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Extended Gram-type determinant, wave and rational solutions to two (3+1)-dimensional nonlinear evolution equations

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ABSTRACT

New exact Grammian determinant solutions to two (3+1)-dimensional nonlinear evolution equations are derived. Extended set of sufficient conditions consisting of linear partial differential equations with variable-coefficients is presented. Moreover, a systematic analysis of linear partial differential equations is used for solving the representative linear systems. The bilinear Bäcklund transformations are also constructed for the equations. Also, as an application of the bilinear Bäcklund transforms, a new class of wave and rational solutions to the equations are explicitly computed.

1. Introduction

Many phenomena in the nonlinear sciences and in the physics can be modeled by a various classes of integrable nonlinear partial differential equations. Consequently, construction of wave solutions of nonlinear evolution equations plays a crucial role in the study of nonlinear sciences and nonlinear phenomena. To obtain the traveling wave solutions for these equations, especially for the higher-dimensional and the coupled nonlinear evolution equations, can make people know the described physical phenomena. Besides traveling wave solutions, another set of interesting multi-exponential wave solutions [10,11] is a linear combination of exponential waves. Nowadays, with the rapid software technology development, solving nonlinear partial differential equations via explicit and symbolic computation is taking an increasing role due to its accuracy, efficiency and its restrained use. To this end, in the open literature, a set of systematic methods have been developed to obtain explicit solutions for nonlinear evolution equation, such as tanh–coth function, sine–cosine function, Jacobi elliptic function method, symmetry method, Weierstrass function method, the F-expansion method, Homotopy perturbation method, variational iteration method [31–33] and so on. However, all methods mentioned above have some restrictions in their applications.

On the other hand, the Hirota bilinear formalism [1,2] has been successfully used in the search for exact solutions of continuous and discrete systems [20–22], and also in the search for new integrable equations by testing for multisoliton solutions or Bäcklund transformations, and even used in constructing *N*-soliton solutions for integrable couplings by perturbation [8]. It is now believed that most integrable systems if not all, could be transformed into bilinear forms by dependent variable transformations. Therefore, one would expect to study most integrable systems within the bilinear formalism. Moreover, various methods have been presented in the last four decades to construct exact solutions for many nonlinear evolution equations, such as Grammian determinant approach [1,23], and Wronskian determinant approach [4,15]. Wronskian determinant, Grammian determinant and Pfaffian solutions to the (3+1)-dimensional generalized *KP* and *BKP* equations were constructed in [6,7].

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In this work, our aim is to investigate two well-known models that are of particular interest in science. The (3+1)-dimensional Jimbo–Miwa equation [30]:

$$u_{xxxy} + 3u_{xx}u_y + 3u_x u_{yx} + 2u_{yt} - 3u_{xz} = 0, \tag{1}$$

and the (3+1)-dimensional nonlinear evolution equation [16]:

$$3w_{xz} - (2w_t + w_{xxx} - 2ww_x)_y + 2(w_x \partial_x^{-1} w_y)_x = 0, \tag{2}$$

where

$$u = u(x,y,z,t), \quad w = w(x,y,z,t), \quad f_x = \frac{\partial f}{\partial x} \quad \text{and} \quad \partial_x^{-1} f = \int f dx.$$

Both of which can be written in terms of the Hirota bilinear operator. Moreover, Eq. (1) was firstly investigated by Jimbo–Miwa and its soliton solutions were obtained in [30]. It is the second member in the entire Kadomtsev–Petviashvili hierarchy. Lately, Wazwaz [12,13] successfully studied one-soliton solutions to Eq. (1) by means of the Tanh–Coth method. He also employed the Hirota's bilinear method to the Jimbo–Miwa equation and confirmed that it is completely integrable and it admits multiple-soliton solutions of any order. In fact Eq. (2) was firstly investigated by Geng (see [16], where it was decomposed into systems of solvable ordinary differential equations with the help of the (1+1)-dimensional *AKNS* equations and algebraic-geometrical solutions for it were explicitly given in terms of the Riemann theta functions). Further, Geng and Ma [14] derived an N-soliton solution of the equation and its Wronskian form by using the Hirota direct method and Wronskian technique. Eq. (2) has Grammian determinant solution in [5] and Eq. (1) has Wronskian solution in [19].

In our present paper, we will show the above (3+1)-dimensional nonlinear evolution equations has a class of extended Grammian determinant solutions, with all generating functions for matrix entries satisfying a linear system of partial differential equations involving variable-coefficient, which guarantee that the Grammian determinant solves the equation. The Jacobi identity for determinants is the key to establish the Grammian formulation [18]. Moreover, a systematic analysis of linear partial differential equations is used to solve the representative linear systems.

A theory of transformation of surfaces initiated by Bäcklund [24] and later developed by Loewner [25] has, in recent years, proved to be of remarkable importance in the analysis of a wide range of physical phenomena, and the successful applications of this transformation theory to nonlinear evolution equations have led to a rekindling of interest in this topic. Perhaps the simplest example of Bäcklund transformation is the Cauchy–Riemann relations. In particular, Bäcklund transformations of the Sine-Gordon equation have generated results of interest in dislocation theory [26], in the study of long Josephson junctions [27], and in the investigation of propagation of long optical pulses through a resonant laser medium [28]. The work by Miura [29] on the Korteweg-de Vries equation has likewise involved the use of a Bäcklund transformation. In 1950, Loewner [25] introduced an important generalization of the concept of Bäcklund transformation. This was in connection with the reduction to canonical form of the well-known hodograph equations of gasdynamics. In Section 5, we would like to present a bilinear Bäcklund transformation for the above (3+1)-dimensional Jimbo–Miwa Eq. (1) and the (3+1)-dimensional nonlinear evolution Eq. (2). Furthermore, we will use the transformation to generate a new class of wave and rational solutions to the same equation.

2. Pfaffian and bilinear form

2.1. Pfaffian

Pfaffians, which may be an unfamiliar word, are closely related to determinants. They are usually defined by the property that the square of a Pfaffian is the determinant of an antisymmetric matrix. This feature leads often to the misunderstanding that a Pfaffian is a special case of a determinant. In fact, it is easy to recognize that the Pfaffians are a generalization of determinants. Therefore, Plücker relations and Jacobi identities, which are identities for determinants, also hold for Pfaffians.

In this paper, we will use Pfaffian identities [3] to search for exact solutions to the nonlinear partial differential equations: (1) and (2).

Let us discuss some basics about the Pfaffian [3]. Let $A = (\alpha_{jk})_{1 \leqslant j,k \leqslant m}$ be a skew-symmetric matrix, in which $\alpha_{j,k} = -\alpha_{k,j}$ for $j,k = 1,2,\ldots,m$. It is known that $\det(A)$ of odd order vanishes but $\det(A)$ of even order $m = 2n$ is the square of a Pfaffian, that is

$$\det(A) = \begin{cases} 0, & \text{if } m \text{ is odd}, \\ \mathrm{Pf}(\alpha_{jk})^2_{1 \leqslant j,k \leqslant m}, & \text{if } m \text{ is even}. \end{cases} \tag{3}$$

We can denote this Pfaffian Pf $(\alpha_{jk})_{1 \leqslant j,k \leqslant 2n}$ by

$$\mathrm{Pf}(\alpha_{jk})_{1 \leqslant j,k \leqslant 2n} = (1,2,3,\ldots,2n). \tag{4}$$

Then, we have

$$\begin{vmatrix} 0 & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ -\alpha_{12} & 0 & \alpha_{23} & \alpha_{24} \\ -\alpha_{13} & -\alpha_{23} & 0 & \alpha_{34} \\ -\alpha_{14} & -\alpha_{24} & -\alpha_{34} & 0 \end{vmatrix} = [\alpha_{12}\alpha_{34} - \alpha_{13}\alpha_{24} + \alpha_{14}\alpha_{23}]^2 \tag{5}$$

$$\equiv (1,2,3,4)^2. \tag{6}$$

Therefore, a second-order ($n = 2$) Pfaffian is given by

$$(1,2,3,4) = (1,2)(3,4) - (1,3)(2,4) + (1,4)(2,3), \tag{7}$$

where

$$(j,k) = \alpha_{j,k} \quad \text{for } j < k. \tag{8}$$

Also, it should be noted that from the antisymmetric $\alpha_{j,k} = -\alpha_{k,j}$, we have

$$(j,k) = -(k,j). \tag{9}$$

In general, we have an expansion rule for a Pfaffian $(1,2,\ldots,2n)$ of order n:

$$(1,2,\ldots,2n) = \sum_{j=2}^{2n} (-1)^j (1,j)(2,3,\ldots,\hat{j},\ldots,2n), \tag{10}$$

where the notation $\hat{j}$ means that the index j is omitted. An alternative expansion reads

$$(1,2,\ldots,2n) = \sum_{j=1}^{2n-1} (-1)^{j-1} (1,2,\ldots,\hat{j},\ldots,2n-1)(j,2n). \tag{11}$$

Repeating the above expansion, we arrive at the summation of products of first-order Pfaffians [3]:

$$(1,2,\ldots,2n) = \sum{}' (-1)^P (i_1,i_2)(i_3,i_4)(i_5,i_6)\ldots(i_{2n-1},i_{2n}), \tag{12}$$

where $\sum'$ means the sum over all possible combinations of pairs selected from $\{1,2,\ldots,2n\}$ that satisfy

$$i_1 < i_2, i_3 < i_4, i_5 < i_6 \ldots, i_{2n-1} < i_{2n}, i_1 < i_3 < \ldots < i_{2n-1}.$$

These first-order Pfaffians (i,j) are called the entries in the Pfaffian. In the above equation,The factor $(-1)^P = +1$ or -1 if the sequence $\{i_k\}_{k=1}^{2n}$ is an even or odd permutation of $1,2,\ldots,2n$.

Moreover, the Pfaffian $(i_1,i_2,\ldots,i_{2n})$ vanishes if $i_l = i_m$ for any pair of m and l chosen from $1,2,\ldots,2n$. Also, the interchange of labels i_l and i_m changes the parity of each permutation in the sum, and thus, the Pfaffian has the skew-symmetric property

$$(i_1,\ldots,i_l,\ldots,i_m,\ldots,i_{2n}) = -(i_1,\ldots,i_m,\ldots,i_l,\ldots,i_{2n}), \tag{13}$$

where $1 \leqslant l < m \leqslant 2n$. The Pfaffian also is denoted conventionally by Caianiello [17]

$$\mathrm{Pf}(\alpha_{i,j})_{1\leqslant i,j\leqslant 2n} = \left| \begin{matrix} \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,2n} \\ & \alpha_{2,3} & \cdots & \alpha_{2,2n} \\ & & \ddots & \vdots \\ & & & \alpha_{2n-1,2n} \end{matrix} \right|, \tag{14}$$

and when $n = 1$, 2, the Pfaffian reads

$$\mathrm{Pf}(\alpha_{i,j})_{1\leqslant i,j\leqslant 2} = \alpha_{1,2} = (1,2),$$
$$\mathrm{Pf}(\alpha_{i,j})_{1\leqslant i,j\leqslant 4} = \alpha_{1,2}\alpha_{3,4} - \alpha_{1,3}\alpha_{2,4} + \alpha_{1,4}\alpha_{2,3} = (1,2,3,4).$$

Moreover, the Pfaffian obeys an expansion rule

$$(\alpha_1,\alpha_2,\ldots,\alpha_{2N}) = \sum_{j=1}^{2N} (\alpha_i,\alpha_j)\Gamma(i,j), \quad 1 \leqslant i \leqslant 2N, \tag{15}$$

with the cofactor $\Gamma(i,j)$ being defined by

$$\Gamma(i,j) = (-1)^{i+j-1}\left(\alpha_2,\ldots,\hat{\alpha}_i,\ldots,\hat{\alpha}_j,\ldots\alpha_{2N}\right), \quad i < j,$$
$$\Gamma(j,i) = -\Gamma(i,j), \quad i > j, \ \Gamma(i,i) = 0,$$

where $\hat{\alpha}_k$ means that the label α_k is omitted. We have several expansion theorems on the Pfaffian. Below we describe two of them, which are relevant to the present paper.

Proposition 1 [1]. *Let A be a* $2n \times 2n$ *skew-symmetric matrix. Then*

$$\mathrm{Pf}(A) = (1,2,\ldots,2n) = \sum_{\sigma} sgn(\sigma)\prod_{i=1}^{n}(\sigma(2i-1),\sigma(2i)), \tag{16}$$

where the summation is taken over all permutations

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & 2n \\ i_1 & i_2 & \ldots & i_{2n} \end{pmatrix}$$

with

$$i_1 < i_2, i_3 < i_4, \ldots, i_{2n-1} < i_{2n}, i_1 < i_3 < \ldots < i_{2n-1},$$

and $sgn(\sigma) = (-1)^{inv(\sigma)}$.

We have several expansion theorems on the Pfaffian. Below we describe two of them, which are relevant to the present work.

Lemma 1 [3]. *Let n be a positive integer. Then*

$$\begin{aligned}(\alpha_1,\alpha_2,1,2,\ldots,2n) = {} & \sum_{j=2}^{2n}(-1)^j(\alpha_1,\alpha_2,1,j)[(2,3,\ldots,\hat{j},\ldots 2n) + (1,j)(\alpha_1,\alpha_2,2,3,\ldots,\hat{j},\ldots 2n)] \\ & -(\alpha_1,\alpha_2)(a_1,a_2,1,2,\ldots,2n),\end{aligned} \tag{17}$$

and

$$(\beta_1,\beta_2,\gamma_1,\gamma_2,1,2,\ldots,2n) = \sum_{j=1}^{2n}\sum_{k=j+1}^{2n}(-1)^{j+k-1}(\beta_1,\beta_2,j,k)\times(\gamma_1,\gamma_2,1,2,\ldots,\hat{j},\ldots,\hat{k},\ldots 2n), \tag{18}$$

provided that

$$(\beta_j,\gamma_k) = 0, \quad \text{for } j,k = 1,2.$$

We shall use Eqs. (17) and (18) to express the derivatives of the Pfaffain by the Pfaffians of lower order. In the next lemma we describe two of the identities of Pfaffians which correspond to the Jacobi identity of determinants.

Lemma 2 [3]. *Let m and n be positive integers. Then*

$$(\alpha_1,\alpha_2,\ldots,\alpha_{2m},1,2,\ldots,2n)(1,2,\ldots,2n) = \sum_{s=2}^{2m}(-1)^s(\alpha_1,\alpha_s,1,2,\ldots,2n)\times(\alpha_1,\alpha_2,\ldots,\hat{\alpha}_s,\ldots,\alpha_{2m},1,2,\ldots,2n), \tag{19}$$

and

$$(\alpha_1,\alpha_2,\ldots,\alpha_{2m-1},1,2,3,\ldots,2n-1)(1,2,\ldots,2n) = \sum_{s=1}^{2m-1}(-1)^{s-1}(\alpha_s,1,\ldots,2n-1)\times(\alpha_1,\alpha_2,\ldots,\hat{\alpha}_s,\ldots,\alpha_{2m-1},1,\ldots,2n). \tag{20}$$

2.2. Bilinear form

In this subsection, we would like to transform the (3+1)-dimensional nonlinear evolution Eqs. (1) and (2) into the bilinear forms by dependent variable transformations.

Through the dependent variable transformations:

$$u = 2(\ln f)_x \quad \text{and} \quad w = -3(\ln f)_{xx}, \tag{21}$$

the above (3+1)-dimensional nonlinear evolution Eqs. (1) and (2) are mapped into the Hirota bilinear equations:

$$(D_x^3 D_y + 2D_y D_t - 3D_x D_z)f\cdot f = 0, \tag{22}$$

where the bilinear differential operator D is defined by

$$D_t^k D_x^n D_y^m \varphi(t,x,y)\cdot\psi(t,x,y) = (\partial_t - \partial_{t'})^k(\partial_x - \partial_{x'})^n(\partial_y - \partial_{y'})^m\varphi(t,x,y)\psi(t',x',y')|_{t'=t,x'=x,y'=y.} \tag{23}$$

We can rewrite Eq. (22) in terms of f as follows

$$(f_{xxxy} + 2f_{yt} - 3f_{xz})f - f_{xxx}f_y - 3f_{xxy}f_x + 3f_{xx}f_{xy} - 2f_y f_t + 3f_x f_z = 0. \tag{24}$$

3. Sufficient conditions and Grammian solutions

3.1. Sufficient conditions

Let us introduce the following Grammian determinant:

$$f_N = \det(a_{ij})_{1\leqslant i,j\leqslant N}, \tag{25}$$

$$a_{ij} = \delta_{ij} + \int^x \phi_i\psi_j dx, \quad i,j = 1,2,\ldots,N, \ \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}, \tag{26}$$

and all the elements $\phi_i = \phi_i(x,y,z,t)$ and $\psi_j = \psi_j(x,y,z,t)$ satisfy the linear differential equations:

$$\begin{cases} \phi_{i,y} = 2\xi(t)\phi_{i,xx}, \\ \phi_{i,z} = 2\xi(t)\phi_{i,xxxx} + \sum_{k=1}^{N}\lambda_{ik}(t)\phi_k, \\ \phi_{i,t} = \phi_{i,xxx} + \sum_{k=1}^{N}\eta_{ik}(t)\phi_k, \\ \psi_{j,y} = -2\xi(t)\psi_{j,xx}, \\ \psi_{j,z} = -2\xi(t)\psi_{j,xxxx} + \sum_{l=1}^{N}\mu_{jl}(t)\psi_l, \\ \psi_{j,t} = \psi_{j,xxx} + \sum_{l=1}^{N}\rho_{jl}(t)\psi_l, \end{cases} \tag{27}$$

where $\xi(t)$, $\lambda_{ik}(t)$, $\mu_{jl}(t)$, $\eta_{ik}(t)$ and $\rho_{jl}(t)$ are arbitrary differentiable functions in t, and $\lambda_{ij}(t) + \mu_{ji}(t) = 0$, $\eta_{ij}(t) + \rho_{ji}(t) = 0$ for $i,j = 1,2,\ldots,N$. In what follows, as an application of the Pfaffian technique, we shall construct new Grammian solutions to the (3+1)-dimensional Jimbo–Miwa Eq. (1) and the (3+1)-dimensional nonlinear evolution Eq. (2).

Theorem 3 (Sufficient condition). *Let $\phi_i(x,y,z,t)$ and $\psi_j(x,y,z,t)$, $i,j = 1,2,\ldots,N$, satisfy (27), then the Grammian determinant f_N defined by (25) solves the Hirota bilinear Eq. (22) and the functions $u = 2(\ln f_N)_x$ and $w = -3(\ln f_N)_{xx}$ solves the (3+1)-dimensional Jimbo–Miwa Eq. (1) and the (3+1)-dimensional nonlinear evolution Eq. (2) respectively.*

Proof. Let us express the Grammian determinant f_N by means of a Pfaffian as

$$f_N = (1,2,\cdots,N,N^*,\cdots,2^*,1^*) = (\bullet), \tag{28}$$

where $(i,j^*) = a_{ij}$ and $(i,j) = (i^*,j^*) = 0$.

To compute the derivatives of the entries (i,j^*) and the Grammian f_N, we introduce new Pfaffian entries

$$\begin{aligned} &(d_n,j^*) = \frac{\partial^n}{\partial x^n}\psi_j, \quad (d_n^*,i) = \frac{\partial^n}{\partial x^n}\phi_i, \\ &(d_m,d_n^*) = (d_n,i) = (d_m^*,j^*) = 0, \quad m,n \geqslant 0, \end{aligned} \tag{29}$$

by using Eqs. (26) and (27), we can get

$$\frac{\partial}{\partial x}(i,j^*) = \phi_i\psi_j = (d_0,d_0^*,i,j^*), \tag{30}$$

$$\frac{\partial}{\partial y}(i,j^*) = \int^x(\phi_{i,y}\psi_j + \phi_i\psi_{j,y})dx = 2\xi(t)\int^x(\phi_{i,xx}\psi_j - \phi_i\psi_{j,xx})dx = 2\xi(t)(\phi_{i,x}\psi_j - \phi_i\psi_{j,x}) = 2\xi(t)[-(d_1,d_0^*,i,j^*) + (d_0,d_1^*,i,j^*)], \tag{31}$$

$$\begin{aligned} \frac{\partial}{\partial t}(i,j^*) &= \int^x(\phi_{i,t}\psi_j + \phi_i\psi_{j,t})dx = \int^x\left[\left(\phi_{i,xxx} + \sum_{k=1}^{N}\eta_{ik}\phi_k\right)\psi_j + \phi_i\left(\psi_{j,xxx} + \sum_{l=1}^{N}\rho_{jl}\psi_l\right)\right]dx \\ &= \int^x[\phi_{i,xxx}\psi_j + \phi_i\psi_{j,xxx}]dx + \sum_{k=1}^{N}\eta_{ik}\left(\delta_{kj} + \int^x\phi_k\psi_j dx\right) + \sum_{l=1}^{N}\rho_{jl}\left(\delta_{il} + \int^x\phi_i\psi_l dx\right) - \sum_{k=1}^{N}\eta_{ik}\delta_{kj} - \sum_{l=1}^{N}\rho_{jl}\delta_{il} \\ &= (\phi_{i,xx}\psi_j - \phi_{i,x}\psi_{j,x} + \phi_i\psi_{j,xx}) + \sum_{k=1}^{N}\eta_{ik}a_{kj} + \sum_{l=1}^{N}\rho_{jl}a_{il} - \eta_{ij} - \rho_{ji} \\ &= [(d_2,d_0^*,i,j^*) - (d_1,d_1^*,i,j^*) + (d_0,d_2^*,i,j^*)] + \sum_{k=1}^{N}\eta_{ik}a_{kj} + \sum_{l=1}^{N}\rho_{jl}a_{il}, \end{aligned} \tag{32}$$

$$\frac{\partial}{\partial z}(i,j^*) = \int^x \left[\left(2\xi\phi_{i,xxxx} + \sum_{k=1}^N \lambda_{ik}\phi_k\right)\right]\psi_j + \phi_i\left[\left(-2\xi\psi_{j,xxxx} + \sum_{l=1}^N \mu_{jl}\psi_l\right)\right]dx = 2\xi(t)\int^x (\phi_{i,xxxx}\psi_j - \phi_i\psi_{j,xxxx})dx$$
$$+ \sum_{k=1}^N \lambda_{ik}\left(\delta_{kj} + \int^x \phi_k\psi_j dx\right) + \sum_{l=1}^N \mu_{jl}\left(\delta_{il} + \int^x \phi_i\psi_l dx\right) - \sum_{k=1}^N \lambda_{ik}\delta_{kj} - \sum_{l=1}^N \mu_{jl}\delta_{il}$$
$$= 2\xi(t)(\phi_{i,xxx}\psi_j - \phi_{i,xx}\psi_{j,x} + \phi_{i,x}\psi_{j,xx} - \phi_i\psi_{j,xxx}) + \sum_{k=1}^N \lambda_{ik}a_{kj} + \sum_{l=1}^N \mu_{jl}a_{il} - \lambda_{ij} - \mu_{ji}$$
$$= 2\xi(t)[(d_0, d_3^*, i, j^*) - (d_1, d_2^*, i, j^*) + (d_2, d_1^*, i, j^*) - (d_3, d_0^*, i, j^*)] + \sum_{k=1}^N \lambda_{ik}a_{kj} + \sum_{l=1}^N \mu_{jl}a_{il}. \tag{33}$$

Therefore, from the above results (30)–(33), we have the following differential formulae for f_N:

$$f_{N,x} = (d_0, d_0^*, \bullet), \tag{34}$$
$$f_{N,y} = 2\xi[-(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)], \tag{35}$$
$$f_{N,t} = (d_2, d_0^*, \bullet) - (d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet) + \sum_{l=1}^N (\eta_{ll} + \rho_{ll})(\bullet), \tag{36}$$
$$f_{N,z} = 2\xi[(d_0, d_3^*, \bullet) - (d_1, d_2^*, \bullet) + (d_2, d_1^*, \bullet) - (d_3, d_0^*, \bullet)] + \sum_{l=1}^N (\lambda_{ll} + \mu_{ll})(\bullet), \tag{37}$$
$$f_{N,xx} = (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet), \tag{38}$$
$$f_{N,xy} = 2\xi[-(d_2, d_0^*, \bullet) + (d_0, d_2^*, \bullet)], \tag{39}$$
$$f_{N,xxx} = (d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet), \tag{40}$$
$$f_{N,xxy} = 2\xi[(d_0, d_3^*, \bullet) + (d_1, d_2^*, \bullet) - (d_2, d_1^*, \bullet) - (d_3, d_0^*, \bullet)], \tag{41}$$
$$f_{N,xz} = 2\xi[(d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet) + (d_2, d_1^*, d_0, d_0^*, \bullet) - (d_1, d_2^*, d_0, d_0^*, \bullet)] + \sum_{l=1}^N (\lambda_{ll} + \mu_{ll})(d_0, d_0^*, \bullet)$$
$$= 2\xi[(d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet) + (d_2, d_1^*, d_0, d_0^*, \bullet) - (d_1, d_2^*, d_0, d_0^*, \bullet)], \tag{42}$$
$$f_{N,yt} = 2\xi[(d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet) + (d_3, d_1^*, \bullet) - (d_1, d_3^*, \bullet) + (d_2, d_0^*, d_0, d_1^*, \bullet) - (d_0, d_2^*, d_1, d_0^*, \bullet)]$$
$$+ 2\xi\sum_{l=1}^N (\eta_{ll} + \rho_{ll})(d_0, d_1^*, \bullet) - 2\xi\sum_{l=1}^N (\eta_{ll} + \rho_{ll})(d_1, d_0^*, \bullet)$$
$$= 2\xi[(d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet) + (d_3, d_1^*, \bullet) - (d_1, d_3^*, \bullet) + (d_2, d_0^*, d_0, d_1^*, \bullet) - (d_0, d_2^*, d_1, d_0^*, \bullet)], \tag{43}$$
$$f_{N,xxxy} = 2\xi[(d_0, d_4^*, \bullet) - (d_4, d_0^*, \bullet) - 2(d_3, d_1^*, \bullet) + 2(d_1, d_3^*, \bullet) + (d_2, d_0^*, d_0, d_1^*, \bullet) - (d_0, d_2^*, d_1, d_0^*, \bullet)], \tag{44}$$

where we have used the abbreviated notation $\bullet = 1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*$. We can now compute that

$$(f_{N,xxxy} + 2f_{N,yt} - 3f_{N,xz})f = 12\xi\left[\frac{1}{2}(d_2, d_0^*, d_0, d_1^*, \bullet) - \frac{1}{2}(d_0, d_2^*, d_1, d_0^*, \bullet) - \frac{1}{2}(d_2, d_1^*, d_0, d_0^*, \bullet) + \frac{1}{2}(d_1, d_2^*, d_0, d_0^*, \bullet)\right](\bullet)$$
$$= 12\xi[(d_2, d_0^*, d_0, d_1^*, \bullet) - (d_0, d_2^*, d_1, d_0^*, \bullet)](\bullet),$$

$$(-f_{N,xxx}f_y - 3f_{N,xxy}f_x - 2f_{N,y}f_{N,t} + 3f_{N,x}f_{N,z})$$
$$= 12\xi\left[\frac{1}{2}(d_2, d_0^*, \bullet)(d_1, d_0^*, \bullet) - \frac{1}{2}(d_2, d_0^*, \bullet)(d_0, d_1^*, \bullet) + \frac{1}{2}(d_0, d_2^*, \bullet)(d_1, d_0^*, \bullet)\right.$$
$$\left. - \frac{1}{2}(d_0, d_2^*, \bullet)(d_0, d_1^*, \bullet) + (d_0, d_0^*, \bullet)(d_2, d_1^*, \bullet) - (d_0, d_0^*, \bullet)(d_1, d_2^*, \bullet)\right],$$

$$(3f_{N,xx}f_{N,xy}) = 12\xi\left[-\frac{1}{2}(d_2, d_0^*, \bullet)(d_1, d_0^*, \bullet) + \frac{1}{2}(d_0, d_2^*, \bullet)(d_1, d_0^*, \bullet) - \frac{1}{2}(d_2, d_0^*, \bullet)(d_0, d_1^*, \bullet) + \frac{1}{2}(d_0, d_2^*, \bullet)(d_0, d_1^*, \bullet)\right].$$

Substituting the above derivatives of f_N into the LHS of Eq. (24), we arrive at

$$(f_{N,xxxy} + 2f_{N,yt} - 3f_{N,xz})f - f_{N,xxx}f_y - 3f_{N,xxy}f_x + 3f_{N,xx}f_{N,xy} - 2f_{N,y}f_{N,t} + 3f_{N,x}f_{N,z}$$
$$= 12\xi[(d_0, d_0^*, d_2, d_1^*, \bullet)(\bullet) - (d_0, d_0^*, \bullet)(d_2, d_1^*, \bullet) - (d_0^*, d_2, \bullet)(d_0, d_1^*, \bullet)]$$
$$- 12\xi[(d_0, d_0^*, d_1, d_2^*, \bullet)(\bullet) - (d_0, d_0^*, \bullet)(d_1, d_2^*, \bullet) - (d_0^*, d_1, \bullet)(d_0, d_2^*, \bullet)] = 0, \tag{45}$$

where we have made use of the known Jacobi identities for determinants. This shows that the Grammian determinant $f_N = (\bullet)$, with the conditions (27) solves the Hirota bilinear Eq. (22) and the functions $u = 2(\ln f_N)_x$ and $w = -3(\ln f_N)_{xx}$ solves the (3+1)-dimensional Jimbo–Miwa Eq. (1) and the (3+1)-dimensional nonlinear evolution Eq. (2) respectively. This ends the proof. □

From the above theorem, we can see that if a set of functions $\phi_i(x,y,z,t)$ and $\psi_j(x,y,z,t)$, $1 \leqslant i,j \leqslant N$, satisfy the conditions (27), then the determinant $f_N = \det(a_{ij})$ is an exact solution to the Hirota bilinear Eq. (22), where the entry a_{ij} is defined in (26). The conditions (27) consist of three linear systems of second-order, third-order and fourth-order partial differential equations. It is rather difficult to solve those linear systems explicitly, but based on the idea used in [9], we can find some special solutions to the present linear systems. Before we proceed to solve (27), let us observe the Grammian determinants and solutions more carefully.

Remark 4. From the compatibility conditions $\phi_{i,yt} = \phi_{i,ty}$, $1 \leqslant i \leqslant N$, we have the equality

$$2\xi_t(t)\phi_{i,xx} = 0, \quad 1 \leqslant i \leqslant N. \tag{46}$$

Therefore, in the following discussion we assume that $\xi(t) = \xi_0$.

Remark 5. From the compatibility conditions $\phi_{i,zt} = \phi_{i,tz}$, $1 \leqslant i \leqslant N$, we have the equality

$$\sum_{k=1}^{N} \lambda_{ik,t}\phi_k = 0, \quad 1 \leqslant i \leqslant N. \tag{47}$$

Therefore, we note that if there is one entry λ_{ij} satisfying $\lambda_{ij,t} \neq 0$, then $\phi_j = 0$. So in the following discussion, we assume that the coefficient matrix $\Lambda = (\lambda_{ij})$ is a real constant matrix.

Remark 6. If the coefficient matrix $\Lambda = (\lambda_{ij})$ is similar to another matrix $\Theta = (\theta_{ij})$ under an invertible constant matrix P, that is $P^{-1}\Lambda P = \Theta$, and if we take the notation $\Phi = (\phi_1, \phi_2, \ldots, \phi_N)^T$, then $\overline{\Phi} = P^{-1}\Phi$ solves

$$\overline{\Phi}_y = 2\xi_0\overline{\Phi}_{xx}, \quad \overline{\Phi}_z - 2\xi_0\overline{\Phi}_{xxxx} = \Theta\overline{\Phi}, \quad \overline{\Phi}_t = \overline{\Phi}_{xxx}. \tag{48}$$

The coefficient matrix $\Theta = (\theta_{ij})_{N\times N}$ is equal to $-\Lambda^T$. If we take the notation $\Psi = (\psi_1, \psi_2, \ldots, \psi_N)^T$, then $\overline{\Psi} = P^T\Psi$ solves

$$\overline{\Psi}_y = 2\xi_0\overline{\Psi}_{xx}, \quad \overline{\Psi}_z - 2\xi_0\overline{\Psi}_{xxxx} = -\Theta^T\overline{\Psi}, \quad \overline{\Psi}_t = \overline{\Psi}_{xxx}. \tag{49}$$

We can rewrite the Grammian matrix (26) to $I + A$, where I is the identity matrix, and the element of matrix A is $A(i,j) = \int^x \phi_i\psi_j dx$. Noting that $\overline{\Phi} = P^{-1}\Phi$ and $\overline{\Psi}^T = \Psi^T P$, we have the following result:

$$\det\left(\delta_{ij} + \int^x \phi_i\psi_j dx\right)_{N\times N} = \det(I + A) = \det(P^{-1}(I + A)P) = \det\left(I + \int^x \overline{\phi}_i\overline{\psi}_j dx\right).$$

It follows that the resultant Grammian determinant solutions to the (3+1)-dimensional Jimbo–Miwa Eq. (1) and the (3+1)-dimensional nonlinear evolution Eq. (2) keep the same under the similar transformation. Therefore, based on the above Remarks, in order to construct Grammian determinant solutions to the (3+1)-dimensional nonlinear partial differential Eq. (1), we only need to consider the reduced case of (27) under $\xi_0 = 1/2$, $\eta_{ij} = \rho_{ij} = 0$ and $d\Lambda/dt = 0$, i.e., the following conditions:

$$\begin{cases} \phi_{i,y} = \phi_{i,xx}, \phi_{i,z} = \phi_{i,xxxx} + \displaystyle\sum_{k=1}^{N} \lambda_{ik}\phi_k, \quad \phi_{i,t} = \phi_{i,xxx}, \\ \psi_{j,y} = -\psi_{j,xx}, \psi_{j,z} = -\psi_{j,xxxx} + \displaystyle\sum_{l=1}^{N} \mu_{jl}\psi_l, \quad \psi_{j,t} = \psi_{j,xxx}, \\ \lambda_{ij} + \mu_{ji} = 0, \quad 1 \leqslant i,j \leqslant N, \end{cases} \tag{50}$$

where λ_{ij} are arbitrary real constants. On the other hand, the Jordan form of a real matrix has the following types of the blocks:

$$\begin{pmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{pmatrix}_{k_i\times k_i}, \tag{51}$$

$$\begin{pmatrix} \Xi_i & & & 0 \\ I_2 & \Xi_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & \Xi_i \end{pmatrix}_{l_i\times l_i}, \quad \Xi_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}, \ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{52}$$

where λ_i, α_i and $\beta_i > 0$ are all real constants. The first type of blocks has the real eigenvalue λ_i with algebraic multiplicity k_i, and the second type of blocks has the complex eigenvalues $\pm\lambda_i = \alpha_i \pm \beta_i\sqrt{-1}$ with algebraic multiplicity l_i. We will construct solutions for the system of differential equations defined by (50), according to the situations of eigenvalues of the coefficient matrix. Based on Remark 4.5.4, all we need to do is to solve a group of subsystems in the sufficient condition on the Grammian determinant solutions, whose coefficient matrixes are of the forms (51) and (52).

4. Some solutions for the representative systems

In this section, we would like to construct some solutions to the associated system of differential equations defined by (50). For a nonzero real eigenvalue λ_i, we start from the eigenfunction $\phi_i(\lambda_i)$ determined by

$$\begin{cases} (\phi_i(\lambda_i))_y = (\phi_i(\lambda_i))_{xx}, \\ (\phi_i(\lambda_i))_z - (\phi_i(\lambda_i))_{xxxx} = \lambda_i \phi_i(\lambda_i), \\ (\phi_i(\lambda_i))_t = (\phi_i(\lambda_i))_{xxx}. \end{cases} \tag{53}$$

Two special solutions to this system in two cases of $\lambda_i > 0$ and $\lambda_i < 0$ are

$$\phi_i(\lambda_i) = e^{\frac{1}{4}\delta_i^2(-2y+(1+\delta_i^2)z)}(C_{1i}\sin\vartheta + C_{2i}\cos\vartheta), \quad \delta_i = \sqrt{\lambda_i},$$

$$\phi_i(\lambda_i) = e^{\frac{1}{4}\delta_i^2(2y-(1+\delta_i^2)z)}(C_{1i}\sinh\vartheta + C_{2i}\cosh\vartheta), \quad \delta_i = \sqrt{-\lambda_i},$$

where

$$\vartheta = \frac{1}{\sqrt{2}}\delta_i x + \frac{1}{2\sqrt{2}}\delta_i^3 t,$$

and C_{1i} and C_{2i} are arbitrary real constants. By an inspection, we find that

$$(\partial_z - \partial_x^4)\begin{pmatrix} \phi_i(\lambda_i) \\ \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i) \\ \vdots \\ \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i) \end{pmatrix} = \begin{pmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{pmatrix}_{k_i\times k_i}\begin{pmatrix} \phi_i(\lambda_i) \\ \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i) \\ \vdots \\ \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i) \end{pmatrix},$$

$$\left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_y = \left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_{xx}, \quad 0 \leqslant j \leqslant k_{i-1},$$

and

$$\left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_t = \left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_{xxx}, \quad 0 \leqslant j \leqslant k_{i-1},$$

where ∂_{λ_i} denotes the derivatives with respect to λ_i and k_i is an arbitrary non-negative integer. Noting that the coefficient matrix (μ_{ij}) of ψ_i satisfies $\mu_{ij} + \lambda_{ij} = 0$, we take

$$\begin{cases} (\psi_i(\lambda_i))_y = -(\psi_i(\lambda_i))_{xx}, \\ (\psi_i(\lambda_i))_z + (\psi_i(\lambda_i))_{xxxx} = -\lambda_i \psi_i(\lambda_i), \\ (\psi_i(\lambda_i))_t = (\psi_i(\lambda_i))_{xxx}. \end{cases} \tag{54}$$

Two special solutions to this system in two cases of $\lambda_i > 0$ and $\lambda_i < 0$ are

$$\psi_i(\lambda_i) = e^{\frac{1}{4}\delta_i^2(2y-(1+\delta_i^2)z)}(D_{1i}\sin\vartheta + D_{2i}\cos\vartheta), \quad \delta_i = \sqrt{\lambda_i},$$

$$\psi_i(\lambda_i) = e^{\frac{1}{4}\delta_i^2(-2y+(1+\delta_i^2)z)}(D_{1i}\sinh\vartheta + D_{2i}\cosh\vartheta), \quad \delta_i = \sqrt{-\lambda_i},$$

where

$$\vartheta = \frac{1}{\sqrt{2}}\delta_i x + \frac{1}{2\sqrt{2}}\delta_i^3 t,$$

and D_{1i} and D_{2i} are arbitrary real constants. Then we can find that

$$(\partial_z + \partial_x^4)\begin{pmatrix} \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\psi_i(\lambda_i) \\ \vdots \\ \frac{1}{1!}\partial_{\lambda_i}\psi_i(\lambda_i) \\ \psi_i(\lambda_i) \end{pmatrix} = \begin{pmatrix} -\lambda_i & -1 & & 0 \\ & -\lambda_i & \ddots & \\ & & \ddots & -1 \\ 0 & & & -\lambda_i \end{pmatrix}_{k_i\times k_i}\begin{pmatrix} \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\psi_i(\lambda_i) \\ \vdots \\ \frac{1}{1!}\partial_{\lambda_i}\psi_i(\lambda_i) \\ \psi_i(\lambda_i) \end{pmatrix}$$

Therefore, through these two sets of eigenfunctions, we can construct Grammian determinant entry and then obtain solutions to the bilinear Eq. (22). For the second type of Jordan blocks of the coefficient matrix, we set a pair of eigenfunctions:

$$\Phi_i(\alpha_i,\beta_i) = (\phi_{i1}(\alpha_i,\beta_i),\phi_{i2}(\alpha_i,\beta_i))^T,$$

is determined by

$$\Phi_{iz} - \Phi_{ixxxx} = \Xi\Phi_i, \quad \Phi_i = \begin{pmatrix}\phi_{i1}(\alpha_i,\beta_i)\\ \phi_{i2}(\alpha_i,\beta_i)\end{pmatrix}, \ \Xi_i = \begin{pmatrix}\alpha_i & -\beta_i\\ \beta_i & \alpha_i\end{pmatrix}, \tag{55}$$

and

$$\left(\phi_{ij}(\alpha_i,\beta_i)\right)_y = \left(\phi_{ij}(\alpha_i,\beta_i)\right)_{xx}, \quad \left(\phi_{ij}(\alpha_i,\beta_i)\right)_t = \left(\phi_{ij}(\alpha_i,\beta_i)\right)_{xxx}, \qquad j=1,2.$$

Again by inspection, one can see that

$$\left(\partial_z - \partial_x^4\right)\begin{pmatrix}\Phi_i\\ \frac{1}{1!}\partial_{\alpha_i}\Phi_i\\ \vdots\\ \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{k_i-1}\Phi_i\end{pmatrix} = \begin{pmatrix}\Xi_i & & & 0\\ I_2 & \Xi_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & \Xi_i\end{pmatrix}_{l_i\times l_i}\begin{pmatrix}\Phi_i\\ \frac{1}{1!}\partial_{\alpha_i}\Phi_i\\ \vdots\\ \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{k_i-1}\Phi_i\end{pmatrix}.$$

Similarly, if

$$\Psi_i(\alpha_i,\beta_i) = (\psi_{i1}(\alpha_i,\beta_i),\psi_{i2}(\alpha_i,\beta_i))^T,$$

is determined by

$$\Psi_{iz} + \Psi_{ixxxx} = -\Xi\Psi_i, \quad \Psi_i = \begin{pmatrix}\psi_{i1}(\alpha_i,\beta_i)\\ \psi_{i2}(\alpha_i,\beta_i)\end{pmatrix}, \ \Xi_i = \begin{pmatrix}\alpha_i & -\beta_i\\ \beta_i & \alpha_i\end{pmatrix}, \tag{56}$$

and

$$\left(\psi_{ij}(\alpha_i,\beta_i)\right)_y = \left(\psi_{ij}(\alpha_i,\beta_i)\right)_{xx}, \quad \left(\psi_{ij}(\alpha_i,\beta_i)\right)_t = \left(\psi_{ij}(\alpha_i,\beta_i)\right)_{xxx}, \qquad j=1,2.$$

Again by inspection, one can see that

$$\left(\partial_z + \partial_x^4\right)\begin{pmatrix}\Psi_i\\ \frac{1}{1!}\partial_{\alpha_i}\Psi_i\\ \vdots\\ \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{k_i-1}\Psi_i\end{pmatrix} = \begin{pmatrix}-\Xi_i & -I_2 & & 0\\ & -\Xi_i & \ddots & \\ & & \ddots & -I_2\\ 0 & & & -\Xi_i\end{pmatrix}_{l_i\times l_i}\begin{pmatrix}\Psi_i\\ \frac{1}{1!}\partial_{\alpha_i}\Psi_i\\ \vdots\\ \frac{1}{(l_i-1)!}\partial_{\alpha_i}^{k_i-1}\Psi_i\end{pmatrix}.$$

One special solution to Eqs. (55) and (56) are given as follows

$$\begin{cases}\phi_{i1} = \exp(x+y+t+(1+\alpha_i)z)\cos(\beta_i z),\\ \phi_{i2} = \exp(x+y+t+(1+\alpha_i)z)\sin(\beta_i z),\\ \psi_{i1} = \exp(x+y+t-(1+\alpha_i)z)\cos(\beta_i z),\\ \psi_{i2} = -\exp(x+y+t-(1+\alpha_i)z)\sin(\beta_i z),\end{cases} \tag{57}$$

where the parameters α_i and β_i are arbitrary constants.

5. Bilinear Bäcklund transformation

5.1. Bäcklund transformation

The Bäcklund transformations are essentially defined as a pair of partial differential relations involving two independent variables and their derivatives which together imply that each one of the dependent variables satisfies separately a partial differential equation. Thus, for example, the transformation

$$L_1(u,u_t,u_x,u_y,u_{xx},\ldots) \quad \text{and} \quad L_2(v,v_t,v_x,v_y,v_{xx},\ldots),$$

would imply that u and v satisfy partial differential equations of the operational form,

$$P(u)=0 \quad \text{and} \quad Q(v)=0.$$

In this paper we would like to present a bilinear Bäcklund transformation for the above (3+1)-dimensional nonlinear evolution equations (1) and (2).

Let us suppose that we have another solution f' to the generalized bilinear Eq. (22):

$$(D_x^3D_y + 2D_yD_t - 3D_xD_z)f' \cdot f' = 0, \tag{58}$$

and then we introduce the key function

$$P = [(D_x^3D_y + 2D_yD_t - 3D_xD_z)f \cdot f]f'^2 - [(D_x^3D_y + 2D_yD_t - 3D_xD_z)f' \cdot f']f^2. \tag{59}$$

If $P = 0$ then f solves the bilinear Eq. (22) if and only if f' solves the bilinear Eq. (22), that is

$$f \text{ is a solution of } (22) \Longleftrightarrow f' \text{ is a solution of } (22).$$

Therefore, if we can obtain, from $P = 0$ by interchanging the dependent variables f and f', a system of bilinear equations that guarantees $P = 0$:

$$\begin{aligned} &F_1(D_t, D_x, D_y, D_z)f \cdot f' = 0,\\ &F_2(D_t, D_x, D_y, D_z)f \cdot f' = 0,\\ &\vdots\\ &F_M(D_t, D_x, D_y, D_z)f \cdot f' = 0, \end{aligned}$$

where the $F_i's$ are polynomials in the indicated variables and M is natural number depending on the complexity of the equation. It is known that the identities of Hirota's bilinear operators are necessarily to split P into a system of polynomials $F_i's$.

Let us now introduce the following useful identities for Hirota's bilinear operators:

$$D_\eta(D_\zeta a \cdot b) \cdot ba = D_\zeta(D_\eta a \cdot b) \cdot ba, \tag{60}$$

$$D_\eta ab \cdot cd = (D_\eta a \cdot d)cb - ad(D_\eta c \cdot b), \tag{61}$$

$$b^2(D_\zeta^2 a \cdot a) - (D_\zeta^2 b \cdot b)a^2 = 2D_\zeta(D_\zeta a \cdot b) \cdot ba, \tag{62}$$

$$b^2(D_\eta D_\zeta a \cdot a) - (D_\eta D_\zeta b \cdot b)a^2 = 2D_\zeta(D_\eta a \cdot b) \cdot ba, \tag{63}$$

and

$$\begin{aligned} b^2(2D_\eta^3D_\zeta a \cdot a) - a^2(2D_\eta^3D_\zeta b \cdot b) = {} & D_\eta(3D_\eta^2D_\zeta a \cdot b) \cdot ba + D_\eta(3D_\eta^2 a \cdot b) \cdot (D_\zeta b \cdot a) + D_\eta(6D_\eta D_\zeta a \cdot b) \cdot (D_\eta b \cdot a)\\ & + D_\zeta(D_\eta^3 a \cdot b) \cdot ba + D_\zeta(3D_\eta^2 a \cdot b) \cdot (D_\eta b \cdot a). \end{aligned} \tag{64}$$

Noting that, the above identities (60)–(63) can be found in [1], and the identity (64) can be obtained by making the independent variable transformation $D_\eta \to D_\eta + \epsilon D_\zeta$ in the bilinear identity [1]

$$b^2(D_\eta^4 a \cdot a) - a^2(D_\eta^4 b \cdot b) = 2D_\eta[(D_\eta^3 a \cdot b) \cdot ba + 3(D_\eta^2 a \cdot b) \cdot (D_\eta b \cdot a)],$$

and comparing the coefficient of ϵ. In the above identities a, b are arbitrary continuous functions of the independent variables η, ζ. For more identities and general exchange formulas you may see [1]. Applying the above identities on Eq. (59) we can obtain

$$f'^2(4D_yD_tf \cdot f) - f^2(4D_yD_tf' \cdot f') = 8D_y(D_tf \cdot f') \cdot f'f, \tag{65}$$

$$f^2(6D_xD_zf' \cdot f') + f'^2(6D_xD_zf \cdot f) = -12D_x(D_zf \cdot f') \cdot f'f, \tag{66}$$

and

$$\begin{aligned} f'^2(2D_x^3D_yf \cdot f) - f^2(2D_x^3D_yf' \cdot f') = {} & D_x(3D_x^2D_yf \cdot f') \cdot f'f + D_x(3D_x^2f \cdot f') \cdot (D_yf' \cdot f) + D_x(6D_xD_yf \cdot f') \cdot (D_xf' \cdot f)\\ & + D_y(D_x^3f \cdot f') \cdot f'f + D_y(3D_x^2f \cdot f') \cdot (D_xf' \cdot f). \end{aligned} \tag{67}$$

Substituting the above results into the right-hand side of Eq. (59) we may obtain

$$\begin{aligned} 2P = {} & 8D_y(D_tf \cdot f') \cdot f'f + D_y(D_x^3f \cdot f') \cdot f'f - 12D_x(D_zf \cdot f') \cdot f'f + D_x(3D_x^2D_yf \cdot f') \cdot f'f + D_x(3D_x^2f \cdot f') \cdot (D_yf' \cdot f)\\ & + D_x(6D_xD_yf \cdot f') \cdot (D_xf' \cdot f) + D_y(3D_x^3f \cdot f') \cdot (D_xf' \cdot f). \end{aligned} \tag{68}$$

Lemma 7. *Let f and f' be arbitrary continuous functions of independent variables x, y, z, t. Then $D_x(D_zf \cdot f') \cdot (D_xf \cdot f') = D_z(D_x^2f \cdot f') \cdot ff' - D_x(D_xD_zf \cdot f') \cdot ff'$.*

Let us now introduce new arbitrary parameters λ, μ, ξ, ϑ and ε_i, $(i = 1, 2, 3)$, into Eq. (68) to obtain

$$\begin{aligned} 2P_\pm = {} & D_y[(8D_t + D_x^3 \pm \mu D_x^2 \pm \lambda D_x \pm \varepsilon_1)f \cdot f'] \cdot f'f + D_x[(3D_x^2D_y - 12D_z \mp \mu D_xD_y \mp \lambda D_y \pm \varepsilon_2)f \cdot f'] \cdot f'f + D_x[(3D_x^2\\ & \pm \vartheta D_y \pm \varepsilon_3)f \cdot f'] \cdot (D_yf' \cdot f) + D_x[(6D_xD_y \mp \mu D_y)f \cdot f'] \cdot (D_xf' \cdot f) + D_y[(3D_x^2 \pm \xi D_x \mp \varepsilon_3)f \cdot f'] \cdot (D_xf' \cdot f). \end{aligned} \tag{69}$$

This is possible because the coefficients of λ, μ, ξ, ϑ and ε_i, $(i = 1, 2, 3)$,

$$\lambda : \pm D_y[D_x f \cdot f'] \cdot f'f \mp D_x[D_y f \cdot f'] \cdot f'f,$$
$$\mu : \pm D_x(D_z f \cdot f') \cdot (D_x f \cdot f') \mp D_z(D_x^2 f \cdot f') \cdot ff' \pm D_x(D_x D_z f \cdot f') \cdot ff'$$
$$\xi : \pm D_y[D_x f \cdot f'] \cdot (D_x f' \cdot f),$$
$$\vartheta : \pm D_x[D_y f \cdot f'] \cdot (D_y f' \cdot f),$$
$$\varepsilon_1 : \pm D_y[ff'] \cdot f'f,$$
$$\varepsilon_2 : \pm D_x[ff'] \cdot f'f,$$
$$\varepsilon_3 : \pm D_x[ff'] \cdot (D_y f' \cdot f) \mp D_y[ff' \cdot (D_x f' \cdot f),$$

are all equal to zero because of Lemma 5, and the properties

$$D_\zeta f \cdot f = 0, \tag{70}$$
$$D_\zeta f \cdot g = -D_\zeta g \cdot f, \tag{71}$$
$$D_\eta(D_\zeta f \cdot g) \cdot gf = D_\zeta(D_\eta f \cdot g) \cdot gf. \tag{72}$$

Then $P_\pm = 0$ if $F_i f \cdot f' = 0, 1 \leqslant i \leqslant 5$, where $F_i's$ can be found from Eq. (69) as follows

$$\begin{cases} F_1 f \cdot f' \equiv (8D_t + D_x^3 \pm \mu D_x^2 \pm \lambda D_x \pm \varepsilon_1) f \cdot f' = 0, \\ F_2 f \cdot f' \equiv (3D_x^2 D_y - 12 D_z \mp \mu D_x D_y \mp \lambda D_y \pm \varepsilon_2) f \cdot f' = 0, \\ F_3 f \cdot f' \equiv (3D_x^2 \pm \vartheta D_y \pm \varepsilon_3) f \cdot f' = 0, \\ F_4 f \cdot f' \equiv (6 D_x D_y \mp \mu D_y) f \cdot f' = 0, \\ F_5 f \cdot f' \equiv (3D_x^2 \pm \xi D_x \mp \varepsilon_3) f \cdot f' = 0. \end{cases} \tag{73}$$

Since the coefficients of λ, μ, ξ, ϑ, ε_1, ε_2 and ε_3 are zero because of Eqs. (70)–(72) and Lemma 5, this shows that Eqs. (73) presents a Bäcklund transformation for the (3+1)-dimensional nonlinear evolution Eqs. (1) and (2).

5.2. Traveling wave solutions

In what follows, as an application of the bilinear Bäcklund transformation (73), we shall construct new solutions to the (3+1)-dimensional soliton Eqs. (2) and (1). For this purpose, we start with $f = 1$, which is the trivial solution of Eq. (22) obviously. Noting that

$$D_\zeta^n \phi \cdot 1 = \frac{\partial^n}{\partial \zeta^n} \phi, \quad n \geqslant 1, \tag{74}$$

then, the bilinear Bäcklund transformation (73) associated with $f = 1$ becomes a system of linear partial differential equations

$$\begin{cases} 8\frac{\partial f'}{\partial t} + \frac{\partial^3 f'}{\partial x^3} \pm \mu \frac{\partial^2 f'}{\partial x^2} \pm \lambda \frac{\partial f'}{\partial x} \pm \varepsilon_1 f' = 0, \\ 3\frac{\partial^3 f'}{\partial x^2 \partial y} - 12\frac{\partial f'}{\partial z} \mp \mu \frac{\partial^2 f'}{\partial x \partial y} \mp \lambda \frac{\partial f'}{\partial y} \pm \varepsilon_2 f' = 0, \\ 3\frac{\partial^2 f'}{\partial x^2} \pm \vartheta \frac{\partial f'}{\partial y} \pm \varepsilon_3 f' = 0, \\ 6\frac{\partial^2 f'}{\partial x \partial y} \mp \mu \frac{\partial f'}{\partial y} = 0, \\ 3\frac{\partial^2 f'}{\partial x^2} \pm \xi \frac{\partial f'}{\partial x} \mp \varepsilon_3 f' = 0. \end{cases} \tag{75}$$

Let us consider a class of exponential wave solutions of the form

$$f' = 1 + \varepsilon e^{kx + ly + mz - \omega t + \zeta^0}, \quad \zeta^0 = const, \tag{76}$$

where ε, k, l, m and ω are constants to be determined. Upon selecting

$$\varepsilon_1 = 0, \quad \varepsilon_2 = 0, \quad \varepsilon_3 = 0. \tag{77}$$

After tedious but straightforward calculations we get

$$\begin{cases} m_\mp = -\frac{1}{12} l(3k^2 \mp \lambda), \quad \omega_\pm = \frac{1}{8} k(7k^2 \pm \lambda), \\ \mu_\pm = \pm 6k, \quad \xi_\mp = \mp 3k, \quad \vartheta_\mp = \mp 3\frac{k^2}{l}. \end{cases} \tag{78}$$

Therefore, we obtain a class of exponential wave solutions to the (3+1)-dimensional bilinear Eq. (22):

$$f'_\pm = 1 + \varepsilon \exp\left(kx + ly - \frac{l(3k^2 \mp \lambda)}{12} z - \frac{k(7k^2 \pm \lambda)}{8} t + \zeta^0\right), \tag{79}$$

where ε, k, l, λ and ζ^0 are arbitrary constants; and

$$u = 2(\ln f'_{\pm})_x, \tag{80}$$

solves the (3+1)-dimensional Jimbo–Miwa Eq. (1), and

$$w = -3(\ln f'_{\pm})_{xx}, \tag{81}$$

solves the (3+1)-dimensional nonlinear evolution Eq. (2).

Let us second consider a class of first-order polynomial solutions

$$f' = kx + ly + mz - \omega t, \tag{82}$$

where k, l, m and ω are constants to be determined. Similarly Upon selecting

$$\varepsilon_1 = 0, \quad \varepsilon_2 = 0, \quad \varepsilon_3 = 0, \tag{83}$$

a direct computation shows that the system (75) becomes

$$\begin{cases} -8\omega \pm \lambda k = 0, \\ -12m \mp \lambda l = 0, \\ \vartheta l = 0, \quad \mu l = 0, \quad \xi k = 0. \end{cases} \tag{84}$$

Again after straightforward calculations we get

$$m_{\mp} = \mp \frac{1}{12}\lambda l, \quad \omega_{\pm} = \pm \frac{1}{12}\lambda k. \tag{85}$$

Therefore, we obtain a class of solutions to the (3+1)-dimensional bilinear Eq. (22):

$$f'_{\pm} = kx + ly \mp \frac{\lambda l}{12} z \mp \frac{1}{12}\lambda kt + \zeta^0, \tag{86}$$

where k, l, λ and ζ^0 are arbitrary constants; and

$$u = 2(\ln f'_{\pm})_x = \frac{2k}{kx + ly + \mp \frac{\lambda l}{12} z \mp \frac{1}{12}\lambda kt + \zeta}, \tag{87}$$

produces a class of rational solutions to the (3+1)-dimensional Jimbo–Miwa Eq. (1), and

$$w = -3(\ln f'_{\pm})_{xx} = \frac{-3k^2}{\left(kx + ly \mp \frac{\lambda l}{12} z \mp \frac{1}{12}\lambda kt + \zeta^0\right)^2}, \tag{88}$$

produces a class of rational solutions to the (3+1)-dimensional nonlinear evolution Eq. (2).

6. Conclusions and remarks

We have built an extended Grammian formulation for the (3+1)-dimensional nonlinear evolution equations:

$$u_{xxxy} + 3u_{xx}u_y + 3u_x u_{yx} + 2u_{yt} - 3u_{xz} = 0,$$

and

$$3w_{xz} - (2w_t + w_{xxx} - 2ww_x)_y + 2(w_x \partial_x^{-1} w_y)_x = 0,$$

The facts used in our construction are the Jacobi identity for determinants. Theorem 3 presents the main results on Grammian solutions, which say that

$$u = 2\frac{\partial}{\partial x}(\ln f_N), \quad f_N = \det(a_{ij})_{1 \leqslant i,j \leqslant 2N},$$

$$w = -3\frac{\partial^2}{\partial x^2}(\ln f_N), \quad f_N = \det(a_{ij})_{1 \leqslant i,j \leqslant 2N},$$

where the elements of f_N are defined by $a_{ij} = \delta_{ij} + \int^x \phi_i \psi_j dx$, $i, j = 1, 2, \ldots, 2N$, with ϕ_i and ψ_j satisfying

$$\phi_{i,y} = 2\xi \phi_i^{(2)}, \quad \phi_{i,z} = 2\xi \phi_i^{(4)} + \sum_{k=1}^{2N} \lambda_{ik}\phi_k, \quad \phi_{i,t} = \phi_i^{(3)} + \sum_{k=1}^{2N} \eta_{ik}\phi_k,$$

$$\psi_{j,y} = -2\xi \psi_j^{(2)}, \quad \psi_{j,z} = -2\xi \psi_j^{(4)} + \sum_{l=1}^{2N} \mu_{jl}\psi_l, \quad \psi_{j,t} = \psi_j^{(3)} + \sum_{l=1}^{2N} \rho_{jl}\psi_l,$$

where ξ, λ, μ, η and ρ are arbitrary continuous function in t, solves the above (3+1)-dimensional nonlinear evolution equations. In Theorem 3, we only considered specific sufficient conditions: (27), though, there is a free set of continuous functions in the conditions. In Section 4, we constructed some solutions for the representative systems of sufficient conditions (27), The conditions (27) are a generalization to one given in [5], which deal with Eq. (2) only, but in this work, we deal with both Eqs. (1) and (2). Actually if $\xi(t) = 1/2$ and the coefficient matrix $= (\lambda_{ij})$ and the coefficient matrix $= (\eta_{ij})$ are zero, then the result of the above theorem boils down to result in [5] on the study of Eq. (2). The bilinear Bäcklund transformations were furnished for the (3+1)-dimensional nonlinear evolution Eqs. (1) and (2), based on the existence of exchange identities for Hirota bilinear operators. In Section 5.2, We constructed a new class of exact wave solutions and a new class of rational solutions to the above (3+1)-dimensional nonlinear evolution Eqs. (1) and (2) of the forms

$$u = 2\frac{\partial}{\partial x}(\ln f'_{\pm}) \quad \text{and} \quad w = -3\frac{\partial^2}{\partial x^2}(\ln f'_{\pm}),$$

respectively.

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