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Double breather and peak solitons of a semi discrete short pulse equation

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In this paper, we define and explore the Darboux and binary Darboux transformations for a semi-discrete short pulse equation. By iterating the binary Darboux transformation, we obtain quasi-Grammian solutions. Furthermore, we derive explicit matrix solutions for the binary Darboux matrix and reduce them to the elementary Darboux matrix. Finally, we plot the dynamics of bright, dark, double breather, and peak soliton solutions.

Keywords: Integrable systems; short pulse equation; quasi-Grammian; binary Darboux transformation; breather; soliton.

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1. Introduction

The integrable equation known as short pulse (SP) equation is introduced by Schafer and Wayne, given by

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1.1}$$

having real-valued function $u = u(x,t)$ for the propagation of ultra-short waves in nonlinear media which can be associated with pseudo spherical surfaces [1–4]. By using hodograph transformation SP equation can be transformed into sine-Gordon equation, coupled dispersionless equation and modified Korteweg–de Vries (mKdV) equation [5–7]. The integrability of SP equation has been studied through different points of view, such as conservation laws [8, 9] existence of bi-Hamiltonian structure [10], existence of Lax pair of Wadati–Konno–Ichikawa (WKI) type [11], soliton solutions, etc. [12–15].

In this paper, we study the Lax pair for semi-discrete SP equation and then study the two Darboux as well as binary Darboux transformation of SP equation. For this purpose, we apply the Darboux matrix on Lax pair of SP equation for both direct and adjoint space to obtain the multi-soliton solutions. Through the iteration of binary Darboux transformation, we derive the general expressions of multi-quasi-Grammian solutions and represent these solutions by using quasideterminant approach. Finally, we calculate the exact solutions for the grammians, bright and dark double breathers and peak solutions for semi-discrete SP equation.

2. Lax Pair

The Lax pair for SP equation is given by

$$\partial_x \Psi = U\Psi = \lambda \begin{pmatrix} 1 & u_x \\ u_x & -1 \end{pmatrix} \Psi, \tag{2.1}$$

$$\partial_t \Psi = V\Psi = \frac{\lambda}{2} \begin{pmatrix} u^2 & u^2 u_x \\ u^2 u_x & -u^2 \end{pmatrix} \Psi + \frac{1}{2} \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix} \Psi + \frac{1}{4\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi, \tag{2.2}$$

where Ψ is an eigenmatrix of 2×2 order. An appropriate hodograph transformation is applied which transforms the independent variables (x, t) into new variables (X, T), i.e.

$$dX = \omega dx + \frac{1}{2}u^2 \omega dt, \quad dT = dt. \tag{2.3}$$

Also the old dynamical variable u transforms into new dynamical variable related by

$$\omega^2 = 1 + (u_x)^2, \tag{2.4}$$

which transforms (1.1) into the new form

$$x_{XT} = -\frac{1}{2}(u^2)_X, \tag{2.5}$$

$$u_{XT} = x_X u, \tag{2.6}$$

where X, T represent the derivatives with respect to X and T. Equations (2.5) and (2.6) can be expressed as the compatibility condition for the following linear system:

$$\Psi_X = E(X,T;\lambda)\Psi = (\lambda \partial_X P)\Psi, \tag{2.7}$$

$$\Psi_T = F(X,T;\lambda)\Psi = \left(Q + \frac{1}{\lambda}R\right)\Psi, \tag{2.8}$$

where the matrices P, Q and R are given by

$$P = \begin{pmatrix} x & u \\ u & -x \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}, \quad R = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.9}$$

The compatibility condition $E_T - F_X + [E, F] = 0$, of the linear system (2.7), (2.8) gives the zero-curvature equation which is equivalent to (2.5), (2.6). Now, the Lax pair of semi-discrete version of (2.5), (2.6) is given by

$$\Psi_{\sigma+1}(\lambda) = E_\sigma(\lambda)\Psi_\sigma(\lambda), \tag{2.10}$$

$$\frac{d}{dT}\Psi_\sigma(\lambda) = F_\sigma(\lambda)\Psi_\sigma(\lambda), \tag{2.11}$$

where the matrices E_σ and F_σ are

$$E_\sigma = I + \lambda(P_{\sigma+1} - P_\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda\begin{pmatrix} x_{\sigma+1} - x_\sigma & u_{\sigma+1} - u_\sigma \\ u_{\sigma+1} - u_\sigma & -(x_{\sigma+1} - x_\sigma) \end{pmatrix}, \tag{2.12}$$

$$F_\sigma = Q_\sigma + \lambda^{-1}R = \begin{pmatrix} 0 & -\dfrac{u_\sigma}{2} \\ \dfrac{u_\sigma}{2} & 0 \end{pmatrix} + \lambda^{-1}\begin{pmatrix} \dfrac{1}{4} & 0 \\ 0 & -\dfrac{1}{4} \end{pmatrix}, \tag{2.13}$$

where $\Psi_\sigma(\lambda)$ is the eigenmatrix of order 2×2 depending on continuous independent variable T and spectral parameter λ, σ in the subscript is discrete index and $x_\sigma(T)$ and $u_\sigma(T)$ are the scalar functions. The compatibility condition of the linear system (2.10), (2.11), i.e. $\frac{d}{dT}E_\sigma + E_\sigma F_\sigma - F_{\sigma+1}E_\sigma = 0$, gives the semi-discrete version of (2.5), (2.6), as

$$\frac{d}{dT}(x_{\sigma+1} - x_\sigma) + \frac{u_{\sigma+1} + u_\sigma}{2}(u_{\sigma+1} - u_\sigma) = 0, \tag{2.14}$$

$$\frac{d}{dT}(u_{\sigma+1} - u_\sigma) - \frac{u_{\sigma+1} + u_\sigma}{2}(x_{\sigma+1} - x_\sigma) = 0. \tag{2.15}$$

For the continuum limit, $\lim_{a\to 0}\frac{x_{\sigma+1}-x_\sigma}{a} = x_X$, $\lim_{a\to 0}\frac{u_{\sigma+1}-u_\sigma}{a} = u_X$, so Eqs. (2.14) and (2.15) reduces to (2.5), (2.6). Further, from the determinant of matrix (2.12), if we take the first integral $(x_{\sigma+1}-x_\sigma)^2+(u_{\sigma+1}-u_\sigma)^2 =$ constant, then sdSP equations (2.14), (2.15) can be equivalent to a chain of Backlund transformations for the mKdV equation

$$\frac{d}{dT}(u_{\sigma+1}-u_\sigma)=\frac{1}{2}\sqrt{c-(u_{\sigma+1}-u_\sigma)^2}(u_{\sigma+1}+u_\sigma). \tag{2.16}$$

Matrix (2.13) is traceless and used for the mKdV zero curvature representation. Similarly, one can show the transformation of sdSP into sine-Gordon equation by introducing the variables

$$x_\sigma=x_0+2\sum_{b=0}^{\sigma-1}\cos\frac{\theta_{b+1}+\theta_b}{2},\quad u_\sigma=-\frac{d}{dT}\theta_\sigma. \tag{2.17}$$

By using the transformation (2.17), Eqs. (2.14), (2.15) can be converted into a chain of Backlund transformations of sine-Gordon equation

$$\frac{d}{dT}(\theta_{\sigma+1}-\theta_\sigma)=2\sin\frac{\theta_{\sigma+1}+\theta_\sigma}{2}. \tag{2.18}$$

3. Darboux Transformation

Darboux transformation is an important technique to calculate solutions of integrable systems (for detail see [14, 16–25]). Now, we define the Darboux transformation on the linear system (2.10), (2.11) by using Darboux matrix $D_\sigma(\lambda)$ to obtain the solitonic solutions. The Darboux matrix transforms the matrix solution from the space V to new space $\widetilde{V}$, i.e.

$$\begin{aligned} D_\sigma(\lambda): V &\to \widetilde{V} \\ :\Psi_\sigma &\to \widetilde{\Psi}_\sigma. \end{aligned} \tag{3.1}$$

The one-fold Darboux transformation on matrix solution Ψ_σ is given by

$$\Psi_\sigma[1]=D_\sigma(\lambda)\Psi_\sigma, \tag{3.2}$$

where $D_\sigma(\lambda)$ is the Darboux matrix. The new transformed solution $\Psi_\sigma[1]$ satisfies the following linear system (2.10), (2.11) as

$$\begin{aligned} \Psi_{\sigma+1}[1]&=E_\sigma[1]\Psi_\sigma[1], \\ \frac{d}{dT}\Psi_\sigma[1]&=F_\sigma[1]\Psi_\sigma[1], \end{aligned} \tag{3.3}$$

having $E_\sigma[1]$ and $F_\sigma[1]$ as,

$$\begin{aligned} E_\sigma[1]&=I+\lambda(P_{\sigma+1}[1]-P_\sigma[1]), \\ F_\sigma[1]&=Q_\sigma[1]+\lambda^{-1}R[1]. \end{aligned} \tag{3.4}$$

Now, we define the Darboux matrix as

$$D_\sigma(\lambda)=\lambda^{-1}I-N_\sigma, \tag{3.5}$$

where N_σ is the auxiliary matrix of order 2×2 which is yet to be obtain and I is the 2×2 identity matrix. The choice for N_σ is $N_\sigma=\Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}$, where Θ_σ is the distinct matrix solution of the linear system (2.10), (2.11) having order 2×2 which can be calculated by using j-eigenvector functions $\Psi_\sigma(\lambda_j)$ evaluated at λ_j, $j = 1, 2$, whereas the matrix Λ is a diagonal matrix of order 2×2 having eigenvalues λ_1, λ_2. Therefore, the matrix Θ_σ can be defined as

$$\Theta_\sigma = (\Psi_\sigma(\lambda_1)|e\rangle_1, \Psi_\sigma(\lambda_2|e\rangle_2)), \tag{3.6}$$

evaluated at

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2). \tag{3.7}$$

By using (3.6), (3.7), the linear system (2.10), (2.11) can be expressed in the matrix form as

$$\Theta_{\sigma+1} = \Theta_\sigma + (P_{\sigma+1} - P_\sigma)\Theta_\sigma\Lambda, \tag{3.8}$$

$$\frac{d}{dT}\Theta_\sigma = Q_\sigma\Theta_\sigma + R\Theta_\sigma\Lambda^{-1}. \tag{3.9}$$

Based upon the above consequences, we can prove the following theorems.

Theorem 1. *Under the action of Darboux transformation* (3.5), *the new solution* (3.4) *has the same form as that of* P_σ *in* (2.10), (2.11), *provided that the matrix* N_σ *has to fulfill the following conditions*:

$$P_\sigma[1] = P_\sigma - N_\sigma, \tag{3.10}$$

$$(N_{\sigma+1} - N_\sigma)N_\sigma = (P_{\sigma+1} - P_\sigma)N_\sigma - N_{\sigma+1}(P_{\sigma+1} - P_\sigma). \tag{3.11}$$

Proof. The relation between the Darboux transformed solution $P_\sigma[1]$ and the untransformed solution P_σ is obtained and expressed in Eq. (3.10). Now, we have to verify that the choice of matrix $N_\sigma = \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}$ satisfies the condition given by (3.11), i.e.

$$\begin{aligned}
(N_{\sigma+1} - N_\sigma)N_\sigma &= \Theta_{\sigma+1}\Lambda^{-1}\Theta_{\sigma+1}^{-1}\Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1} - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}\Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1} \\
&\quad + \Theta_{\sigma+1}\Lambda^{-1}\Theta_\sigma^{-1}\Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1} - \Theta_{\sigma+1}\Lambda^{-1}\Theta_{\sigma+1}^{-1}\Theta_{\sigma+1}\Lambda^{-1}\Theta_\sigma^{-1}, \\
&= (\Theta_{\sigma+1}\Lambda^{-1}\Theta_\sigma^{-1} - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1})\Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1} \\
&\quad - \Theta_{\sigma+1}\Lambda^{-1}\Theta_{\sigma+1}^{-1}(\Theta_{\sigma+1}\Lambda^{-1}\Theta_\sigma^{-1} - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}), \\
&= (P_{\sigma+1} - P_\sigma)N_\sigma - N_{\sigma+1}(P_{\sigma+1} - P_\sigma),
\end{aligned}$$

which is Eq. (3.11). So, the proof is complete. □

Theorem 2. *Under the action of Darboux transformation* (3.5), *the new solution* (3.4) *has the same form as that of* Q_σ, R *in* (2.10), (2.11), *provided that the matrix* N_σ *has to fulfill the following conditions*:

$$Q_\sigma[1] = Q_\sigma + [R, N_\sigma], \tag{3.12}$$

$$R[1] = R, \tag{3.13}$$

$$\frac{d}{dT}N_\sigma = [Q_\sigma, N_\sigma] + [R, N_\sigma]N_\sigma. \tag{3.14}$$

Proof. The relation between the Darboux transformed solution $Q_\sigma[1]$, $R[1]$ and the untransformed solution Q_σ, R is obtained and expressed in Eqs. (3.12) and (3.13). Now, we have to verify that the choice of matrix $N_\sigma = \Theta_\sigma \Lambda^{-1}\Theta_\sigma^{-1}$ satisfies the condition given by (3.14). For this, we operate $\frac{d}{dT}$ on matrix N_σ, as

$$\begin{aligned}
\frac{d}{dT}N_\sigma &= \left(\frac{d}{dT}\Theta_\sigma\right)\Lambda^{-1}\Theta_\sigma^{-1} + \Theta_\sigma\Lambda^{-1}\left(\frac{d}{dT}\Theta_\sigma^{-1}\right),\\
&= \left(\frac{d}{dT}\Theta_\sigma\right)\Lambda^{-1}\Theta_\sigma^{-1} + \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}\left(\frac{d}{dT}\Theta_\sigma\right)\Theta_\sigma^{-1},\\
&= (Q_\sigma\Theta_\sigma + R\Theta_\sigma\Lambda^{-1})\Lambda^{-1}\Theta_\sigma^{-1} - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}(Q_\sigma\Theta_\sigma + R\Theta_\sigma\Lambda^{-1})\Theta_\sigma^{-1},\\
&= [Q_\sigma, N_\sigma] + [R, N_\sigma]N_\sigma.
\end{aligned}$$

which is Eq. (3.11). So, the proof is complete. □

Remark 1. Thus, the matrix $N_\sigma = \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}$ is the good choice which satisfies the conditions imposed by the Darboux transformation. So, the Darboux transformation preserves the system i.e. if Ψ_σ, P_σ, Q_σ and R, respectively, are the solutions of the linear system (2.10), (2.11), therefore $\Psi_\sigma[1], P_\sigma[1], Q_\sigma[1]$ and $R[1]$ are also the solutions of the same equations.

By using (3.10), the Darboux transformation on solutions x_σ and u_σ can be expressed as

$$x_\sigma[1] = x_\sigma - N_{\sigma,11}, \quad u_\sigma[1] = u_\sigma - N_{\sigma,12}. \tag{3.15}$$

Now, further we present the solutions by using quasideterminants (for detail see [33, 34]) defined as

$$\begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & \boxed{Z_{22}} \end{vmatrix} = Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}.$$

So, we can express the Darboux matrix on solution Ψ_σ as

$$\begin{aligned}
\Psi_\sigma[1] \equiv D_\sigma(\lambda)\Psi_\sigma &= (\lambda^{-1}I - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1})\Psi_\sigma,\\
&= \begin{vmatrix} \Theta_\sigma & \Psi_\sigma \\ \Theta_\sigma\Lambda^{-1} & \boxed{\lambda^{-1}\Psi_\sigma} \end{vmatrix}.
\end{aligned}\tag{3.16}$$

For the next iteration of Darboux transformation, take $\Theta_{\sigma,1}$, $\Theta_{\sigma,2}$ are the two distinct solutions of the linear system (3.8), (3.9) at $\Lambda = \Lambda_1^{-1}$ and $\Lambda = \Lambda_2^{-1}$, respectively. The two-fold Darboux transformation on $\Psi_\sigma[2]$ is given as

$$\begin{aligned}
\Psi_\sigma[2] &= (\lambda^{-1}I - N_\sigma[2])(\lambda^{-1}I - N_\sigma[1])\Psi_\sigma,\\
&= (\lambda^{-1}I - N_\sigma[2])\Psi_\sigma[1],
\end{aligned}\tag{3.17}$$

where one- and two-fold Darboux transformed matrix $N_\sigma[1]$, $N_\sigma[2]$ in terms of one- and two-fold Darboux transformed particular matrix solution $\Theta_\sigma[1]$, $\Theta_\sigma[2]$, respectively, are given by $N_\sigma[1] = \Theta_\sigma[1]\Lambda_1^{-1}(\Theta_\sigma[1])^{-1}$, $N_\sigma[2] = \Theta_\sigma[2]\Lambda_2^{-1}(\Theta_\sigma[2])^{-1}$. Also, $\Theta_\sigma[2]$ is written as

$$\begin{aligned}\Theta_\sigma[2] &= (\Theta_{\sigma,2}\Lambda_2^{-1} - N_\sigma[1]\Theta_{\sigma,2}),\\ &= \begin{vmatrix} \Theta_{\sigma,1} & \Theta_{\sigma,2} \\ \Theta_{\sigma,1}\Lambda_1^{-1} & \boxed{\Theta_{\sigma,2}\Lambda_2^{-1}} \end{vmatrix}.\end{aligned} \tag{3.18}$$

By using (3.16), (3.18) in (3.17), we get

$$\begin{aligned}\Psi_\sigma[2] &= \lambda^{-1}\begin{vmatrix} \Theta_{\sigma,1} & \Psi_\sigma \\ \Theta_{\sigma,1}\Lambda_1^{-1} & \boxed{\lambda^{-1}\Psi_\sigma} \end{vmatrix} - \begin{vmatrix} \Theta_{\sigma,1} & \Theta_{\sigma,2} \\ \Theta_{\sigma,1}\Lambda_1^{-1} & \boxed{\Theta_{\sigma,2}\Lambda_2^{-1}} \end{vmatrix}\Lambda_2^{-1}\\ &\quad\times\begin{vmatrix} \Theta_{\sigma,1} & \Theta_{\sigma,2} \\ \Theta_{\sigma,1}\Lambda_1^{-1} & \boxed{\Theta_{\sigma,2}\Lambda_2^{-1}} \end{vmatrix}^{-1}\begin{vmatrix} \Theta_{\sigma,1} & \Psi_\sigma \\ \Theta_{\sigma,1}\Lambda_1^{-1} & \boxed{\lambda^{-1}\Psi_\sigma} \end{vmatrix},\\ &= \begin{vmatrix} \Theta_{\sigma,1}\Lambda_1^{-1} & \lambda^{-1}\Psi_\sigma \\ \Theta_{\sigma,1}\Lambda_1^{-2} & \boxed{\lambda^{-2}\Psi_\sigma} \end{vmatrix} - \begin{vmatrix} \Theta_{\sigma,1}\Lambda_1^{-1} & \Theta_{\sigma,2}\Lambda_2^{-1} \\ \Theta_{\sigma,1}\Lambda_1^{-2} & \boxed{\Theta_{\sigma,2}\Lambda_2^{-2}} \end{vmatrix}\\ &\quad\times\begin{vmatrix} \Theta_{\sigma,1}\Lambda_1^{-1} & \Theta_{\sigma,2}\Lambda_2^{-1} \\ \Theta_{\sigma,1} & \boxed{\Theta_{\sigma,2}} \end{vmatrix}^{-1}\begin{vmatrix} \Theta_{\sigma,1}\Lambda_1^{-1} & \lambda^{-1}\Psi_\sigma \\ \Theta_{\sigma,1} & \boxed{\Psi_\sigma} \end{vmatrix},\\ &= \begin{vmatrix} \Theta_{\sigma,1} & \Theta_{\sigma,2} & \Psi_\sigma \\ \Theta_{\sigma,1}\Lambda_1^{-1} & \Theta_{\sigma,2}\Lambda_2^{-1} & \lambda^{-1}\Psi_\sigma \\ \Theta_{\sigma,1}\Lambda_1^{-2} & \Theta_{\sigma,2}\Lambda_2^{-2} & \boxed{\lambda^{-2}\Psi_\sigma} \end{vmatrix},\end{aligned}$$

where we have used homological relation in second step and noncommutative Jacobi identity in the last step.[a] Similarly, K-fold Darboux transformation is given by

[a]For a general quasideterminant expanded about $N \times N$ matrix D, we have

$$\begin{vmatrix} E & F & G \\ H & A & B \\ J & C & \boxed{D} \end{vmatrix} = \begin{vmatrix} E & G \\ J & \boxed{D} \end{vmatrix} - \begin{vmatrix} E & F \\ J & \boxed{C} \end{vmatrix}\begin{vmatrix} E & F \\ H & \boxed{A} \end{vmatrix}^{-1}\begin{vmatrix} E & G \\ H & \boxed{B} \end{vmatrix}.$$

From the noncommutative Jacobi identity, we get the homological relation

$$\begin{vmatrix} E & F & G \\ H & A & \boxed{B} \\ J & C & D \end{vmatrix} = \begin{vmatrix} E & F & O \\ H & A & \boxed{O} \\ J & C & I \end{vmatrix}\begin{vmatrix} E & F & G \\ H & A & B \\ J & C & \boxed{D} \end{vmatrix},$$

where O and I denote the null and identity matrices, respectively.

$$\Psi_\sigma[K] = \begin{vmatrix} \Theta_{\sigma,1} & \Theta_{\sigma,2} & \cdots & \Theta_{\sigma,K} & \Psi_\sigma \\ \Theta_{\sigma,1}\Lambda_1^{-1} & \Theta_{\sigma,2}\Lambda_2^{-1} & \cdots & \Theta_{\sigma,K}\Lambda_K^{-1} & \lambda^{-1}\Psi_\sigma \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_{\sigma,1}\Lambda_1^{-(K-1)} & \Theta_{\sigma,2}\Lambda_2^{-(K-1)} & \cdots & \Theta_{\sigma,K}\Lambda_K^{-(K-1)} & \lambda^{-(K-1)}\Psi_\sigma \\ \Theta_{\sigma,1}\Lambda_1^{-K} & \Theta_{\sigma,2}\Lambda_2^{-K} & \cdots & \Theta_{\sigma,K}\Lambda_K^{-K} & \boxed{\lambda^{-K}\Psi_\sigma} \end{vmatrix}. \tag{3.19}$$

Now, Eq. (3.10) can be written as

$$\begin{aligned} P_\sigma[1] &= P_\sigma - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}, \\ &= P_\sigma + \begin{vmatrix} \Theta_\sigma & I \\ \Theta_\sigma\Lambda^{-1} & \boxed{O} \end{vmatrix}. \end{aligned} \tag{3.20}$$

The result can be generalized to K-times Darboux transformation as

$$P_\sigma[K] = P_\sigma + \begin{vmatrix} \Theta_{\sigma,1} & \Theta_{\sigma,2} & \cdots & \Theta_{\sigma,K} & O \\ \Theta_{\sigma,1}\Lambda_1^{-1} & \Theta_{\sigma,2}\Lambda_2^{-1} & \cdots & \Theta_{\sigma,K}\Lambda_K^{-1} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_{\sigma,1}\Lambda_1^{-(K-1)} & \Theta_{\sigma,2}\Lambda_2^{-(K-1)} & \cdots & \Theta_{\sigma,K}\Lambda_K^{-(K-1)} & I \\ \Theta_{\sigma,1}\Lambda_1^{-K} & \Theta_{\sigma,2}\Lambda_2^{-K} & \cdots & \Theta_{\sigma,K}\Lambda_K^{-K} & \boxed{O} \end{vmatrix}. \tag{3.21}$$

In soliton theory, we have also used the adjoint pair of matrix eigenvalue problems. So, for the adjoint pair taking adjoint of the linear system (2.10) and (2.11), we get

$$\Phi_{\sigma+1} = \Phi_\sigma(I + \mu(P_{\sigma+1}^\dagger - P_\sigma^\dagger)), \tag{3.22}$$

$$\frac{d}{dT}\Phi_\sigma = \Phi_\sigma(Q_\sigma^\dagger + \mu^{-1}R^\dagger), \tag{3.23}$$

In Eqs. (3.22) and (3.23) μ is a spectral parameter and Φ_σ represents an adjoint matrix eigenfunction defined by adjoint Lax pair (3.22) and (3.23). The compatibility condition of this adjoint Lax pair does not generate any new conditions except the original zero curvature equation. Such type of connection has also been used in Riemann–Hilbert problems [32]. The Darboux matrix $D_\sigma(\mu)$ transforms the matrix solution Φ_σ in space $V^\dagger$ to new matrix field solution $\widetilde{\Phi}_\sigma$ in adjoint space $\widetilde{V^\dagger}$, i.e.

$$\begin{aligned} D_\sigma(\mu) : V^\dagger &\to \widetilde{V}^\dagger \\ : \Phi_\sigma &\to \widetilde{\Phi}_\sigma. \end{aligned} \tag{3.24}$$

Now, we can write Darboux transformation on matrix solution Φ_σ as

$$\Phi_\sigma[1] \equiv D_\sigma(\mu)\Phi_\sigma = (\mu^{-1}I - M_\sigma)\Phi_\sigma, \tag{3.25}$$

where M_σ is the 2×2 matrix which is to be determined and I is 2×2 identity matrix. The covariance of the linear system under the Darboux transformation requires that the new solution $\widetilde{\Phi}_\sigma$ satisfies the linear system (3.22) and (3.23), given by

$$\begin{aligned}\Phi_{\sigma+1}[1] &= \Phi_\sigma[1](I+\mu(P^\dagger_{\sigma+1}[1]-P^\dagger_\sigma[1])),\\ \frac{d}{dT}\Phi_\sigma[1] &= \Phi_\sigma[1]\left(Q^\dagger_\sigma[1]+\frac{1}{\mu}R^\dagger[1]\right).\end{aligned} \tag{3.26}$$

Now, applying the Darboux transformation (3.25) on (3.26), we get the Darboux transformed matrix functions $P^\dagger_\sigma, Q^\dagger_\sigma$ and $R^\dagger$ as

$$\begin{aligned}P^\dagger_\sigma[1] &= P^\dagger_\sigma - M_\sigma,\\ Q^\dagger_\sigma[1] &= Q^\dagger_\sigma + [R^\dagger, M_\sigma],\\ R^\dagger[1] &= R^\dagger.\end{aligned} \tag{3.27}$$

The matrix M_σ can be constructed from the eigenmatrices of the linear system and we take M_σ as $M_\sigma = \Omega_\sigma F^{-1}\Omega_\sigma^{-1}$, where $F=\mathrm{diag}(\mu_1,\mu_2)$ is the eigenmatrix. The particular matrix solution Ω_σ of the linear system (3.22) and (3.23) is an invertible 2×2 matrix which is given as

$$\Omega_\sigma = (\Phi_\sigma(\mu_1)|e\rangle_1,\ \Phi_\sigma(\mu_2|e\rangle_2)). \tag{3.28}$$

Each column $|\Phi_\sigma\rangle_j = \Phi_\sigma(\mu_j)|e_j\rangle$ in Ω_σ is a column solution of the linear system (3.22), (3.23). The K-fold Darboux transformation on matrix solution and matrix function Φ_σ, $P^\dagger_\sigma$ can be expressed as

$$\Phi_\sigma[K] = \begin{vmatrix} \Omega_{\sigma,1} & \Omega_{\sigma,2} & \cdots & \Omega_{\sigma,K} & \Phi_\sigma \\ \Omega_{\sigma,1}F_1^{-1} & \Omega_{\sigma,2}F_2^{-1} & \cdots & \Omega_{\sigma,K}F_K^{-1} & \mu^{-1}\Phi_\sigma \\ \Omega_{\sigma,1}F_1^{-2} & \Omega_{\sigma,2}F_2^{-2} & \cdots & \Omega_{\sigma,K}F_K^{-2} & \mu^{-2}\Phi_\sigma \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Omega_{\sigma,1}F_1^{-K} & \Omega_{\sigma,2}F_2^{-K} & \cdots & \Omega_{\sigma,K}F_K^{-K} & \boxed{\mu^{-K}\Phi_\sigma} \end{vmatrix}. \tag{3.29}$$

Similarly, the quasideterminants of $P^\dagger_\sigma[K]$ is written as

$$P^\dagger_\sigma[K] = P_\sigma + \begin{vmatrix} \Omega_{\sigma,1} & \Omega_{\sigma,2} & \cdots & \Omega_{\sigma,K} & O \\ \Omega_{\sigma,1}F_1^{-1} & \Omega_{\sigma,2}F_2^{-1} & \cdots & \Omega_{\sigma,K}F_K^{-1} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Omega_{\sigma,1}F_1^{-(K-1)} & \Omega_{\sigma,2}F_2^{-(K-1)} & \dots & \Omega_{\sigma,K}F_K^{-(K-1)} & I \\ \Omega_{\sigma,1}F_1^{-K} & \Omega_{\sigma,2}F_2^{-K} & \cdots & \Omega_{\sigma,K}F_K^{-K} & \boxed{O} \end{vmatrix}. \tag{3.30}$$

Equations (3.29) and (3.30) are the Kth quasideterminant solutions of the SP equation for the adjoint space.

4. Binary Darboux Transformation

For the binary Darboux transformation (for detail see [26–31]), consider the hat space $\widehat{V}$ which is the copied version of direct space V, so that the corresponding solutions are $\widehat{\Psi}_\sigma \in \widehat{V}$. Therefore, the linear system can be written as

$$\begin{aligned}\widehat{\Psi}_{\sigma+1} &= \widehat{\Psi}_\sigma + \lambda(\widehat{P}_{\sigma+1} - \widehat{P}_\sigma)\widehat{\Psi}_\sigma,\\ \frac{d}{dT}\widehat{\Psi}_\sigma &= \widehat{Q}_\sigma\widehat{\Psi}_\sigma + \lambda^{-1}\widehat{R}_\sigma\widehat{\Psi}_\sigma,\end{aligned} \tag{4.1}$$

where the matrices $\widehat{P}_\sigma, \widehat{Q}_\sigma$ and $\widehat{R}_\sigma$ are given by

$$\widehat{P}_\sigma = \begin{pmatrix} \widehat{x}_\sigma & \widehat{u}_\sigma \\ \widehat{u}_\sigma & -\widehat{x}_\sigma \end{pmatrix}, \quad \widehat{Q}_\sigma = \frac{1}{2}\begin{pmatrix} 0 & -\widehat{u}_\sigma \\ \widehat{u}_\sigma & 0 \end{pmatrix}, \quad \widehat{R} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2}$$

The particular solutions for the direct and adjoint spaces are $\Theta_\sigma, \Omega_\sigma$, respectively. So, the corresponding solutions for hat space are $\widehat{\Theta}_\sigma \in \widehat{V}$ and $\widehat{\Phi}_\sigma \in \widehat{V}^\dagger$. Also assuming that $i(\widehat{\Theta}_\sigma) \in \widetilde{V}^\dagger$, then one can write the transformation as

$$D_\sigma^{(-1)\dagger}(\lambda) : V^\dagger \to \widetilde{V}^\dagger. \tag{4.3}$$

Since $\Phi_\sigma \in V^\dagger$, we have

$$i(\widehat{\Theta}_\sigma) = D_\sigma^{(-1)\dagger}(\lambda)\Phi_\sigma. \tag{4.4}$$

Also from $D_\sigma^\dagger(\lambda)(i(\Theta_\sigma)) = 0$, we obtain $i(\Theta_\sigma) = \Theta_\sigma^{(-1)\dagger}$ and similarly $i(\widehat{\Theta}_\sigma) = \widehat{\Theta}_\sigma^{(-1)\dagger}$. So, we can write

$$\widehat{\Theta}_\sigma^{(-1)\dagger} = D_\sigma^{(-1)\dagger}(\lambda)\Phi_\sigma,$$

and

$$\widehat{\Theta}_\sigma = (D_\sigma^{(-1)\dagger}(\lambda)\Phi_\sigma)^{(-1)\dagger}, \tag{4.5}$$

where $D_\sigma(\lambda) = \lambda^{-1}I - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}$. By using the value of $D_\sigma(\lambda)$ in Eq. (4.5), we get

$$\begin{aligned}\widehat{\Theta}_\sigma &= ((\lambda^{-1}I - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1})^{(-1)\dagger}\Phi_\sigma)^{(-1)\dagger},\\ &= (\lambda^{-1}I - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1})\Phi_\sigma^{(-1)\dagger},\\ &= \Theta_\sigma(\lambda^{-1}I - \Lambda^{-1})\Theta_\sigma^{-1}\Phi_\sigma^{(-1)\dagger},\\ &= \Theta_\sigma(\lambda^{-1}I - \Lambda^{-1})(\Phi_\sigma^\dagger\Theta_\sigma)^{-1},\\ &= \Theta_\sigma\Delta(\Theta_\sigma, \Phi_\sigma)^{-1},\end{aligned} \tag{4.6}$$

where the algebraic potential $\Delta(\Theta_\sigma, \Phi_\sigma)$ is defined as

$$\Delta(\Theta_\sigma, \Phi_\sigma) = (\Phi_\sigma^\dagger\Theta_\sigma)(\lambda^{-1}I - \Lambda^{-1})^{-1}. \tag{4.7}$$

Similarly, for the adjoint space the matrix $\widehat{\Omega}_\sigma$ can be written as

$$\widehat{\Omega}_\sigma = \Omega_\sigma\Delta(\Psi_\sigma, \Omega_\sigma)^{(-1)\dagger}, \tag{4.8}$$

where

$$\Delta(\Psi_\sigma,\Omega_\sigma) = -(\lambda^{-1}I - F^{(-1)\dagger})^{-1}(\Omega_\sigma^\dagger \Psi_\sigma). \tag{4.9}$$

By expressing Eqs. (4.7) and (4.9) in the form of matrix for the solutions Θ_σ and Ω_σ, we get

$$F^{(-1)\dagger}\Delta(\Theta_\sigma,\Omega_\sigma) - \Delta(\Theta_\sigma,\Omega_\sigma)\Lambda^{-1} = \Omega_\sigma^\dagger \Theta_\sigma, \tag{4.10}$$

where Δ matrix is given by

$$\Delta(\Theta_\sigma,\Omega_\sigma) = \frac{\langle \Theta_\sigma \,|\, \Omega_\sigma \rangle}{F^{(-1)\dagger} - \Lambda^{-1}}. \tag{4.11}$$

Now, we define Darboux transformation in hat space

$$\widehat{D}_\sigma(\lambda) \equiv (\lambda^{-1}I - \widehat{N}_\sigma) = (\lambda^{-1}I - \widehat{\Theta}_\sigma F^{(-1)\dagger}\widehat{\Theta}_\sigma^{-1}), \tag{4.12}$$

where the action of Darboux transformation is

$$\widehat{D}_\sigma(\lambda)\widehat{\Psi}_\sigma = \Psi_\sigma[1], \tag{4.13}$$

which is equivalent to the Darboux transformation in direct space. Now, we use the definition of binary Darboux transformation which relates two solutions $\widehat{\Psi}_\sigma$ and Ψ_σ as

$$\widehat{D}_\sigma(\lambda)\widehat{\Psi}_\sigma = D_\sigma(\lambda)\Psi_\sigma, \tag{4.14}$$

Based upon the above results, we can prove the following theorems.

Theorem 3. *Under the action of binary Darboux transformation* (4.14) *and by using* (4.12), *the new transformed matrix solution* $\widehat{\Psi}_\sigma$ *has the following form*:

$$\widehat{\Psi}_\sigma = \Psi_\sigma - \Theta_\sigma \Delta(\Theta_\sigma,\Omega_\sigma)^{-1}\Delta(\Psi_\sigma,\Omega_\sigma), \tag{4.15}$$

Proof. The definition of binary Darboux transformation (4.14) implies

$$\widehat{\Psi}_\sigma = \widehat{D}_\sigma^{-1}(\lambda)D_\sigma(\lambda)\Psi_\sigma,$$

Now, using the values of $\widehat{D}_\sigma(\lambda)$ and $D_\sigma(\lambda)$, we get

$$\begin{aligned}\widehat{\Psi}_\sigma &= (\lambda^{-1}I - \widehat{\Theta}_\sigma F^{(-1)\dagger}\widehat{\Theta}_\sigma^{-1})^{-1}(\lambda^{-1}I - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1})\Psi_\sigma,\\ &= \widehat{\Theta}_\sigma(\lambda^{-1}I - F^{(-1)\dagger})^{-1}\widehat{\Theta}_\sigma^{-1}\Theta_\sigma(\lambda^{-1}I - \Lambda^{-1})\Theta_\sigma^{-1}\Psi_\sigma,\end{aligned}$$

using Eq. (4.6) in above expression, we get

$$\begin{aligned}\widehat{\Psi}_\sigma &= \Theta_\sigma\Delta(\Theta_\sigma,\Phi_\sigma)^{-1}(\lambda^{-1}I - F^{(-1)\dagger})^{-1}\\ &\quad\times \Delta(\Theta_\sigma,\Phi_\sigma)\Theta_\sigma^{-1}\Theta_\sigma(\lambda^{-1}I - \Lambda^{-1})\Theta_\sigma^{-1}\Psi_\sigma,\\ &= \Theta_\sigma\Delta(\Theta_\sigma,\Phi_\sigma)^{-1}(\lambda^{-1}I - F^{(-1)\dagger})^{-1}\\ &\quad\times (\lambda^{-1}\Delta(\Theta_\sigma,\Phi_\sigma) - \Delta(\Theta_\sigma,\Phi_\sigma)\Lambda^{-1})\Theta_\sigma^{-1}\Psi_\sigma,\end{aligned} \tag{4.16}$$

by substituting the value of $\Delta(\Theta_\sigma, \Phi_\sigma)\Lambda^{-1}$ from Eq. (4.10) in expression (4.16), we get

$$\begin{aligned}\widehat{\Psi}_\sigma &= \Theta_\sigma \Delta(\Theta_\sigma, \Phi_\sigma)^{-1}(\lambda^{-1}I - F^{(-1)\dagger})^{-1} \\ &\quad \times (\lambda^{-1}\Delta(\Theta_\sigma, \Phi_\sigma)\Theta_\sigma^{-1} - F^{(-1)\dagger}\Delta(\Theta_\sigma, \Phi_\sigma)\Theta_\sigma^{-1} + \Omega_\sigma^\dagger)\Psi_\sigma, \\ &= (\lambda^{-1}I - F^{(-1)\dagger})^{-1}(\lambda^{-1}I - F^{(-1)\dagger})\left(I + \frac{\Theta_\sigma\Delta(\Theta_\sigma, \Phi_\sigma)^{-1}\Omega_\sigma^\dagger}{\lambda^{-1}I - F^{(-1)\dagger}}\right)\Psi_\sigma, \\ &= \Psi_\sigma + \Theta_\sigma\Delta(\Omega_\sigma, \Phi_\sigma)^{-1}(\lambda^{-1}I - F^{(-1)\dagger})^{-1}\Omega_\sigma^\dagger\Psi_\sigma.\end{aligned}$$

Using Eq. (4.9)

$$\widehat{\Psi}_\sigma = \Psi_\sigma - \Theta_\sigma\Delta(\Theta_\sigma, \Omega_\sigma)^{-1}\Delta(\Psi_\sigma, \Omega_\sigma),$$

which is Eq. (4.15). So, the proof is complete. □

Theorem 4. *Under the action of binary Darboux transformation* (4.14) *and by using* (4.12), *the new transformed matrix solution* $\widehat{\Psi}_\sigma$ *has the following form*:

$$\widehat{P}_\sigma = P_\sigma + \Theta_\sigma\Delta(\Theta_\sigma, \Omega_\sigma)^{-1}\Omega_\sigma^\dagger. \tag{4.17}$$

Proof. Applying the definition of binary Darboux transformation (4.14) on the solution of sdSP equation P_σ, written as

$$\begin{aligned}\widehat{P}_\sigma - \widehat{\Theta}_\sigma F^{(-1)\dagger}\widehat{\Theta}_\sigma^{-1} &= P_\sigma - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}, \\ \widehat{P}_\sigma &= P_\sigma - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1} + \widehat{\Theta}_\sigma F^{(-1)\dagger}\widehat{\Theta}_\sigma^{-1},\end{aligned}$$

by using Eq. (4.6), we get

$$\widehat{P}_\sigma = P_\sigma - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1} + \Theta_\sigma\Delta^{-1}F^{(-1)\dagger}\Delta\Theta_\sigma^{-1},$$

by substituting Eq. (4.10), we get

$$\begin{aligned}\widehat{P}_\sigma &= P_\sigma - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1} + \Theta_\sigma\Delta^{-1}(\Omega_\sigma^\dagger\Theta_\sigma + \Delta\Lambda^{-1})\Theta_\sigma^{-1}, \\ &= P_\sigma - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1} + \Theta_\sigma\Delta^{-1}\Omega_\sigma^\dagger\Theta_\sigma\Theta_\sigma^{-1} + \Theta_\sigma\Delta^{-1}\Delta\Lambda^{-1}\Theta_\sigma^{-1}, \\ &= P_\sigma - \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1} + \Theta_\sigma\Delta^{-1}\Omega_\sigma^\dagger + \Theta_\sigma\Lambda^{-1}\Theta_\sigma^{-1}, \\ &= P_\sigma + \Theta_\sigma\Delta^{-1}\Omega_\sigma^\dagger,\end{aligned}$$

which is Eq. (4.17). So, the proof is complete. □

Now, expression (4.15) can be expressed in terms of quasideterminant as

$$\widehat{\Psi}_\sigma = \begin{vmatrix} \Delta(\Theta_\sigma, \Omega_\sigma) & \Delta(\Psi_\sigma, \Omega_\sigma) \\ \Theta_\sigma & \boxed{\Psi_\sigma} \end{vmatrix}. \tag{4.18}$$

This is known as the quasi-Grammian solution of the system. Similarly, expression (4.17) can be written in the form of quasideterminant as

$$\begin{aligned}\widehat{P}_\sigma &= P_\sigma - (O - \Theta_\sigma \Delta(\Theta_\sigma, \Omega_\sigma)^{-1}\Omega_\sigma^\dagger),\\ &= P_\sigma - \begin{vmatrix} \Delta(\Theta_\sigma, \Omega_\sigma) & \Omega_\sigma^\dagger \\ \Theta_\sigma & \boxed{O} \end{vmatrix}.\end{aligned}$$

We can calculate the Kth iteration of $\widehat{P}_\sigma$ through the iteration of binary Darboux transformation given by

$$\widehat{P}_\sigma = P_\sigma - \begin{vmatrix} \Delta(\Theta_{\sigma,1}, \Omega_{\sigma,1}) & \cdots & \Delta(\Theta_{\sigma,K}, \Omega_{\sigma,1}) & \Omega_{\sigma,1}^\dagger \\ \vdots & \cdots & \vdots & \vdots \\ \Delta(\Theta_{\sigma,1}, \Omega_{\sigma,K}) & \cdots & \Delta(\Theta_{\sigma,K}, \Omega_{\sigma,K}) & \Omega_{\sigma,K}^\dagger \\ \Theta_{\sigma,1} & \cdots & \Theta_{\sigma,K} & \boxed{I} \end{vmatrix}. \tag{4.19}$$

The quasideterminant solutions (4.19) are called the quasi-Grammian solutions of sdSP equation.

Remark 2. Therefore, by the use of binary Darboux transformation, we can derive the quasi-grammian solutions for the semi-discrete SP equation. Also, the potential Δ can be written in terms of quasideterminants. So, by developing the binary Darboux transformation in terms of spectral parameters, we can obtain the expression of matrix solutions in terms of Grammian-type quasideterminants, which have different forms as we have derived by elementary Darboux transformation. Applications of binary Darboux transformation are to calculate the exact solutions, lump and breather solutions can significantly improve our understanding of nonlinear waves.

5. Exact Solutions

In this section, we calculate the expressions for the grammian and peak soliton solutions of sdSP equation by using binary Darboux transformation. To obtain an explicit expression, we define a gauge transformation on linear system (2.10) and (2.11) which is a constant matrix has the form

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \tag{5.1}$$

also the matrices E_σ and F_σ transforms as

$$\begin{aligned} E_\sigma &\to \widetilde{E}_\sigma = W^{-1}E_\sigma W = I + \lambda(\widetilde{P}_{\sigma+1} - \widetilde{P}_\sigma),\\ F_\sigma &\to \widetilde{F}_\sigma = W^{-1}F_\sigma W = \widetilde{Q}_\sigma + \lambda^{-1}\widetilde{R}, \end{aligned} \tag{5.2}$$

where

$$\widetilde{P}_\sigma = \begin{pmatrix} 0 & x_\sigma + iu_\sigma \\ x_\sigma - iu_\sigma & 0 \end{pmatrix},$$
$$\widetilde{Q}_\sigma = \begin{pmatrix} -\frac{u_\sigma}{2} & 0 \\ 0 & \frac{u_\sigma}{2} \end{pmatrix}, \quad \widetilde{R} = \frac{1}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.3}$$

By using gauge transformation, the linear system (2.10) and (2.11) becomes

$$\begin{pmatrix} U_{\sigma+1} \\ V_{\sigma+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} U_\sigma \\ V_\sigma \end{pmatrix} + \lambda \begin{pmatrix} 0 & x_{\sigma+1} - x_\sigma + i(u_{\sigma+1} - u_\sigma) \\ x_{\sigma+1} - x_\sigma - i(u_{\sigma+1} - u_\sigma) & 0 \end{pmatrix}\begin{pmatrix} U_\sigma \\ V_\sigma \end{pmatrix}, \tag{5.4}$$

$$\frac{d}{dT}\begin{pmatrix} U_\sigma \\ V_\sigma \end{pmatrix} = \begin{pmatrix} -\frac{u_\sigma}{2} & 0 \\ 0 & \frac{u_\sigma}{2} \end{pmatrix}\begin{pmatrix} U_\sigma \\ V_\sigma \end{pmatrix} + \frac{\lambda^{-1}}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_\sigma \\ V_\sigma \end{pmatrix}. \tag{5.5}$$

To compute expressions for the solution, we take a seed solution, i.e. $x_{\sigma+1} - x_\sigma = a \neq 0$ and $u_\sigma = 0$, where a is a real constant, so the solutions U_σ and V_σ of the linear system (5.4) and (5.5) can be computed as

$$U_\sigma(\lambda) = (1 + a\lambda)^\sigma e^{T/4\lambda} + i(1 - a\lambda)^\sigma e^{-T/4\lambda},$$
$$V_\sigma(\lambda) = (1 + a\lambda)^\sigma e^{T/4\lambda} - i(1 - a\lambda)^\sigma e^{-T/4\lambda}.$$

Now, we define the particular matrix solution Θ_σ to the linear system (5.4) and (5.5) of sdSP equation for $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}$ can be written as

$$\Theta_\sigma = \begin{pmatrix} U_\sigma(\lambda) & U_\sigma(\overline{\lambda}) \\ V_\sigma(\lambda) & -V_\sigma(\overline{\lambda}) \end{pmatrix}. \tag{5.6}$$

Similarly, for the adjoint space, Ω_σ can be defined as

$$\Omega_\sigma = \begin{pmatrix} Y_\sigma(\mu) & Y_\sigma(\overline{\mu}) \\ Z_\sigma(\mu) & -Z_\sigma(\overline{\mu}) \end{pmatrix}, \tag{5.7}$$

where

$$Y_\sigma(\mu) = (1 + a\mu)^\sigma e^{T/4\mu} + i(1 - a\mu)^\sigma e^{-T/4\mu},$$
$$Z_\sigma(\mu) = (1 + a\mu)^\sigma e^{T/4\mu} - i(1 - a\mu)^\sigma e^{-T/4\mu}.$$

Starting from the definition of $\Delta(\Theta_\sigma, \Omega_\sigma)$ given in (4.11) and using (5.6) and (5.7), we get

$$\widehat{N}_\sigma = \Theta_\sigma \Delta^{-1}(\Theta_\sigma, \Omega_\sigma)\Omega_\sigma^\dagger = \frac{1}{K_\sigma}\begin{pmatrix} \widehat{N}_{\sigma,11} & \widehat{N}_{\sigma,12} \\ \widehat{N}_{\sigma,21} & \widehat{N}_{\sigma,22} \end{pmatrix},$$

$$= \frac{1}{K_\sigma}\begin{pmatrix} \begin{array}{l} \frac{A_\sigma U_\sigma(\lambda) Z_\sigma(\mu)}{\overline{\mu} - \overline{\lambda}} + \frac{C_\sigma U_\sigma(\overline{\lambda}) Z_\sigma(\mu)}{\lambda - \overline{\mu}} \\ \quad + \frac{B_\sigma U_\sigma(\lambda) Z_\sigma(\overline{\mu})}{\overline{\lambda} - \mu} + \frac{D_\sigma U_\sigma(\overline{\lambda}) Z_\sigma(\overline{\mu})}{\mu - \lambda} \end{array} & \begin{array}{l} \frac{A_\sigma U_\sigma(\lambda) Y_\sigma(\mu)}{\overline{\mu} - \overline{\lambda}} - \frac{C_\sigma U_\sigma(\overline{\lambda}) Y_\sigma(\mu)}{\lambda - \overline{\mu}} \\ \quad + \frac{B_\sigma U_\sigma(\lambda) Y_\sigma(\overline{\mu})}{\overline{\lambda} - \mu} - \frac{D_\sigma U_\sigma(\overline{\lambda}) Y_\sigma(\overline{\mu})}{\mu - \lambda} \end{array} \\ \begin{array}{l} \frac{A_\sigma V_\sigma(\lambda) Z_\sigma(\mu)}{\overline{\mu} - \overline{\lambda}} - \frac{C_\sigma V_\sigma(\overline{\lambda}) Z_\sigma(\mu)}{\lambda - \overline{\mu}} \\ \quad + \frac{B_\sigma V_\sigma(\lambda) Z_\sigma(\overline{\mu})}{\overline{\lambda} - \mu} - \frac{D_\sigma V_\sigma(\overline{\lambda}) Z_\sigma(\overline{\mu})}{\mu - \lambda} \end{array} & \begin{array}{l} \frac{A_\sigma V_\sigma(\lambda) Y_\sigma(\mu)}{\overline{\mu} - \overline{\lambda}} + \frac{C_\sigma V_\sigma(\overline{\lambda}) Y_\sigma(\mu)}{\lambda - \overline{\mu}} \\ \quad + \frac{B_\sigma V_\sigma(\lambda) Y_\sigma(\overline{\mu})}{\overline{\lambda} - \mu} + \frac{D_\sigma V_\sigma(\overline{\lambda}) Y_\sigma(\overline{\mu})}{\mu - \lambda} \end{array} \end{pmatrix}, \tag{5.8}$$

where

$$\begin{aligned} A_\sigma &= Z_\sigma(\overline{\mu}) U_\sigma(\overline{\lambda}) + Y_\sigma(\overline{\mu}) V_\sigma(\overline{\lambda}), \\ B_\sigma &= Z_\sigma(\mu) U_\sigma(\overline{\lambda}) - Y_\sigma(\mu) V_\sigma(\overline{\lambda}), \\ C_\sigma &= Z_\sigma(\overline{\mu}) U_\sigma(\lambda) - Y_\sigma(\overline{\mu}) V_\sigma(\lambda), \\ D_\sigma &= Z_\sigma(\mu) U_\sigma(\lambda) + Y_\sigma(\mu) V_\sigma(\lambda). \\ K_\sigma &= \frac{(Z_\sigma(\mu) U_\sigma(\lambda) + Y_\sigma(\mu) V_\sigma(\lambda))(Z_\sigma(\overline{\mu}) U_\sigma(\overline{\lambda}) + Y_\sigma(\overline{\mu}) V_\sigma(\overline{\lambda}))}{(\mu - \lambda)(\overline{\mu} - \overline{\lambda})} \\ &\quad - \frac{(Z_\sigma(\mu) U_\sigma(\overline{\lambda}) - Y_\sigma(\mu) V_\sigma(\overline{\lambda}))(Z_\sigma(\overline{\mu}) U_\sigma(\lambda) - Y_\sigma(\overline{\mu}) V_\sigma(\lambda))}{(\mu - \overline{\lambda})(\overline{\mu} - \lambda)}. \end{aligned}$$

The expressions (5.8) are presented in Figs. 1 and 2, which represent the traveling of two double breather, bright and dark breather together and their interaction. Further, we reduce the expressions of binary Darboux transformation into elementary Darboux transformation.

5.1. *Reduction*

Now, for reduction substitute $\overline{\mu} = \lambda$, $\mu = \overline{\lambda}$, then $A_\sigma = D_\sigma = U_\sigma(\overline{\lambda})V_\sigma(\lambda) + U_\sigma(\lambda)V_\sigma(\overline{\lambda})$, $B_\sigma = 0 = C_\sigma$. Also, $K_\sigma = A_\sigma^2/(\lambda - \overline{\lambda})^2$. So, expression (5.8) becomes

$$\widehat{N}_\sigma = \frac{1}{K_\sigma} \times \begin{pmatrix} \frac{A_\sigma U_\sigma(\lambda) V_\sigma(\overline{\lambda})}{\lambda - \overline{\lambda}} + \frac{A_\sigma U_\sigma(\overline{\lambda}) V_\sigma(\lambda)}{\overline{\lambda} - \lambda} & \frac{2A_\sigma U_\sigma(\lambda) U_\sigma(\overline{\lambda})}{\lambda - \overline{\lambda}} \\ \frac{2A_\sigma V_\sigma(\lambda) V_\sigma(\overline{\lambda})}{\lambda - \overline{\lambda}} & \frac{A_\sigma U_\sigma(\lambda) V_\sigma(\overline{\lambda})}{\lambda - \overline{\lambda}} + \frac{A_\sigma U_\sigma(\overline{\lambda}) V_\sigma(\lambda)}{\overline{\lambda} - \lambda} \end{pmatrix}.$$

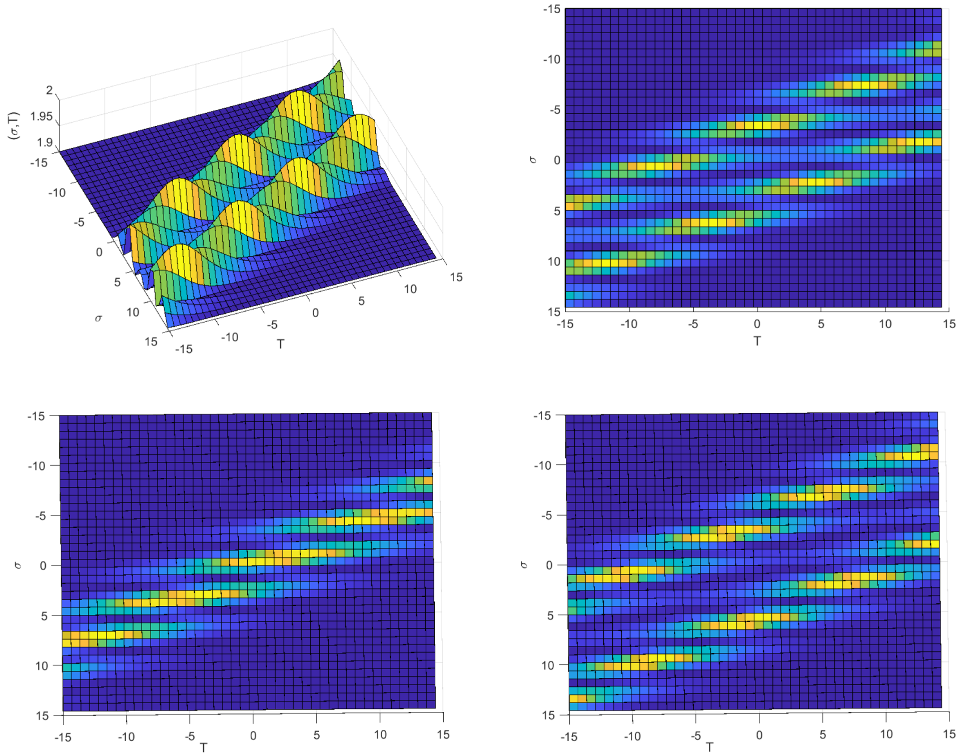

Fig. 1. Dynamics of $\hat{N}_{\sigma,11}=\hat{N}_{\sigma,22}$: for numerical values $\lambda = 1.9 + 0.71\mathrm{i}$, $\mu = 1.8 + 0.5\mathrm{i}$, $a = 2$ Parallel traveling of two double breathers, interaction and after interaction.

(5.9)

The expressions (5.9) represent the double bright and dark breathers shown in Figs. 3 and 4.

Now, further substituting $\overline{\lambda} = -\lambda$, we get the solutions of the elementary Darboux transformation as

$$\widehat{N}_{\sigma,11} = -\widehat{N}_{\sigma,22} = \frac{2(U_\sigma(\lambda)U_\sigma(\lambda) - V_\sigma(\lambda)V_\sigma(\lambda))}{U_\sigma(\lambda)U_\sigma(\lambda) + V_\sigma(\lambda)V_\sigma(\lambda)},$$

$$\widehat{N}_{\sigma,12} = \widehat{N}_{\sigma,21} = \frac{4\lambda U_\sigma(\lambda)V_\sigma(\lambda)}{U_\sigma(\lambda)U_\sigma(\lambda) + V_\sigma(\lambda)V_\sigma(\lambda)}.$$

By using Eq. (3.15), we can express

$$x_\sigma[1] = x_\sigma - \frac{2(U_\sigma(\lambda)U_\sigma(\lambda) - V_\sigma(\lambda)V_\sigma(\lambda))}{U_\sigma(\lambda)U_\sigma(\lambda) + V_\sigma(\lambda)V_\sigma(\lambda)},$$

$$u_\sigma[1] = -\frac{4\lambda U_\sigma(\lambda)V_\sigma(\lambda)}{U_\sigma(\lambda)U_\sigma(\lambda) + V_\sigma(\lambda)V_\sigma(\lambda)}.$$

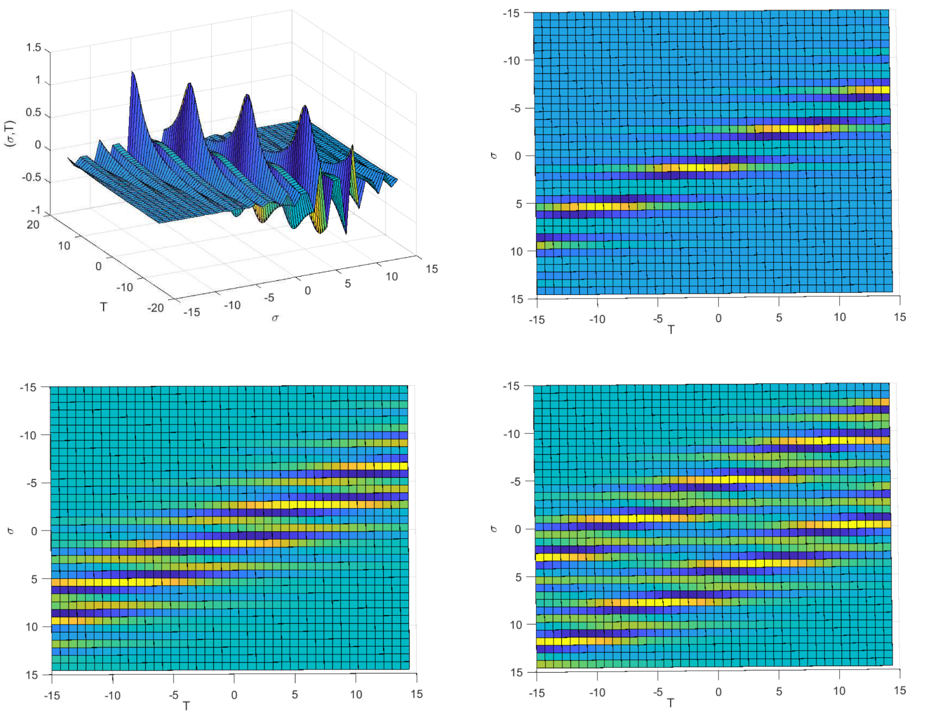

Fig. 2. Dynamics of $\hat{N}_{\sigma,21} = \hat{N}_{\sigma,12}$: for numerical values $\lambda = 2.9 + 0.71\mathrm{i}$, $\mu = 1.8 + 0.5\mathrm{i}$, $a = 2$, Traveling of bright and dark breather together, interaction and after interaction.

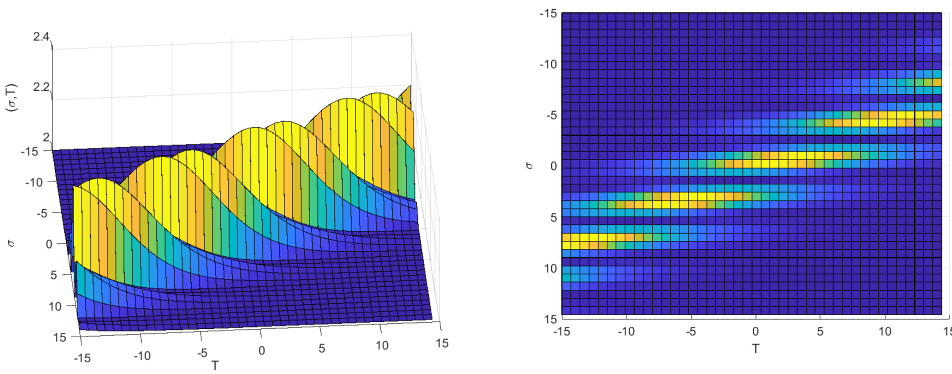

Fig. 3. Dynamics of double bright breather: for numerical values $\lambda = 2.9 + 0.71\mathrm{i}$, $a = 2$.

The above expressions represent the bright and dark together breather, peak soliton solutions depicted in Figs. 5 and 6. Thus, we have derived the expressions for two double bright and dark breathers solutions by using binary Darboux transformation and reduced them to elementary Darboux transformation and obtain the peak solutions for the sdSP equation.

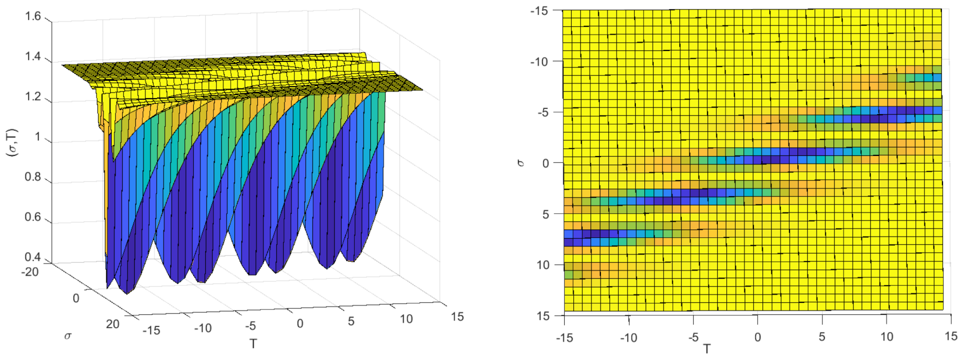

Fig. 4. Dynamics of double dark breather: for numerical values $\lambda = 2.9 + 0.71\mathrm{i}$, $a = 2$.

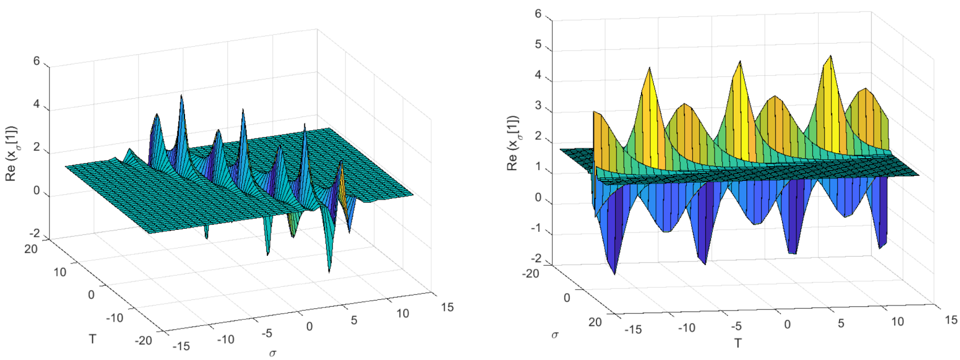

Fig. 5. Dynamics of breather solution: for numerical values $\lambda = 0.9$, $a = 2$.

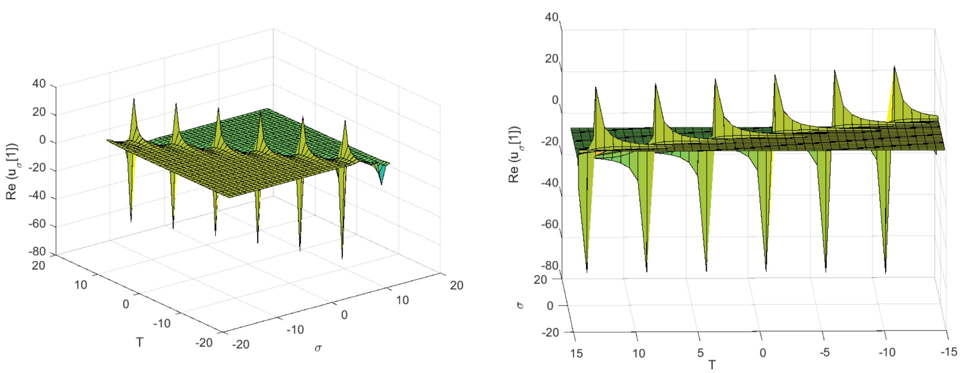

Fig. 6. Dynamics of peak solutions: for numerical values $\lambda = 1.9$, $a = 3$.

6. Conclusion

In this paper, we studied the Lax pair for a semi-discrete SP equation using a suitable hodograph transformation. By applying appropriate transformations, we converted the semi-discrete SP equation into semi-discrete versions of the mKdV and sine-Gordon equations. We then developed two types of Darboux transformations, including the binary Darboux transformation, and applied them to calculate peak and quasi-Grammian solutions. The explicit expressions for these solutions were derived. Finally, we reduced the binary Darboux transformation solutions to elementary Darboux transformation solutions. The bright and dark double breather solutions and peak solutions for the semi-discrete SP equation were plotted.

The breather solutions may play an important role in studying the propagation of ultra-SPs in optical fibers. This work can be extended in various interesting directions, such as studying the discrete SP equation and other semi-discrete integrable systems to calculate multi-Grammian, breather, and soliton solutions. It would also be interesting to study discrete rogue and hump wave solutions for semi-discrete integrable systems. Additionally, extending the theory of binary Darboux transformation to nonlocal integrable models would be valuable.

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