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Symmetry and Integrability of $\mathcal{PT}$-Symmetric Semi-discrete Short pulse equation: A study of Rogue, Breather, and Soliton Solutions

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Abstract

Spontaneous symmetry breaking and $\mathcal{PT}$-symmetry attracts the modern researcher due to its implementation in many fields such as microwave propagation, nonlinear optics. This article studies the $\mathcal{PT}$-symmetric semi-discrete short pulse equation ($\mathcal{PT}$-sdSPE) that can be viewed as a cognate to the Ablowitz-Ladik lattice in the ultra-short-pulse regime. The Lax pair of the system is constructed and demonstrated that one can obtain a variety of new integrable models by symmetry reductions. Furthermore, quasi-grammian solutions of $\mathcal{PT}$-sdSPE are presented using the binary Darboux transformation. Finally, as an explicit example, symmetry preserving and non-preserving grammians, rogue, breather and soliton solutions are celebrated.

Keywords $\mathcal{PT}$-symmetric semi-discrete short pulse · Symmetry preserving and broken solutions · Breather · Binary Darboux transformation

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1 introduction

The short pulse equation (SPE) is a well-known system which is used to describe the propagation of ultra-short optical pulses in quartz fibers [1, 2]. The mathematical formulation of this equation is

$$\beta_{xt} = \beta + \frac{1}{6}\left(\beta^3\right)_{xx}. \tag{1}$$

The model initially arise during the study of pseudo-spherical surfaces [3] and gain its importance due to possess some of key features such as, integrability, infinite many conservation laws [4, 5], the Wadati-Konno-Ichikawa (WKI) type Lax pair [6] and bi-Hamiltonian structure [7, 8]. The transformation of SPE into the sine-Gorden equation is studied by hodograph transformation in [6, 7, 9]. In addition, using the Riemann-Hilbert method, the long-term asymptotic behaviors of SPE are calculated in [10, 11]. Its breather solutions, periodic solutions and multi-soliton solutions have been explored in [9, 12, 13].

The parity-time ($\mathcal{PT}$) symmetry means that the system remains invariant under the space and time transformation. Basically, the theory of $\mathcal{PT}$ -symmetry promoted the advancement of quantum mechanics by improving the concept of non-Hermitian Hamiltonian. The Hermitian Hamiltonian corresponds to real energy levels, whereas non-Hermitian Hamiltonian systems contain complex energy levels. Basically, ($\mathcal{PT}$)-symmetric systems lie between Hermitian and non-Hermitian systems. Bender and Boettcher introduced that a non-Hermitian Hamiltonian contains real eigenvalues only if the system remains invariant under $\mathcal{PT}$ symmetry [14]. This theory can also be extended to systems having complex eigenvalues by assuming complex potential which has to satisfy the following condition $V^*(-x) = V(x)$, whereas the non-Hermitian Hamiltonian $H = p + V(x)$ ("p" is a momentum operator) must be $\mathcal{PT}$ symmetric. $\mathcal{PT}$-symmetric system plays a vital role in mathematical physics [15–19].

Now, we discuss nonlocal problems which is a consequence of $\mathcal{PT}$- symmetry. Nonlocal systems remains invariant under the transformation of reverse space-time. In a few years, many researchers have explored non-local systems. Ablowitz and Musslimani [15] proposed the first $\mathcal{PT}$-symmetric nonlocal Schrodinger equation which was solved by technique inverse scattering transform. Some of the well-familiar nonlocal integrable models include nonlocal sine-Gordon equation, nonlocal Sasa-Satsuma equation, nonlocal modified Korteweg-de Vries equation, nonlocal Manakov system [20–22].

The Darboux transformation (DT) for $\mathcal{PT}$-semi-discrete SPE (sdSPE) is studied and presented soliton solutions in [35] but in this manuscript, we define and explore binary Darboux transformation (BDT) of $\mathcal{PT}$-sdSPE for the first time which is the novelty of this work and also not found in literature. Further, we present our solutions in terms of quasi-grammians instead of writing in determinants. We present naturally occurring quasi-grammian solutions from BDT here. At the end, we present the symmetry preserving and non-preserving grammians, rogue, breather and soliton solutions.

Generally, nonlocal models have been shown to investigate the systems having balanced gain and loss, non-reciprocal media and bidirectional wave propagation which are vital characteristics of modern optical systems such as $\mathcal{PT}$-symmetric waveguides or metamaterials. Thus, $\mathcal{PT}$-sdSPE investigated in this article can be interpreted as a theoretical model for such non-Hermitian nonlinear systems, where the spatial and time reversal symmetry plays a crucial role in the system dynamics. The $\mathcal{PT}$-sdSPE equation has vital applications in studying the ultra-short pulse dynamics in engineered optical settings having nonlocal response features whereas our primary goal in this article is to investigate the mathematical structure and exact solutions of the given equation.

The pattern of the article in which we would like to explore the discrete Darboux transformation (DT) and BDT for $\mathcal{PT}$-sdSPE is as follows. In section 2, we would review the Lax pair of the system and its $\mathcal{PT}$-symmetry preservence. In Section 3, we give a brief discussion of DT for the system and present the conditions imposed by DT on the matrices in Theorem 1, 2. In Section 4, we study BDT for $\mathcal{PT}$-sdSPE. In Section 5, we give general expressions of quasi-grammian solutions of the system in Theorem 3 and also present the quasi-grammians in terms of scalar variables by using BDT. In Section 6, the dynamics of solutions grammians, traveling of bright breather and rogue solutions are given. The concluding remarks are summarized in Section 7.

2 Lax Pair

The Lax pair for semi-discrete SPE is given by

$$\Psi_{n+1} \equiv H_n \Psi_n = (I + \eta E_n)\, \Psi_n, \tag{2}$$

$$\frac{d}{dt}\Psi_n \equiv K_n \Psi_n = \left(\frac{1}{4\eta}\sigma_3 + \frac{1}{2}F_n\right)\Psi_n, \tag{3}$$

where $I = diag(1, 1)$, $\sigma_3 + diag(1, -1)$, also E_n and F_n are 2×2 matrices defined as

$$E_n = \begin{pmatrix} \Delta_n\alpha_n & \Delta_n\beta_n \\ \Delta_n\gamma_n & -\Delta_n\alpha_n \end{pmatrix}, \tag{4}$$

$$F_n = \begin{pmatrix} 0 & -\beta_n \\ \gamma_n & 0 \end{pmatrix}, \tag{5}$$

here α, β and γ are space and time dependent dynamical variables. The compatibility condition $\frac{d}{dt}H_n + H_n K_n - K_{n+1} H_n = 0$, of the linear system (2)-(3) give rise

$$\begin{aligned} \frac{d}{dt}\left(\alpha_{n+1} - \alpha_n\right) + \frac{1}{2}\left(\beta_{n+1}\gamma_{n+1} - \beta_n\gamma_n\right) &= 0, \\ \frac{d}{dt}\left(\beta_{n+1} - \beta_n\right) - \frac{1}{2}\left(\alpha_{n+1} - \alpha_n\right)\left(\beta_{n+1} - \beta_n\right) &= 0, \\ \frac{d}{dt}\left(\gamma_{n+1} - \gamma_n\right) - \frac{1}{2}\left(\alpha_{n+1} - \alpha_n\right)\left(\gamma_{n+1} - \gamma_n\right) &= 0. \end{aligned} \tag{6}$$

Eq. (6) is known as generalized sdSPE which gives different integrable systems under different symmetry reductions. For example, we take $\gamma_n = \beta_n^*$, Eq. (6) transforms to complex semi-discrete SPE [23] i.e.,

$$\frac{d}{dt}\left(\alpha_{n+1}-\alpha_n\right)+\frac{1}{2}\left(|\beta_{n+1}|^2-|\beta_n|^2\right)=0,$$
$$\frac{d}{dt}\left(\beta_{n+1}-\beta_n\right)-\frac{1}{2}\left(\alpha_{n+1}-\alpha_n\right)\left(\beta_{n+1}-\beta_n\right)=0. \tag{7}$$

Now, if we take $\alpha_{-n}(-t) = -\alpha_n(t)$, $\gamma_n(t) = \beta_{-n}(-t)$, we get the $\mathcal{PT}$-symmetric nonlocal semi-discrete SP equation

$$\begin{aligned}
&\frac{d}{dt}\left(\alpha_{n+1}(t)-\alpha_n(t)\right)\\
&\quad+\frac{1}{2}\left(\beta_{n+1}(t)\beta_{-(n+1)}(-t)-\beta_n(t)\beta_{-n}(-t)\right)=0,\\
&\frac{d}{dt}\left(\beta_{n+1}(t)-\beta_n(t)\right)\\
&\quad-\frac{1}{2}\left(\alpha_{n+1}(t)-\alpha_n(t)\right)\left(\beta_{n+1}(t)+\beta_n(t)\right)=0,\\
&\frac{d}{dt}\left(\beta_{-(n+1)}(-t)-\beta_{-n}(-t)\right)\\
&\quad-\frac{1}{2}\left(\alpha_{n+1}(t)-\alpha_n(t)\right)\left(\beta_{-(n+1)}(-t)+\beta_{-n}(-t)\right)=0.
\end{aligned} \tag{8}$$

By applying symmetry reduction $t \to -t$ and $n \to -n$, the system (8) and the potential $V_n(t) = \beta_n(t)\beta_{-n}(-t)$ are invariant. In the coming section, we give brief overview of DT.

3 Darboux Transformation

The DT is a very powerful solution generation technique because it involves purely algebraic algorithms. Basically, it is a gauge transformation of linear systems (for details, see [24–28]). Now, we define DT on Ψ_n as

$$\Psi_n[1] = \left(\eta^{-1}I - S_n\right)\Psi_n, \tag{9}$$

where $S_n = R_n\Xi^{-1}R_n^{-1}$ and $I = diag(1, 1)$, also $R_n = \left(\left|r_n^{(1)}\right\rangle, \left|r_n^{(2)}\right\rangle\right)$ is distinct matrix solution of (2)-(3) evaluated at

$$\Xi = \mathrm{diag}(\eta_1, \eta_2). \tag{10}$$

The Lax Eqs. (2)-(3) can be expressed as matrix form

$$R_{n+1} = R_n + E_n R_n \Xi, \tag{11}$$

$$\frac{d}{dt} R_n = \frac{\sigma_3 R_n \Xi^{-1}}{4} + \frac{F_n R_n}{2}. \tag{12}$$

Theorem 1 *By operating DT, the transformed matrix $E_n[1]$ sustains the original form of E_n as defined in Eq. (4), provided that the auxiliary matrix R_n should satisfy*

$$E_n[1] = E_n - S_{n+1} + S_n, \tag{13}$$

$$(S_{n+1} - S_n) S_n = E_n S_n - S_{n+1} E_n. \tag{14}$$

Proof The transformation between $E_n[1]$ and E_n is developed and expressed in Eq. (13). So, we have to verify the condition (14) using Eq. (11) and $S_n = R_n \Xi^{-1} R_n^{-1}$. For this consider

$$\begin{aligned}
(S_{n+1} - S_n)\, S_n &= \left(R_{n+1} \Xi^{-1} R_{n+1}^{-1} - R_n \Xi^{-1} R_n^{-1}\right) S_n, \\
&= \Big(R_{n+1} \Xi^{-1} R_{n+1}^{-1} - R_{n+1} \Xi^{-1} R_{n+1}^{-1} R_{n+1} R_n^{-1} \\
&\quad + R_{n+1} \Xi^{-1} R_n^{-1} - R_n \Xi^{-1} R_n^{-1}\Big) S_n, \\
&= \left(S_{n+1} \left\{I - R_{n+1} R_n^{-1}\right\} + \{R_{n+1} - R_n\} \Xi^{-1} R_n^{-1}\right) S_n, \\
&= \left(S_{n+1} \left\{-E_n S_n^{-1}\right\} + \left\{E_n R_n R_n^{-1}\right\}\right) S_n, \\
&= E_n S_n - S_{n+1} E_n.
\end{aligned}$$

□

This is exactly Eq. (14).

Theorem 2 *By operating DT, the transformed matrix $F_n[1]$ sustains the original form of F_n as defined in Eq. (5), provided that the auxiliary matrix R_n should satisfy*

$$F_n[1] = F_n + \frac{1}{2}[\sigma_3, S_n], \tag{15}$$

$$\frac{d}{dt} S_n = \frac{1}{2}[F_n, S_n] + \frac{1}{4}[\sigma_3, S_n] S_n, \tag{16}$$

Proof The transformation between $F_n[1]$ and F_n is developed and expressed in Eq. (15). So, we have to verify the condition (16) using Eq. (12) and $S_n = R_n \Xi^{-1} R_n^{-1}$.

For this operating $\frac{d}{dt}$ on S_n, as

$$\begin{aligned}\frac{d}{dt}S_n &= \frac{d}{dt}\left(R_n \Xi^{-1} R_n^{-1}\right),\\ &= \frac{d}{dt}R_n\left(\Xi^{-1}R_n^{-1}\right) - R_n\Xi^{-1}R_n^{-1}\left(\frac{d}{dt}R_n\right)R_n^{-1},\\ &= \frac{\sigma_3 S_n S_n}{4} + \frac{F_n S_n}{2} - \frac{S_n\sigma_3 S_n}{4} - \frac{S_n F_n}{2},\\ &= \frac{1}{4}[\sigma_3, S_n]S_n + \frac{1}{2}[F_n, S_n].\end{aligned}$$

□

This is similar to Eq. (16).

Remark 1 Thus, matrix $S_n = R_n \Xi^{-1} R_n^{-1}$ proves the best choice which satisfies all the conditions imposed by DT. This is the preservation of the system, that is, if Ψ_n, E_n, F_n are the solutions of the Lax pair (2), (3), therefore $\Psi_n[1]$, $E_n[1]$, $F_n[1]$ are also the solutions of the same equations.

By using Eq. (13), we calculate transformations on scalar dynamical variables as

$$\alpha_n[1] = \alpha_n - S^{(1)}_{n,11}, \tag{17}$$

$$\beta_n[1] = \beta_n - S^{(1)}_{n,12}, \tag{18}$$

$$\gamma_n[1] = \gamma_n - S^{(1)}_{n,21}. \tag{19}$$

In terms of quasi-determinants (for detail see [36]), result (9) can be expressed as

$$\begin{aligned}\Psi_n[1] &= \left(\lambda^{-1}I - R_n\Xi^{-1}R_n^{-1}\right)\Psi_n,\\ &= \begin{vmatrix} R_n & \Psi_n \\ R_n\Xi^{-1} & \boxed{\eta^{-1}\Psi_n}\end{vmatrix}.\end{aligned}$$

The K-fold DT on Ψ_n is given by

$$\Psi_n[K] = \begin{vmatrix} R_{n,1} & R_{n,2} & \cdots & R_{n,K} & \Psi_n \\ R_{n,1}\Xi_1^{-1} & R_{n,2}\Xi_2^{-1} & \cdots & R_{n,K}\Xi_K^{-1} & \eta^{-1}\Psi_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{n,1}\Xi_1^{-(K-1)} & R_{n,2}\Xi_2^{-(K-1)} & \cdots & R_{n,K}\Xi_K^{-(K-1)} & \eta^{-(K-1)}\Psi_n \\ R_{n,1}\Xi_1^{-K} & R_{n,2}\Xi_2^{-K} & \cdots & R_{n,K}\Xi_K^{-K} & \boxed{\eta^{-K}\Psi_n}\end{vmatrix}. \tag{20}$$

In the form of quasideterminants, result (13) can be written as

$$\begin{aligned} E_n\,[1] &= E_n - (R_{n+1}\,\Xi^{-1} R_{n+1}^{-1} - R_n\,\Xi^{-1} R_n^{-1}),\\ &= E_n + \left(\begin{vmatrix} R_{n+1} & I \\ R_{n+1}\,\Xi^{-1} & \boxed{O} \end{vmatrix} - \begin{vmatrix} R_n & I \\ R_n\,\Xi^{-1} & \boxed{O} \end{vmatrix} \right). \end{aligned} \tag{21}$$

The generalization to K-th iteration can be written as

$$E_n[K] = E_n + \left(\begin{vmatrix} R_{n+1,1} & R_{n+1,2} & \cdots & R_{n+1,K} & I \\ R_{n+1,1}\,\Xi_1^{-1} & R_{n+1,2}\,\Xi_2^{-1} & \cdots & R_{n+1,K}\,\Xi_K^{-1} & O \\ R_{n+1,1}\,\Xi_1^{-2} & R_{n+1,2}\,\Xi_2^{-2} & \ldots & R_{n+1,K}\,\Xi_K^{-2} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{n+1,1}\,\Xi_1^{-K} & R_{n+1,2}\,\Xi_2^{-K} & \cdots & R_{n+1,K}\,\Xi_K^{-K} & \boxed{O} \end{vmatrix} - \begin{vmatrix} R_{n,1} & R_{n,2} & \cdots & R_{n,K} & I \\ R_{n,1}\,\Xi_1^{-1} & R_{n,2}\,\Xi_2^{-1} & \cdots & R_{n,K}\,\Xi_K^{-1} & O \\ R_{n,1}\,\Xi_1^{-2} & R_{n,2}\,\Xi_2^{-2} & \ldots & R_{n,K}\,\Xi_K^{-2} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ R_{n,1}\,\Xi_1^{-K} & R_{n,2}\,\Xi_2^{-K} & \cdots & R_{n,K}\,\Xi_K^{-K} & \boxed{O} \end{vmatrix} \right).$$

In next section, we define and explore BDT for $\mathcal{PT}$-sdSPE.

4 Binary Darboux Transformation

In order to calculate expression for BDT, we first define the adjoint spectral problem by taking adjoint of linear system (2), (3), we get

$$\Phi_{n+1} = -\Phi_n \left(I + \zeta E_n^{\dagger} \right), \tag{22}$$

$$\frac{d}{dt}\Phi_n = -\Phi_n \left(\frac{1}{4\zeta}\sigma_3 + \frac{1}{2}F_n^{\dagger} \right), \tag{23}$$

where ζ is a spectral parameter and Φ_n is an eigenfunction matrix for adjoint space $(V^{\dagger})$. Now, consider hat space $\widehat{V}$ for BDT, consider the copied version of direct space known as hat space i.e., $\widehat{\Psi}_n \in \widehat{W}$ (for details, see [29–34]). The linear system for the hat space is defined as

$$\widehat{\Psi}_{n+1} = (I + \eta \widehat{E}_n)\widehat{\Psi}_n, \tag{24}$$

$$\frac{d}{dt}\Psi_n = \left(\frac{1}{4\eta}\sigma_3 + \frac{1}{2}\widehat{F}_n \right) \widehat{\Psi}_n. \tag{25}$$

In order to define BDT, consider two standard DT's $\widehat{D}_n(\eta)$ and $D_n(\eta)$ whose action is to transform matrix solutions $\widehat{\Psi}_n$ and Ψ_n respectively onto a same matrix solution

$\Psi_n[1]$, we can write as

$$D_n(\lambda)\Psi_n = \widehat{D}_n(\lambda)\widehat{\Psi}_n. \tag{26}$$

Now, we can write operator of BDT as $\widehat{B}_n = \widehat{D}_n^{-1}(\eta)D_n(\eta)$ which is the definition of BDT. Now operate $\widehat{B}_n$ on matrix solution, $\widehat{\Psi}_n = \widehat{D}_n^{-1}(\eta)D_n(\eta)\Psi_n$. For detail discussion of BDT, assume $i(\widehat{R}_n)$ and Φ_n belongs to adjoint space, we can write

$$i(\widehat{R}_n) = D_n^{(-1)\dagger}(\eta)\Phi_n.$$

Also, we get $iR_n = R_n^{(-1)\dagger}$ from the condition $D_n^{\dagger}(\eta)(iR_n) = 0$. Similarly, $i\widehat{R}_n = \widehat{R}_n^{(-1)\dagger}$, so we can write the transformation on $\widehat{R}_n$ as

$$\widehat{R}_n = R_n G_n^{-1}(R_n, \Phi_n), \tag{27}$$

where G_n is the potential eigenfunction defined as

$$G_n(R_n, \Phi_n) = (\Phi_n^{\dagger} R_n)(\eta^{-1}I - \Xi^{-1})^{-1}. \tag{28}$$

Similar results can be derived for adjoint space

$$\widehat{P}_n = P_n G_n^{(-1)\dagger}(\Psi_n, P_n),$$

where

$$G_n(\Psi_n, P_n) = -(\eta^{-1}I - F^{(-1)\dagger})^{-1}(P_n^{\dagger}\Psi_n). \tag{29}$$

Whereas in the above expressions R_n and P_n are distinct matrix solutions for direct and adjoint spaces. We write Eqs. (28), (29) in terms of matrix and get the result

$$F^{(-1)\dagger}G_n(R_n, P_n) - G_n(R_n, P_n)\Xi^{-1} = P_n^{\dagger}R_n. \tag{30}$$

In order to get the general expression of BDT, putting the values of $D_n(\eta)$, $\widehat{D}_n(\eta)$ and also using Eqs. (27), (30) in the definition of $\widehat{B}_n$, we get

$$\widehat{B}_n = I + R_n G_n^{-1}(R_n, \Phi_n)(\eta^{-1}I - F^{(-1)\dagger})^{-1}P_n^{\dagger}.$$

Now, we are able to write BDT for eigenfunctions Ψ_n and Φ_n as

$$\widehat{\Psi}_n = \Psi_n - R_n G_n^{-1}(R_n, P_n)G_n(\Psi_n, P_n),$$
$$\widehat{\Phi}_n = \Phi_n - P_n G_n^{(-1)\dagger}(R_n, P_n)G_n^{\dagger}(R_n, \Phi_n).$$

In terms of quasideterminants

$$\widehat{\Psi}_n = \begin{vmatrix} G_n(R_n, P_n) & G_n(\Psi_n, P_n) \\ R_n & \boxed{\Psi_n} \end{vmatrix},$$

$$\widehat{\Phi}_n = \begin{vmatrix} G_n^\dagger(R_n, P_n) & G_n^\dagger(R_n, \Phi_n) \\ P_n & \boxed{\Phi_n} \end{vmatrix}.$$

These results can be generalized to K-times of BDT by using the properties of quasideterminants.

5 Quasi-Grammian Solutions of $\mathcal{PT}$-Symmetric Semi-Discrete Short Pulse Equation

Now, we obtain the quasi-grammian solution for the $\mathcal{PT}$-sdSPE.

Theorem 3 *By operating BDT described in (26), the transformed matrix $\widehat{E}_n[1]$ has the form*

$$\widehat{E}_n = E_n + R_{n+1}G_{n+1}^{-1}P_{n+1}^\dagger - R_nG_n^{-1}P_n^\dagger. \tag{31}$$

Proof Applying the definition of BDT on matrix solution of $\mathcal{PT}$-sdSPE E_n as

$$\widehat{E}_n - \widehat{R}_{n+1} + \widehat{R}_n = E_n - R_{n+1} + R_n,$$
$$\widehat{E}_n = E_n + \widehat{S}_{n+1}F^{(-1)\dagger}\widehat{S}_{n+1}^{-1} - \widehat{S}_nF^{(-1)\dagger}\widehat{S}_n^{-1} - R_{n+1}\Xi^{-1}R_{n+1}^{-1} + R_n\Xi^{-1}R_n^{-1}.$$

Using the result (27), we get

$$\widehat{E}_n = E_n - R_{n+1}\Xi^{-1}R_{n+1}^{-1} + R_{n+1}G_{n+1}^{-1}F^{(-1)\dagger}G_{n+1}R_{n+1}^{-1}$$
$$+R_n\Xi^{-1}R_n^{-1} - R_nG_n^{-1}F^{(-1)\dagger}G_nR_n^{-1}.$$

Now, using the value of $F^{(-1)\dagger}G_n$ from expression (30), after solving we obtain

$$\widehat{E}_n = E_n + R_{n+1}G_{n+1}^{-1}P_{n+1}^\dagger - R_nG_n^{-1}P_n^\dagger \tag{32}$$

□

This is exactly (31) which completes the proof.

Now, (31) can be presented in terms of quasideterminant as

$$\widehat{E}_n = E_n - \begin{vmatrix} G_{n+1} & P_{n+1}^\dagger \\ R_{n+1} & \boxed{O} \end{vmatrix} + \begin{vmatrix} G_n & P_n^\dagger \\ R_n & \boxed{O} \end{vmatrix},$$
$$= E_n - \widehat{\mathcal{N}}_{n+1} + \widehat{\mathcal{N}}_n. \tag{33}$$

By using Eq. (33), we write results of BDT for discrete scalar variables

$$
\begin{aligned}
\widehat{\alpha}_n &= \alpha_n - \widehat{\mathcal{N}}_{n,11},\\
\widehat{\beta}_n &= \beta_n - \widehat{\mathcal{N}}_{n,12},\\
\widehat{\gamma}_n &= \gamma_n - \widehat{\mathcal{N}}_{n,21},
\end{aligned} \tag{34}
$$

where $\widehat{\mathcal{N}}_n = R_n G_n^{-1} P_n^\dagger$. By the iteration of BDT, we can calculate K-times iteration of $\widehat{E}_n$ expressed as

$$
\widehat{E}_n[K] = E_n - \left(\begin{vmatrix} G_{n+1}(R_{n+1,1}, P_{n+1,1}) & \cdots & G_{n+1}(R_{n+1,K}, P_1) & P_1^\dagger \\ \vdots & \cdots & \vdots & \vdots \\ G_{n+1}(R_{n+1,1}, P_{n+1,K}) & \cdots & G_{n+1}(R_{n+1,K}, P_{n+1,K}) & P_K^\dagger \\ R_{n+1,1} & \cdots & R_{n+1,K} & \boxed{O} \end{vmatrix} - \begin{vmatrix} G_n(R_{n,1}, P_{n,1}) & \cdots & G_n(R_{n,K}, P_1) & P_1^\dagger \\ \vdots & \cdots & \vdots & \vdots \\ G_n(R_{n,1}, P_{n,K}) & \cdots & G_n(R_{n,K}, P_{n,K}) & P_K^\dagger \\ R_{n,1} & \cdots & R_{n,K} & \boxed{O} \end{vmatrix} \right). \tag{35}
$$

For convenience, we write quasi-grammian in terms of 2×2 matrix, as

$$
\widehat{E}_n = E_n - \left\{ \begin{pmatrix} \begin{vmatrix} G_{n+1} & P_{n+1,1}^\dagger \\ R_{n+1,1} & \boxed{O} \end{vmatrix} & \begin{vmatrix} G_{n+1} & P_{n+1,2}^\dagger \\ R_{n+1,1} & \boxed{O} \end{vmatrix} \\ \begin{vmatrix} G_{n+1} & P_{n+1,1}^\dagger \\ R_{n+1,2} & \boxed{O} \end{vmatrix} & \begin{vmatrix} G_{n+1} & P_{n+1,2}^\dagger \\ R_{n+1,2} & \boxed{O} \end{vmatrix} \end{pmatrix} - \begin{pmatrix} \begin{vmatrix} G_n & P_{n,1}^\dagger \\ R_{n,1} & \boxed{O} \end{vmatrix} & \begin{vmatrix} G_n & P_{n,2}^\dagger \\ R_{n,1} & \boxed{O} \end{vmatrix} \\ \begin{vmatrix} G_n & P_{n,1}^\dagger \\ R_{n,2} & \boxed{O} \end{vmatrix} & \begin{vmatrix} G_n & P_{n,2}^\dagger \\ R_{n,2} & \boxed{O} \end{vmatrix} \end{pmatrix} \right\}.
$$

Now, the quasi-grammian expressions for the discrete scalar variables are

$$
\widehat{\alpha}_n = \alpha_n - \begin{vmatrix} G_n & P_{n,1}^\dagger \\ R_{n,1} & \boxed{O} \end{vmatrix}, \tag{36}
$$

$$
\widehat{\beta}_n = \beta_n - \begin{vmatrix} G_n & P_{n,2}^\dagger \\ R_{n,1} & \boxed{O} \end{vmatrix}, \tag{37}
$$

$$
\widehat{\gamma}_n = \gamma_n - \begin{vmatrix} G_n & P_{n,1}^\dagger \\ R_{n,2} & \boxed{O} \end{vmatrix}. \tag{38}
$$

Remark 2 These are the quasi-grammian solutions for the $\mathcal{PT}$-sdSPE in terms of discrete scalar variables. Hence, by using BDT, we calculate quasi-grammian solutions of $\mathcal{PT}$-sdSPE in the form of quasideterminants which are not similar with the solutions as calculated with classical DT [35].

6 Explicit Solutions

Here, we compute the explicit expressions for the dynamics of breather and rogue solitons solutions of $\mathcal{PT}$-sdSPE. We start from the simple solution of general sdSPE by taking $\alpha_{n+1} - \alpha_n = a$ or $\alpha_n = na$ and $\beta_n = \gamma_n = 0$. The linear system (2)-(3) becomes

$$\begin{pmatrix} Y_{n+1} \\ Z_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + a\eta & 0 \\ 0 & 1 - a\eta \end{pmatrix} \begin{pmatrix} Y_n \\ Z_n \end{pmatrix}, \tag{39}$$

$$\frac{d}{dt} \begin{pmatrix} Y_n \\ Z_n \end{pmatrix} = \begin{pmatrix} \frac{1}{4\eta} & 0 \\ 0 & -\frac{1}{2\eta} \end{pmatrix} \begin{pmatrix} Y_n \\ Z_n \end{pmatrix}. \tag{40}$$

By solving the linear system (39), (40), we get

$$Y_n(\eta) = (1 + a\eta)^n \, exp \left(\frac{1}{4\eta} \right) t, \tag{41}$$

$$Z_n(\eta) = (1 - a\eta)^n \, exp \left(-\frac{1}{4\eta} \right) t. \tag{42}$$

The distinct matrix solution R_n of $\mathcal{PT}$-sdSPE can be defined as

$$R_n = \begin{pmatrix} Y_n(\eta) & Z_n(\overline{\eta}) \\ Z_n(\eta) & -Y_n(\overline{\eta}) \end{pmatrix}. \tag{43}$$

Similarly, for the adjoint space P_n can be written as

$$P_n = \begin{pmatrix} U_n\,(\zeta) & V_n(\overline{\zeta}) \\ V_n\,(\zeta) & -U_n\,(\overline{\zeta}) \end{pmatrix}, \tag{44}$$

where

$$U_n(\zeta) = (1 + a\zeta)^n \, exp \left(\frac{1}{4\zeta} \right) t, \tag{45}$$

$$V_n(\zeta) = (1 - a\zeta)^n \, exp \left(-\frac{1}{4\zeta} \right) t. \tag{46}$$

Now using Eqs. (43), (44) in the definition of $G_n(R_n, P_n)$, we get

$$G_n(R_n, P_n) = \begin{pmatrix} \frac{A_n+B_n}{\zeta-\eta} & \frac{C_n-D_n}{\zeta-\overline{\eta}} \\ \\ \frac{\overline{C}_n-\overline{D}_n}{\overline{\zeta}-\eta} & \frac{\overline{A}_n+\overline{B}_n}{\overline{\zeta}-\overline{\eta}} \end{pmatrix}, \tag{47}$$

where

$$\begin{aligned} A_n &= (1+a\zeta)^n\,(1+a\eta)^n \exp\left(\frac{1}{4\zeta}+\frac{1}{4\eta}\right)t, \\ B_n &= (1-a\zeta)^n\,(1-a\eta)^n \exp\left(-\frac{1}{4\zeta}-\frac{1}{4\eta}\right)t, \\ C_n &= (1+a\zeta)^n\,(1+a\overline{\eta})^n \exp\left(\frac{1}{4\zeta}+\frac{1}{4\overline{\eta}}\right)t, \\ D_n &= (1-a\zeta)^n\,(1-a\overline{\eta})^n \exp\left(-\frac{1}{4\zeta}-\frac{1}{4\overline{\eta}}\right)t. \end{aligned}$$

Now, consider

$$\begin{aligned} \widehat{\mathcal{N}}_n &= R_n G_n^{-1}(R_n, P_n) P_n^{\dagger} = \frac{1}{L_n}\begin{pmatrix} \widehat{\mathcal{N}}_{n,11} & \widehat{\mathcal{N}}_{n,12} \\ \\ \widehat{\mathcal{N}}_{n,21} & \widehat{\mathcal{N}}_{n,22} \end{pmatrix}, \\ &= \frac{1}{L_n}\begin{pmatrix} \frac{\overline{A}_n+\overline{B}_n}{\overline{\zeta}-\overline{\eta}} Y_n(\eta)U_n(\zeta) - \frac{\overline{C}_n-\overline{D}_n}{\overline{\zeta}-\eta} Y_n(\overline{\eta})U_n(\zeta) & \frac{\overline{A}_n+\overline{B}_n}{\overline{\zeta}-\overline{\eta}} Y_n(\eta)V_n(\zeta) - \frac{\overline{C}_n-\overline{D}_n}{\overline{\zeta}-\eta} Y_n(\overline{\eta})V_n(\zeta) \\ \\ -\frac{C_n-D_n}{\zeta-\overline{\eta}} Y_n(\eta)U_n(\overline{\zeta}) + \frac{A_n+B_n}{\zeta-\eta} Y_n(\overline{\eta})U_n(\overline{\zeta}) & +\frac{C_n-D_n}{\zeta-\overline{\eta}} Y_n(\eta)V_n(\overline{\zeta}) - \frac{A_n+B_n}{\zeta-\eta} Y_n(\overline{\eta})V_n(\overline{\zeta}) \\ \frac{\overline{A}_n+\overline{B}_n}{\overline{\zeta}-\overline{\eta}} Z_n(\eta)U_n(\zeta) + \frac{\overline{C}_n-\overline{D}_n}{\overline{\zeta}-\eta} Z_n(\overline{\eta})U_n(\zeta) & \frac{\overline{A}_n+\overline{B}_n}{\overline{\zeta}-\overline{\eta}} Z_n(\eta)V_n(\zeta) + \frac{\overline{C}_n-\overline{D}_n}{\overline{\zeta}-\eta} Z_n(\overline{\eta})V_n(\zeta) \\ \\ -\frac{C_n-D_n}{\zeta-\overline{\eta}} Z_n(\eta)U_n(\overline{\zeta}) - \frac{A_n+B_n}{\zeta-\eta} Z_n(\overline{\eta})U_n(\overline{\zeta}) & -\frac{C_n-D_n}{\zeta-\overline{\eta}} Z_n(\eta)V_n(\overline{\zeta}) + \frac{A_n+B_n}{\zeta-\eta} Z_n(\overline{\eta})V_n(\overline{\zeta}) \end{pmatrix}, \end{aligned} \tag{48}$$

where

$$L_n = \frac{(A_n+B_n)\left(\overline{A}_n+\overline{B}_n\right)}{(\zeta-\eta)\left(\overline{\zeta}-\overline{\eta}\right)} - \frac{(C_n-D_n)\left(\overline{C}_n-\overline{D}_n\right)}{(\zeta-\overline{\eta})\left(\overline{\zeta}-\eta\right)}.$$

The solutions in terms of discrete scalar variables

$$\widehat{\alpha}_n = a - \widehat{\mathcal{N}}_{n,11}, \tag{49}$$

$$\widehat{\beta}_n = -\widehat{\mathcal{N}}_{n,12}, \tag{50}$$

$$\widehat{\gamma}_n = -\widehat{\mathcal{N}}_{n,21}. \tag{51}$$

The dynamics of symmetry preserving and non-preserving grammians and rogue solutions for $\mathcal{PT}$-sdSPE are presented in Figs. 1, 2 and 3.

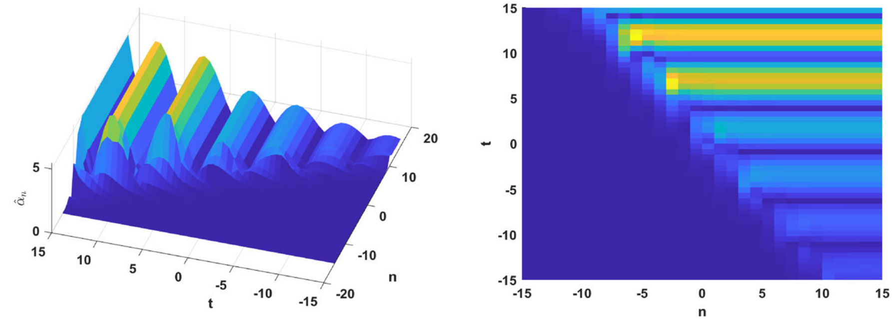

Fig. 1 Dynamics of $\widehat{\alpha}_n$: Travelling of symmetry non-preserving unstable grammian for ζ=0.8+0.7i, η=0.6+0.5i, a=3

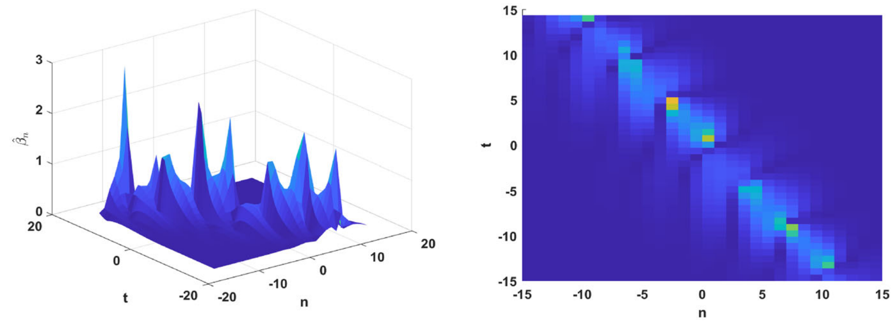

Fig. 2 Dynamics of $\widehat{\beta}_n$: Travelling of symmetry non-preserving decaying rogue for ζ=0.4+0.7i, η=0.9-0.5i, a=3

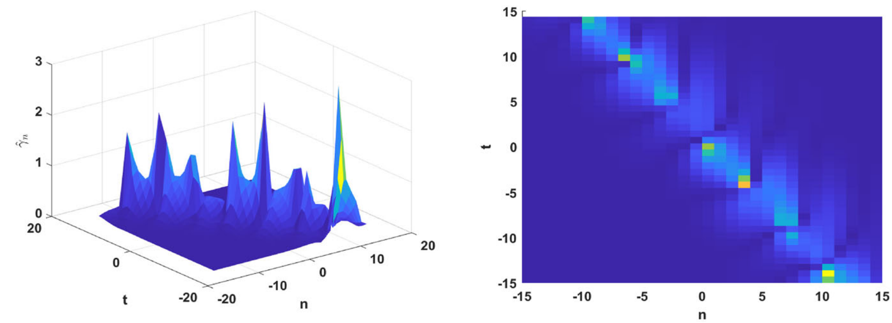

Fig. 3 Dynamics of $\widehat{\gamma}_n$: Travelling of symmetry preserving growing rogue for ζ=0.4+0.7i, η=0.9-0.5i, a=3

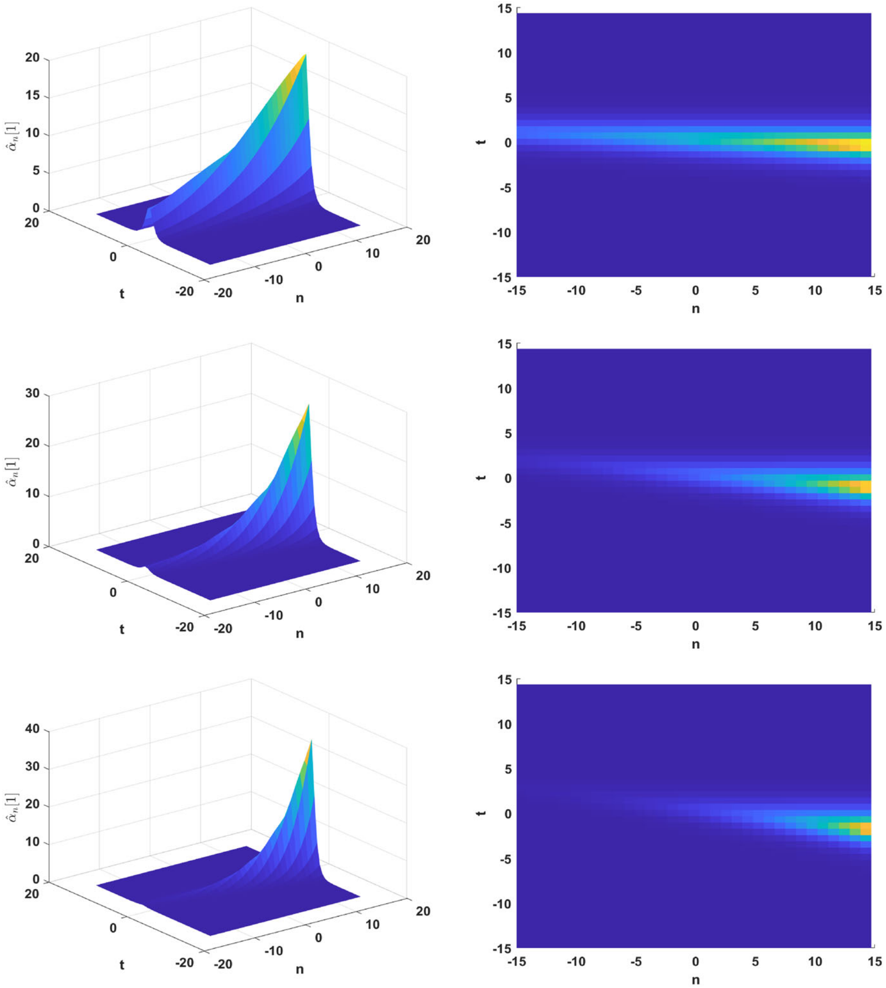

Fig. 4 Dynamics of $\alpha_n[1]$: Symmetry non-preserving unstable soliton for ζ=2.4+4.7i, 2.4+2.7i, 2.4+1.7i, a=3

6.1 Reduction

Now, by substituting $\zeta = \overline{\eta}, \overline{\zeta} = \eta$, we get the explicit expressions for the classical DT as

$$A_n = \overline{A}_n = (1 + a\eta)^n (1 + a\overline{\eta})^n \exp\left(\frac{1}{4\eta} + \frac{1}{4\overline{\eta}}\right) t,$$

$$B_n = \overline{B}_n = (1 - a\eta)^n (1 - a\overline{\eta})^n \exp\left(-\frac{1}{4\eta} - \frac{1}{4\overline{\eta}}\right) t.$$

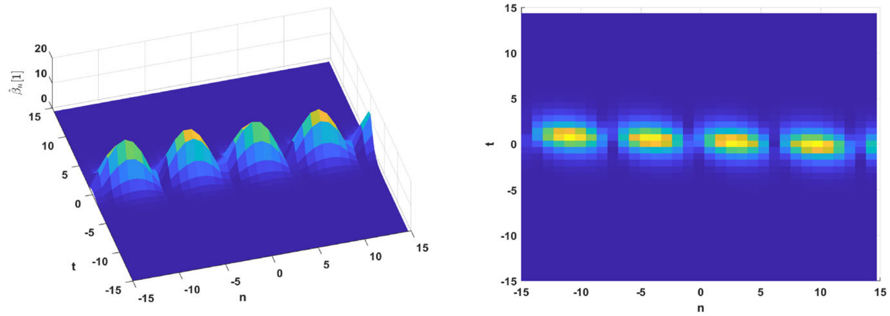

Fig. 5 Dynamics of $\beta_n[1]$: Travelling of symmetry non-preserving decaying breather for ζ=2.4+4.7i, a=3

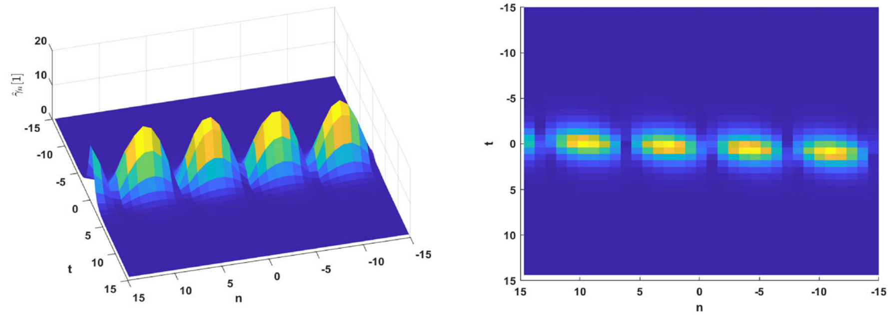

Fig. 6 Dynamics of $\gamma_n[1]$: Travelling of symmetry preserving growing breather for ζ=2.4+4.7i, a=3

Also $L_n = (A_n^2 + B_n^2)/(\eta - \overline{\eta})^2$ and $U_n(\zeta) = Y_n(\overline{\eta})$, $U_n(\overline{\zeta}) = Y_n(\eta)$, $V_n(\zeta) = Z_n(\overline{\eta})$, $V_n(\overline{\zeta}) = Z_n(\eta)$. The terms C_n and D_n are ignored due to divergence.

$$
\begin{aligned}
\widehat{\alpha}_n &= \alpha_n[1] = a - \frac{2(\eta - \overline{\eta})Y_n(\eta)Y_n(\overline{\eta})}{A_n + B_n},\\
\widehat{\beta}_n &= \beta_n[1] = -(\eta - \overline{\eta})\frac{Y_n(\eta)Z_n(\overline{\eta}) + Y_n(\overline{\eta})Z_n(\eta)}{Y_n(\eta)Y_n(\overline{\eta}) + Z_n(\eta)Z_n(\overline{\eta})},\\
\widehat{\gamma}_n &= \gamma_n[1] = (\eta - \overline{\eta})\frac{Y_n(\eta)Z_n(\overline{\eta}) + Y_n(\overline{\eta})Z_n(\eta)}{Y_n(\eta)Y_n(\overline{\eta}) + Z_n(\eta)Z_n(\overline{\eta})}.
\end{aligned} \tag{52}
$$

The expressions (52) represent the symmetry-non-preserving solitons and breather shown in Figs. 4, 5 and 6. So, we have computed the grammians solutions and breather type symmetry preserving and breaking solutions by BDT and reduce them to classical DT solutions.

7 Concluding Remarks

This article explored the semi-discrete version of $\mathcal{PT}$-symmetric SPE. We present how under symmetry transformation generalized-sdSPE transform to complex sdSPE and $\mathcal{PT}$-symmetric sdSPE. By calculating soliton solutions using DT, a systematic framework for BDT was constructed and applied to the $\mathcal{PT}$-symmetric sdSPE leading explicit quasi-grammian solutions.

The quasi-grammians expressions in terms of scalar variables were derived. The classical DT uses a single eigenfunction to generate new solutions whereas BDT uses a pair of eigenfunctions (direct and adjoint) to iteratively build solutions, preserving integrability and constructs a new potential solution by linking two solutions of the linear system. To the best of our knowledge, this is the first application of BDT to the $\mathcal{PT}$-symmetric sdSPE, highlighting the adaptability of this method to nonlocal integrable systems. The resulting quasi-grammian solutions exhibit structurally stable and rich dynamics, extending the family of known solutions beyond what is accessible by using BDT. In addition, explicit expressions for the system were derived. Finally, we reduced the BDT solutions to classical DT solutions. We presented the dynamics of the solutions like spontaneous symmetry breaking grammian, rogue, breather and soliton.

Recently, the study of dispersive regularization of wave gain much attention for the researchers. The dispersive regularization of breaking wave yields the generation of dispersive shock wave (DSW). The phenomenon of DSW exerts a critical influence on nonlinear dynamics in various nonlinear fields, and simulating this complex physical process remains a significant challenge [37–39].

BDT provided a precise method to obtain solutions where grammian and breather solutions that can significantly improve our understanding of nonlinear waves. In particular, breather solutions played a crucial role in modeling the propagation of ultra-short pulses in optical fibers. Exploring the BDT to $\mathcal{PT}$-symmetric fully discrete SPE multi-component system might lead more general solution classes, which would be compelling research direction.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

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