. Dan. Vidensk. Selsk.* **2**(4) (1963) 3–39.
[5] K. Hayashi and T. Nakano, Extended translation invariance and associated gauge fields, *Prog. Theor. Phys.* **38** (1967) 491–507.
[6] K. Hayashi, Gravitational interactions of the proton and the electron: the possible existence of a massless scalar particle, *Il Nuovo Cimento A* **16** (1973) 639–673.

[7] K. Hayashi, The gauge theory of the translation group and underlying geometry, *Phys. Lett. B* **69** (1977) 441–444.

[8] S. Capozziello and V. Faraoni, *Beyond Einstein Gravity*: *A Survey of Gravitational Theories for Cosmology and Astrophysics*, Fundamental Theories of Physics, Vol. 170 (Springer, New York, 2011).

[9] M. Sharif and M. F. Shamir, Exact solutions of Bianchi-type I and V spacetimes in the $f(R)$ theory of gravity, *Class. Quantum Grav.* **26** (2009) 235020.

[10] S. K. Maurya, Anisotropic stars with a modified polytropic equation of state, *Eur. Phys. J. C* **79** (2019) 958–972.

[11] S. K. Maurya, Charged conformal stars and equation of state, F. Tello-Ortiz and M. K. Jasim, *Eur. Phys. J. C* **80** (2020) 918–934.

[12] S. K. Maurya, A. Errehymy, K. N. Singh, F. Tello-Ortiz and M. Daoud, Self-gravitating electrically charged anisotropic strange star model, *Phys. Dark Univ.* **30** (2020) 100640–100654.

[13] S. K. Maurya and F. Tello-Ortiz, Anisotropic fluid spheres in the framework of $f(R,T)$ gravity theory, *Phys. Dark Univ.* **29** (2020) 100577–100595.

[14] S. K. Maurya, F. Tello-Ortiz and M. Govender, Complexity-free anisotropic solution of Buchdahl's model and energy exchange between relativistic fluids by extended gravitational decoupling, *Fortschr. Phys.* **69** (2021) 2100099–2100113.

[15] S. K. Maurya, A. S. Al Kindi, M. R. Al Hatmi and R. Nag, Gravitationally decoupled charged anisotropic solutions in Rastall gravity, *Results Phys.* **29** (2021) 104674–104684.

[16] S. K. Maurya, F. Tello-Ortiz and S. Ray, *Phys. Dark Univ.* **31** (2021) 100753–100766.

[17] S. Nojiri and S. D. Odintsov, Unified cosmic history in modified gravity: from $f(R)$ theory to Lorentz non-invariant models, *Phys. Rep.* **505** (2011) 59–144.

[18] S. Nojiri and S. D. Odintsov, Introduction to modified gravity and gravitational alternative for dark energy, *Int. J. Geom. Methods Mod. Phys.* **4** (2007) 115–145.

[19] M. F. Shamir and T. Naz, Stellar structures in $f(G)$ gravity with Tolman–Kuchowicz spacetime, *Phys. Dark Univ.* **27** (2020) 100472.

[20] M. Zubair and G. Abbas, Some interior models of compact stars in $f(R)$ gravity, *Astrophys. Space Sci.* **361** (2016) 342.

[21] S. K. Maurya *et al.*, Study of anisotropic strange stars in gravity: An embedding approach under the simplest linear functional of the matter-geometry coupling, *Phys. Rev. D* **100** (2019) 044014.

[22] G. Mustafa, M. Zubair, S. Waheed and T. Xia, Realistic stellar anisotropic model satisfying Karmarker condition in $f(R,T)$ gravity, *Eur. Phys. J. C* **80** (2020) 26.

[23] S. Waheed, G. Mustafa, M. Zubair and A. Ashraf, Physically acceptable embedded class-I compact stars in modified gravity with Karmarkar condition, *Symmetry* **12** (2020) 962, doi:10.3390/sym12060962.

[24] M. Rahaman, Ksh. N. Singh, A. Errehymy, F. Rahaman and M. Daoud, Anisotropic Karmarkar stars in $f(R,T)$-gravity, *Eur. Phys. J. C* **80** (2020) 272.

[25] M. Zubair, G. Abbas and I. Noureen, Possible formation of compact stars in $f(R,T)$ gravity, *Astrophys. Space Sci.* **361** (2016) 8.

[26] S. Bahamonde, M. Zubair and G. Abbas, Thermodynamics and cosmological reconstruction in $f(T,B)$ gravity, *Phys. Dark Univ.* **19** (2018) 78–90.

[27] S. Bahamonde, J. L. Said and M. Zubair, Solar system tests in modified teleparallel gravity, *J. Cosmol. Astropart. Phys.* **2020**(10) (2020) 024.

[28] R. Saleem, F. Kramat and M. Zubair, Interior solutions of compact stars in $f(T,\mathcal{T})$ gravity under Karmarkar condition, *Phys. Dark Univ.* **30** (2020) 100592.

[29] I. G. Salako, M. Khlopov, S. Ray, M. Z. Arouko, P. Saha and U. Debnath, Study on anisotropic strange stars in $f(T, \mathcal{T})$ gravity, *Universe* **6** (2020) 167.
[30] A. Ashraf, Z. Zhang, A. Ditta and G. Mustafa, A study of anisotropic spheres in modified gravity via embedding approach, *Ann. Phys.* **422** (2020) 168322.
[31] A. Ditta, A. Ashraf, Z. Zhang and G. Mustafa, Modeling of anisotropic spheres in extended teleparallel theory with matter coupling, *Chin. J. Phys.* **69** (2021) 240–252.
[32] M. Zubair, A. Ditta, G. Abbas and R. Saleem, Physical aspects of anisotropic compact stars in gravity with off diagonal tetrad, *Chin. Phys. C* **45** (2021) 085102.
[33] C. Corda, Interferometric detection of gravitational waves: The definitive test for general relativity, *Int. J. Mod. Phys. D* **18**(14) (2009) 2275–2282.
[34] N. Nazavari, K. Saaidi and A. Mohammadi, Wormhole solution in modified Teleparallel-Rastall gravity and energy conditions, *Gen. Relativ. Gravit.* **55**(3) (2023) 45.
[35] M. Zubair, S. Waheed, M. F. Jamal and G. Mustafa, Realistic anisotropic Karmarkar stars in Rastall gravitational framework, *Results Phys.* **29** (2021) 104787.
[36] S. Capozziello, M. De Laurentis and V. Faraoni, A bird's eye view of $f(R)$-gravity, *Open Astron. J.* **3** (2010) 49–72.
[37] A. V. Astashenok, S. Capozziello and S. D. Odintsov, Nonperturbative models of quark stars in $f(R)$ gravity, *Phys. Lett. B* **742** (2015) 160–166.
[38] S. Capozziello, A. Stabile and A. Troisi, Spherically symmetric solutions in $f(R)$ gravity via the Noether symmetry approach, *Class. Quantum Grav.* **24** (2007) 2153.
[39] S. Capozziello, M. Capriolo and L. Caso, Weak field limit and gravitational waves in $f(T, B)$ teleparallel gravity, *Eur. Phys. J. C* **80** (2020) 156.
[40] S. Capozziello and M. De Laurentis, Extended theories of gravity, *Phys. Rep.* **509** (2011) 167–321.
[41] F. Bajardi and S. Capozziello, $f(G)$ Noether cosmology, *Eur. Phys. J. C* **80** (2020) 704.
[42] F. Bajardi, K. F. Dialektopoulos and S. Capozziello, Higher dimensional static and spherically symmetric solutions in extended Gauss–Bonnet Gravity, *Symmetry* **12** (2020) 372.
[43] M. D. Laurentis, M. Paolella and S. Capozziello, Cosmological inflation in $f(R, G)$ gravity, *Phys. Rev. D* **91** (2015) 083531.
[44] R. Ruderman, Pulsars: structure and dynamics, *Annu. Rev. Astron. Astrophys.* **10** (1972) 427–476.
[45] R. Bowers and E. Liang, Anisotropic spheres in general relativity, *Astrophys. J.* **188** (1974) 657–665.
[46] K. Dev and M. Gleiser, Anisotropic stars: exact solutions, *Gen. Relativ. Gravit.* **34** (2002) 1793–1818.
[47] K. Dev and M. Gleiser, Anisotropic stars II: stability, *Gen. Relativ. Gravit.* **35** (2003) 1435–1457.
[48] S. K. Maurya, Y. K. Gupta, S. Ray and D. Deb, Generalised model for anisotropic compact stars, *Eur. Phys. J. C* **76** (2016) 693.
[49] S. K. Maurya, Y. K. Gupta, B. Dayanandan and S. Ray, A new model for spherically symmetric anisotropic compact star, *Eur. Phys. J. C* **76** (2016) 266.
[50] S. K. Maurya, Y. K. Gupta and S. Ray, All spherically symmetric charged anisotropic solutions for compact stars, *Eur. Phys. J. C* **77** (2017) 360.
[51] K. N. Singh, N. Pradhan and N. Pant, New interior solution describing relativistic fluid sphere, *Pramana — J. Phys.* **89** (2017) 23.

[52] S. K. Maurya, A. Banerjee and S. Hansraj, Role of pressure anisotropy on relativistic compact stars, *Phys. Rev. D* **97** (2018) 044022.
[53] S. K. Maurya, S. D. Maharaj and D. Deb, Generalized anisotropic models for conformal symmetry, *Eur. Phys. J. C* **79** (2019) 170.
[54] S. K. Maurya, A. Banerjee, M. K. Jasim, J. Kumar, A. K. Prasad and A. Pradhan, Anisotropic compact stars in the Buchdahl model: A comprehensive study, *Phys. Rev. D* **99** (2019) 044029.
[55] L. Herrera and N. O. Santos, Local anisotropy in self-gravitating systems, *Phys. Rep.* **286** (1997) 53–130.
[56] K. Lake, All static spherically symmetric perfect-fluid solutions of Einstein's equations, *Phys. Rev. D* **67** (2003) 104015.
[57] L. Herrera, J. Ospino and A. D. Prisco, All static spherically symmetric anisotropic solutions of Einstein's equations, *Phys. Rev. D* **77** (2008) 027502.
[58] B. C. Paul and R. Deb, Relativistic solutions of anisotropic compact objects, *Astrophys. Space Sci.* **354** (2014) 421–430.
[59] G. Mustafa, T.-C. Xia and M. F. Shamir, Realistic solutions of fluid spheres in $f(G, T)$ Gravity under Karmarkar condition, *Ann. Phys.* **413** (2020) 168059.
[60] G. Mustafa and T.-C. Xia, Embedded class solutions of an anisotropic object in Rastall gravity, *Int. J. Mod. Phys. A* **35** (2020) 2050109.
[61] G. Mustafa, M. F. Shamir and X. T.-Cheng, Physically viable solutions of anisotropic spheres in $f(R, G)$ gravity satisfying the Karmarkar condition, *Phys. Rev. D* **101** (2020) 104013.
[62] P. Rastall, Generalization of the Einstein theory, *Phys. Rev. D* **6**(12) (1972) 3357.
[63] N. Tamanini and C. G. Boehmer, Good and bad tetrads in $f(T)$ gravity, *Phys. Rev. D* **86**(4) (2012) 044009.
[64] C. G. Boehmer, A. Mussa and N. Tamanini, Existence of relativistic stars in $f(T)$ gravity, *Class. Quantum Grav.* **28**(24) (2011) 245020.
[65] M. H. Daouda, M. E. Rodrigues and M. J. S. Houndjo, Anisotropic fluid for a set of non-diagonal tetrads in $f(T)$ gravity, *Phys. Lett. B* **715**(1–3) (2012) 241–245.
[66] R. C. Tolman, Static solutions of Einstein's field equations for spheres of fluid, *Phys. Rev.* **55** (1939) 364–373.
[67] J. R. Oppenheimer and G. M. Volkoff, On massive neutron cores, *Phys. Rev.* **55** (1939) 374–381.
[68] H. Heintzmann and W. Hillebrandt, Neutron stars with an anisotropic equation of state: mass, redshift and stability, *Astron. Astrophys.* **38** (1975) 51–55.
[69] Ch. C. Moustakidis, The stability of relativistic stars and the role of the adiabatic index, *Gen. Relativ. Gravit.* **49** (2017) 68.
[70] H. Abreu, H. Hernandez and L. A. Nunez, Sound speeds, cracking and the stability of self-gravitating anisotropic compact objects, *Class. Quantum Grav.* **24** (2007) 4631.
[71] H. A. Buchdahl, General relativistic fluid spheres, *Phys. Rev. D* **116** (1959) 1027–1034.