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Quasi-grammians and breathers of short pulse equation

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ABSTRACT

In this paper, we define and explore the Darboux as well as binary Darboux transformation for short pulse equation. Through the iteration of binary Darboux transformation, we obtain the quasi-grammian solutions. Further, we explore the explicit matrix solutions for the binary Darboux matrix and study their reduction into elementary Darboux matrix. At the end, the dynamics of bright and dark breather, soliton, loop soliton and rogue type solutions are given as an explicit examples.

1. Introduction

The integrable equation known as short pulse (SP) equation is introduced by Schafer and Wayne, given by

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \tag{1.1}$$

having real-valued function $u = u(x, t)$ investigated firstly for pseudo spherical surfaces [1,2]. Later on, Schafer and Wyne mathematically modeled the transmission of ultra short optical signals in nonisotropic nonlinear medium as SP equation [3]. The SP equation is basically an alternative to the cubic nonlinear Schrodinger (NLS) equation. It is also noticed that SP equation can better approximate the solutions of the Maxwell's equations [4]. By using the hodograph transformation SP equation can be transformed into sine-Gordon equation, coupled dispersionless equation and modified Korteweg-de Vries (mKdV) equation [5–7]. The integrability aspects of SP equation has been studied through different point of views, such as conservation laws [8,9] existence of bi-Hamiltonian structure [10], existence of Lax pair of wadati-Konno-Ichikawa (WKI) type [11], soliton solutions etc [12–19]. Also by using Riemann Hilbert method long-term asymptotic behaviors of SP equation were calculated in [20].

Many methods such as B*ä*cklund transformation [15], Hirota method [21–24], Darboux transformation [14], binary Darboux transformation [25] etc., have been employed to calculate the exact solutions of many nonlinear evolution equations. Among all of these, binary Darboux transformation is very effective technique.

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In this paper, we study standard binary Darboux transformation (BDT) for the first time which is the novelty of this article. Further, we express our solutions in the form of quasi-grammians rather than determinants. We present here the quasi-grammian solutions which are naturally arise from BDT. At the end, we present the interaction between breather and kink solitons with the grammians. Also, dark and bright one-soliton, breather and loop soliton solutions for SP equation are presented.

This paper is organized as follows. In Section 2, we discuss the brief description of the Lax pair of SP equation. In Section 3, we study the Darboux transformation. In Section 4, we define standard BDT for SP equation. In Section 5, by using BDT we calculate quasi-grammian solutions of SP equation. In Section 6, breather, one-soliton and loop soliton solutions are presented for zero seed solution. Also, for the non-zero seed solution periodic rogue type solutions are shown. The concluding remarks is written in Section 7.

2. Lax pair

The Lax pair for SP equation is given by

$$\partial_x \Psi = U\Psi = \lambda \begin{pmatrix} 1 & u_x \\ u_x & -1 \end{pmatrix} \Psi, \tag{2.1}$$

$$\partial_t \Psi = V\Psi = \frac{1}{2\lambda} \begin{pmatrix} u^2 & u^2 u_x \\ u^2 u_x & -u^2 \end{pmatrix} \Psi + \frac{1}{2} \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix} \Psi + \frac{\lambda}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi. \tag{2.2}$$

An appropriate hodograph transformation is applied which transforms the independent variables (x, t) into new variables (X, T), i.e.,

$$dX = \omega dx + \frac{1}{2} u^2 \omega dt, \qquad dT = dt. \tag{2.3}$$

Also the old dynamical variable u transforms into new dynamical variable related by

$$\omega^2 = 1 + \left(u_x\right)^2, \tag{2.4}$$

which transforms (1.1) into new equations

$$x_{XT} = -\frac{1}{2}(u^2)_X, \tag{2.5}$$

$$u_{XT} = x_X u, \tag{2.6}$$

where X, T in subscript represents the derivatives with respect to the variables X, T. The equation (2.5)-(2.6) can be expressed as the compatibility condition for the following linear system

$$\Psi_X = E(X, T; \lambda)\Psi = \left(\lambda \partial_X P\right)\Psi, \tag{2.7}$$

$$\Psi_T = F(X, T; \lambda)\Psi = \left(Q + \frac{1}{\lambda} R\right)\Psi, \tag{2.8}$$

where the matrices P, Q and R are given as

$$P = \begin{pmatrix} x & u \\ u & -x \end{pmatrix}, Q = \frac{1}{2} \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix}, R = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}. \tag{2.9}$$

The zero-curvature condition $E_T - F_X + [E, F] = 0$, of linear system (2.7), (2.8) gives the equation of motion which is equivalent to (2.5), (2.6).

3. Darboux transformation

Darboux transformation (DT) is an important technique to calculate solutions of integrable systems (for detail see [14,26–35]). Now, we define DT $D(\lambda)$ to obtain the solitonic solutions. The matrix solution transforms under the action of DT from space V to Darboux transformed space $\widetilde{V}$, i.e.,

$$\begin{aligned} D(\lambda) &: V \longrightarrow \widetilde{V} \\ &: \Psi \longrightarrow \widetilde{\Psi}. \end{aligned} \tag{3.1}$$

The one-fold DT on matrix solution Ψ is given by

$$\Psi[1] = D(\lambda)\Psi, \tag{3.2}$$

where $D(\lambda)$ represents the Darboux matrix. The Darboux transformed eigenfunction $\Psi[1]$ satisfies the following linear system (2.7), (2.8) as

$$\begin{aligned} \Psi_X[1] &= E[1]\Psi[1], \\ \Psi_T[1] &= F[1]\Psi[1], \end{aligned} \tag{3.3}$$

having the form of $E[1]$ and $F[1]$ as,

$$E[1] = \lambda \partial_X P[1],$$

$$F[1]=Q[1]+\frac{1}{\lambda}R[1]. \tag{3.4}$$

Also, the matrices $P[1]$, $Q[1]$ and $R[1]$ are given by

$$P[1]=\begin{pmatrix} x[1] & u[1] \\ u[1] & -x[1] \end{pmatrix}, Q[1]=\frac{1}{2}\begin{pmatrix} 0 & -u[1] \\ u[1] & 0 \end{pmatrix}, R[1]=\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}, \tag{3.5}$$

here $x[1]$ and $u[1]$ represents the Darboux transformed solutions of SP equation given by

$$x[1]_{XT}=-\frac{1}{2}(u^2[1])_X, \quad u[1]_{XT}=x[1]_X u[1]. \tag{3.6}$$

Now, the Darboux matrix is define as

$$D(\lambda)=\lambda^{-1}I-N, \tag{3.7}$$

where N is 2×2 auxiliary matrix of order 2×2, also I is the identity matrix having order 2×2. The choice for N is $N=\Gamma\Lambda^{-1}\Gamma^{-1}$, where Γ is the particular matrix solution of the linear system (2.7), (2.8) having 2×2 order which can be calculated by using j-eigenvector functions $\Psi(\lambda_j)|e\rangle_j$ determined at λ_j, $j=1,2$, where Λ is a diagonal matrix having 2×2 order with eigenvalues λ_1,λ_2. So, the matrix Γ can be defined as

$$\Gamma=\left(\Psi(\lambda_1)|e\rangle_1, \Psi(\lambda_2)|e\rangle_2\right), \tag{3.8}$$

evaluated at

$$\Lambda=\mathrm{diag}(\lambda_1,\lambda_2). \tag{3.9}$$

Now, by applying DT on linear system (3.4), we get the Darboux transformed matrices $P[1]$, $Q[1]$ and $R[1]$ as

$$P[1]=P-N, \tag{3.10}$$

$$Q[1]=Q+[R,N], \tag{3.11}$$

$$R[1]=R, \tag{3.12}$$

As the key feature of DT is to preserve the system i.e., if Ψ, P, Q and R, are the solutions of the Lax pair (2.7), (2.8), therefore $\Psi[1], P[1], Q[1]$ and $R[1]$ also satisfies the same equations.

4. Binary Darboux transformation

For the binary Darboux transformation (BDT) (for detail see [25,36–39]), consider the hat space which is the copied version of direct space, so that the corresponding solution $\widehat{\Psi}$ belongs to the hat space. Therefore, the linear system can be written as

$$\begin{aligned}\widehat{\Psi}_X &= \left(\lambda\partial_X\widehat{P}\right)\widehat{\Psi},\\ \widehat{\Psi}_T &= \left(\widehat{Q}+\frac{1}{\lambda}\widehat{R}\right)\widehat{\Psi},\end{aligned} \tag{4.1}$$

where the matrices $\widehat{P},\widehat{Q}$ and $\widehat{R}$ are given by

$$\widehat{P}=\begin{pmatrix} \hat{x} & \hat{u} \\ \hat{u} & -\hat{x} \end{pmatrix}, \widehat{Q}=\frac{1}{2}\begin{pmatrix} 0 & -\hat{u} \\ \hat{u} & 0 \end{pmatrix}, \widehat{R}=\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2}$$

Let D and $\widehat{D}$ be the two standard DT's which maps two matrix solutions Ψ and $\widehat{\Psi}$ onto a common matrix solution $\Psi[1]$ such that

$$\widehat{D}(\lambda)\widehat{\Psi}=D(\lambda)\Psi. \tag{4.3}$$

Then one can define BDT as $\widehat{B}=\widehat{D}^{-1}(\lambda)D(\lambda)$ such that the matrix solution can be given by $\widehat{\Psi}=\widehat{D}^{-1}(\lambda)D(\lambda)\Psi$. Now, assuming $i(\widehat{\Gamma})$ and Φ belongs to adjoint space, one can write $i(\widehat{\Gamma})=D^{(-1)\dagger}(\lambda)\Phi$. From $D^{\dagger}(\lambda)(i(\Gamma))=0$, we get $i(\Gamma)=\Gamma^{(-1)\dagger}$, similarly $i(\widehat{\Gamma})=\widehat{\Gamma}^{(-1)\dagger}$. So, we can derive the condition on $\widehat{\Gamma}$ i.e.,

$$\widehat{\Gamma}=\Gamma\Delta(\Gamma,\Phi)^{-1}, \tag{4.4}$$

where the eigenfunction potential Δ is given by

$$\Delta(\Gamma,\Phi)=(\Phi^{\dagger}\Gamma)(\lambda^{-1}I-\Lambda^{-1})^{-1}. \tag{4.5}$$

Similarly, the eigenfunction potential for adjoint space is given by

$$\Delta(\Psi,\Pi)=-(\lambda^{-1}I-F^{(-1)\dagger})^{-1}(\Pi^{\dagger}\Psi). \tag{4.6}$$

also

$$\widehat{\Pi}=\Pi\Delta(\Psi,\Pi)^{(-1)\dagger}, \tag{4.7}$$

where Γ, Π are the particular matrix solutions for direct and adjoint spaces. For the particular matrix solutions Γ, Π we express Eqs. (4.5) and (4.6) in the form of matrix, we get

$$F^{(-1)\dagger}\Delta(\Gamma,\Pi)-\Delta(\Gamma,\Pi)\Lambda^{-1}=\Pi^{\dagger}\Gamma. \tag{4.8}$$

Now, we derive the general expressions for BDT by using the values of $\widehat{D}(\lambda)$, $D(\lambda)$ and also with the help of (4.4) and (4.8), we get

$$\widehat{B}=I+\Gamma\Delta(\Gamma,\Phi)^{-1}(\lambda^{-1}I-F^{(-1)\dagger})^{-1}\Pi^{\dagger}.$$

Now, we are able to define BDT for the eigenfunctions Ψ and Φ as

$$\widehat{\Psi}=\Psi-\Gamma\Delta(\Gamma,\Pi)^{-1}\Delta(\Psi,\Pi), \tag{4.9}$$

$$\widehat{\Phi}=\Phi-\Pi\Delta(\Gamma,\Pi)^{(-1)\dagger}\Delta(\Gamma,\Phi)^{\dagger}. \tag{4.10}$$

Now, we express the solutions by using quasideterminants (for detail see [40,41])

$$\widehat{\Psi}=\begin{vmatrix}\Delta(\Gamma,\Pi) & \Delta(\Psi,\Pi)\\ \Gamma & \boxed{\Psi}\end{vmatrix}, \tag{4.11}$$

$$\widehat{\Phi}=\begin{vmatrix}\Delta(\Gamma,\Pi)^{\dagger} & \Delta(\Gamma,\Phi)^{\dagger}\\ \Pi & \boxed{\Phi}\end{vmatrix}. \tag{4.12}$$

5. Quasi-grammian solutions of short pulse equation

In order to calculate quasi-grammian solutions, applying the definition of BDT on the solution P of SP equation, we get

$$\widehat{P}-\widehat{\Gamma}F^{(-1)\dagger}\widehat{\Gamma}^{-1}=P-\Gamma\Lambda^{-1}\Gamma^{-1},$$

$$\widehat{P}=P-\Gamma\Lambda^{-1}\Gamma^{-1}+\widehat{\Gamma}F^{(-1)\dagger}\widehat{\Gamma}^{-1}, \tag{5.1}$$

by using Eq. (4.4), we get

$$\widehat{P}=P-\Gamma\Lambda^{-1}\Gamma^{-1}+\Gamma\Delta^{-1}F^{(-1)\dagger}\Delta\Gamma^{-1},$$

by substituting Eq. (4.8), we get

$$\begin{aligned}\widehat{P}&=P-\Gamma\Lambda^{-1}\Gamma^{-1}+\Gamma\Delta^{-1}\left(\Pi^{\dagger}\Gamma+\Delta\Lambda^{-1}\right)\Gamma^{-1},\\ &=P-\Gamma\Lambda^{-1}\Gamma^{-1}+\Gamma\Delta^{-1}\Pi^{\dagger}\Gamma\Gamma^{-1}+\Gamma\Delta^{-1}\Delta\Lambda^{-1}\Gamma^{-1},\\ &=P-\Gamma\Lambda^{-1}\Gamma^{-1}+\Gamma\Delta^{-1}\Pi^{\dagger}+\Gamma\Lambda^{-1}\Gamma^{-1},\\ &=P+\Gamma\Delta^{-1}\Pi^{\dagger},\\ &=P-\left(O-\Gamma\Delta^{-1}\Pi^{\dagger}\right).\end{aligned}$$

In terms of quasideterminants,

$$\widehat{P}=P-\begin{vmatrix}\Delta(\Gamma,\Pi) & \Pi^{\dagger}\\ \Gamma & \boxed{O}\end{vmatrix}.$$

We can calculate the K-th iteration of $\widehat{P}$ through the iteration of BDT given by

$$\widehat{P}=P-\begin{vmatrix}\Delta(\Gamma_1,\Pi_1) & \cdots & \Delta(\Gamma_K,\Pi_1) & \Pi_1^{\dagger}\\ \vdots & \cdots & \vdots & \vdots\\ \Delta(\Gamma_1,\Pi_K) & \cdots & \Delta(\Gamma_K,\Pi_K) & \Pi_K^{\dagger}\\ \Gamma_1 & \cdots & \Gamma_K & \boxed{I}\end{vmatrix} \tag{5.2}$$

The quasideterminant solutions (5.2) are called the quasi-grammian solutions of SP equation. For convenience, quasi-grammians in terms of matrix of order 2×2 are expressed as

$$\widehat{P}=P-\begin{pmatrix}\begin{vmatrix}\Delta(\Gamma,\Pi) & \Pi_1^{\dagger}\\ \Gamma_1 & \boxed{O}\end{vmatrix} & \begin{vmatrix}\Delta(\Gamma,\Pi) & \Pi_2^{\dagger}\\ \Gamma_1 & \boxed{O}\end{vmatrix}\\ \begin{vmatrix}\Delta(\Gamma,\Pi) & \Pi_1^{\dagger}\\ \Gamma_2 & \boxed{O}\end{vmatrix} & \begin{vmatrix}\Delta(\Gamma,\Pi) & \Pi_2^{\dagger}\\ \Gamma_2 & \boxed{O}\end{vmatrix}\end{pmatrix}.$$

Now, we write the quasi-grammian expressions for x and u as

$$\hat{x}=X-\begin{vmatrix}\Delta(\Gamma,\Pi) & \Pi_1^{\dagger}\\ \Gamma_1 & \boxed{O}\end{vmatrix},$$

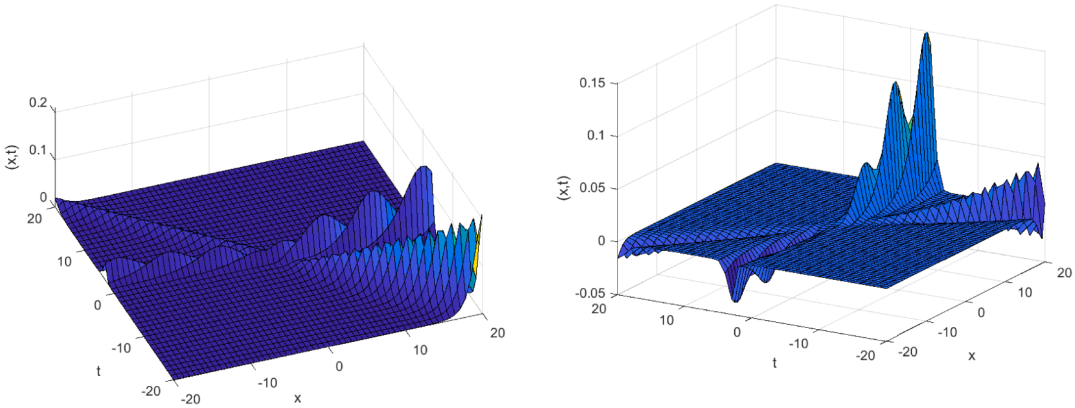

Fig. 1. Dynamics of $\hat{x}$: for numerical values $\lambda = 0.5 + 0.09i$, $\mu = 0.02 + 0.02i$,.

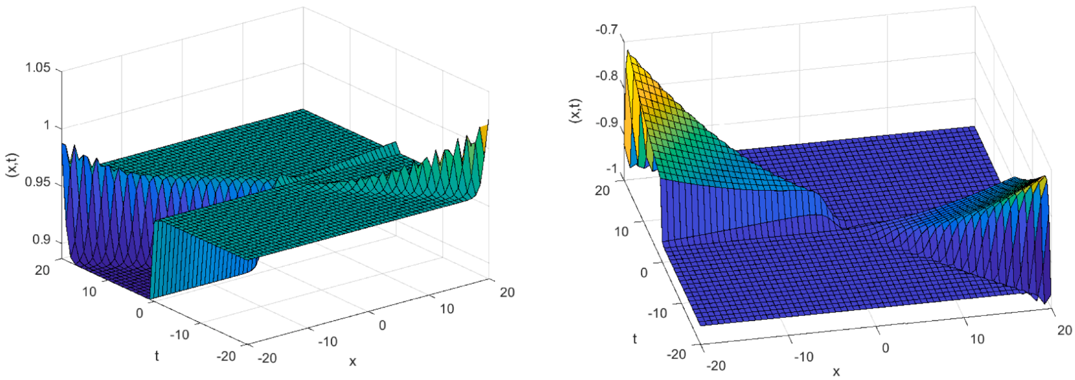

Fig. 2. Dynamics of $\hat{u}$: for numerical values $\lambda = 0.5 + 0.092i$, $\mu = 0.25 + 0.021i$.

$$
\begin{aligned}
&= X + \begin{vmatrix} \Delta(\Gamma,\Pi) & \Pi_2^{\dagger} \\ \Gamma_2 & \boxed{O} \end{vmatrix}, \\
\hat{u} &= u - \begin{vmatrix} \Delta(\Gamma,\Pi) & \Pi_2^{\dagger} \\ \Gamma_1 & \boxed{O} \end{vmatrix}, \\
&= u - \begin{vmatrix} \Delta(\Gamma,\Pi) & \Pi_1^{\dagger} \\ \Gamma_2 & \boxed{O} \end{vmatrix}.
\end{aligned}
$$

Therefore, by using BDT we can derive the quasi-grammian solutions for SP equation. The key feature of this method is that it could significantly enhance our approach towards nonlinear waves because it gives the different solutions from the DT. By using BDT we can calculate grammain and breather solutions.

6. Exact solutions

In this section, we obtain the expressions for the grammian and soliton solutions of SP equation by using BDT. In order to calculate the explicit expression, we take the seed solution $u = 0$ and $\partial_X x = 1$ (or $x = X$), so the linear system (2.7), (2.8) has the following form

$$\Psi_X = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi, \tag{6.1}$$

$$\Psi_T = \frac{1}{4\lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi. \tag{6.2}$$

The linear system (6.1), (6.2) leads to the matrix solution

$$\Psi(X,T;\lambda) = \begin{pmatrix} e^{\zeta} & 0 \\ 0 & e^{\zeta} \end{pmatrix}, \tag{6.3}$$

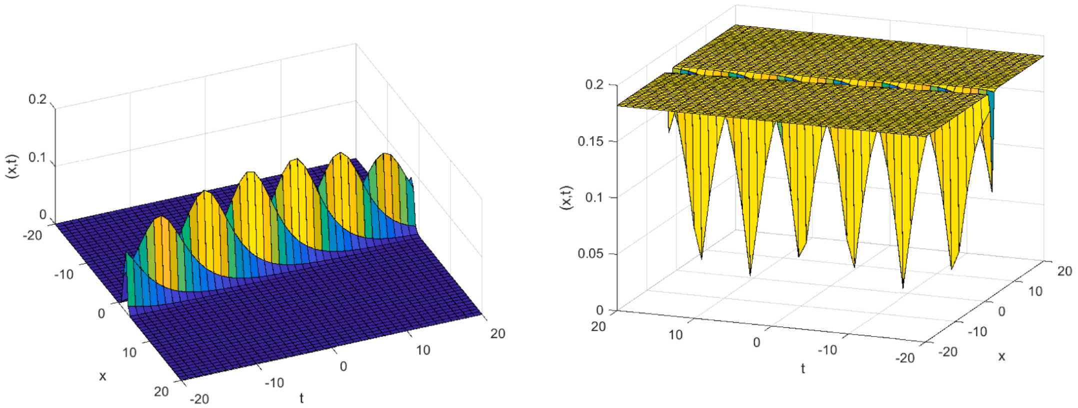

Fig. 3. Bright and dark breather for numerical value $\lambda = 1.5 + 0.091i$.

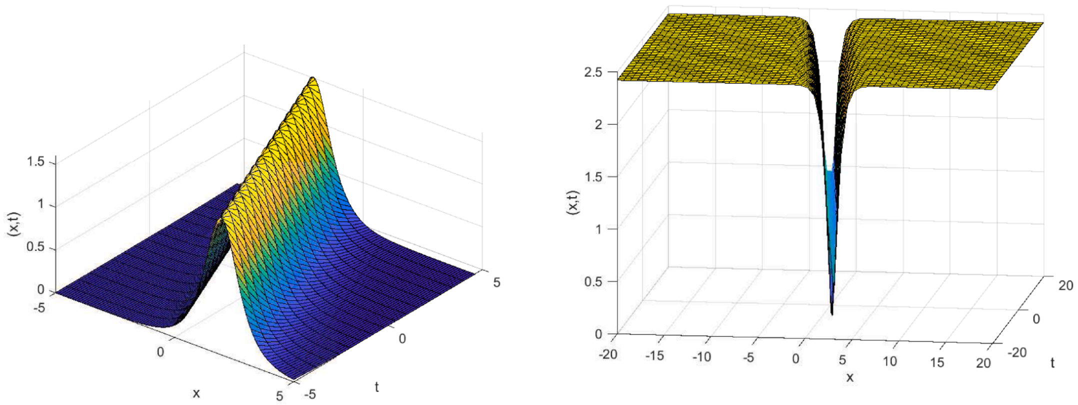

Fig. 4. Bright and dark soliton for numerical values $\lambda = 0.4 + 0.22i$.

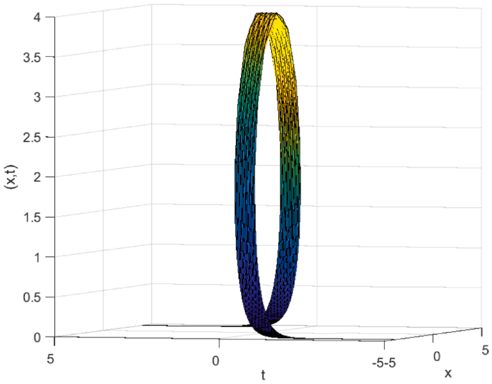

Fig. 5. Loop soliton for numerical value $\lambda = 0.9 + 0.02i$.

where $\zeta = \lambda X + \frac{1}{4\lambda}T$. The distinct matrix solution Γ is defined as

$$\Gamma = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\zeta} & e^{-\bar{\zeta}} \\ -e^{-\zeta} & e^{\bar{\zeta}} \end{pmatrix}. \tag{6.4}$$

Similarly, distinct matrix solution Π is written as

$$\Pi = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\eta} & e^{-\bar{\eta}} \\ -e^{-\eta} & e^{\bar{\eta}} \end{pmatrix}, \tag{6.5}$$

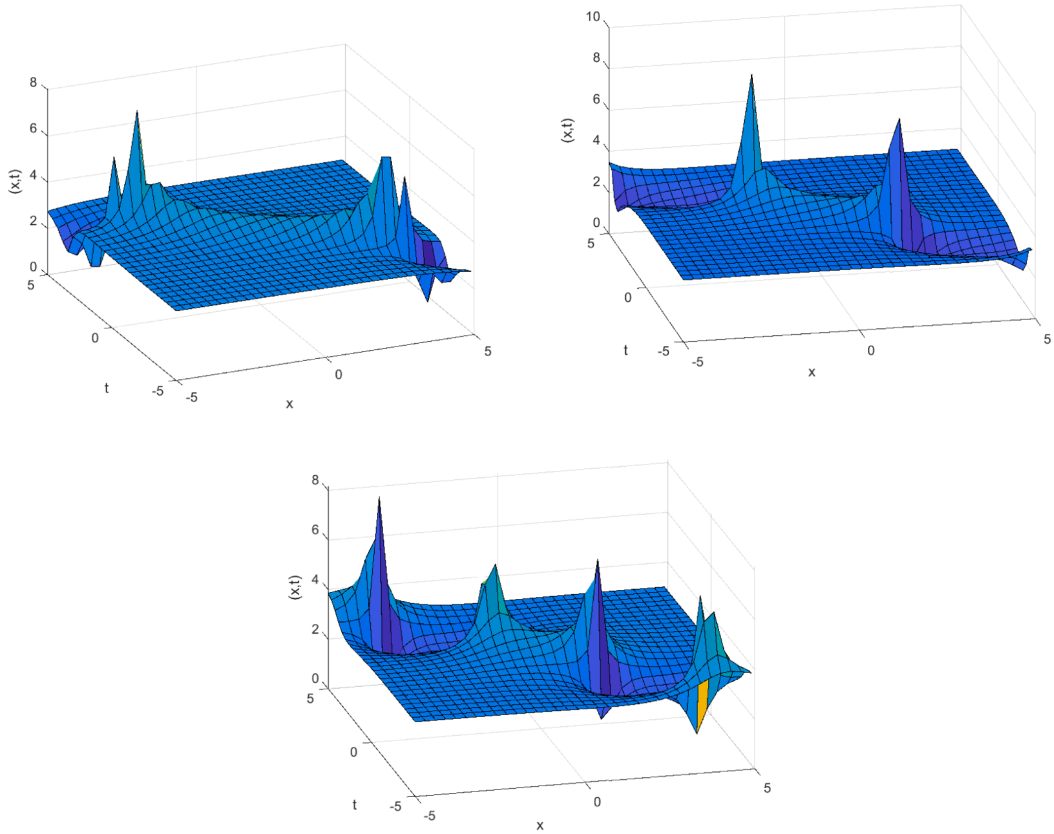

Fig. 6. Dynamics of nonzero background $u[1]$.

where $\eta = \mu X + \frac{1}{4\mu}T$. Now, by using (6.4), (6.5), we get the form of $\Delta(\Gamma, \Pi)$, i.e.,

$$\Delta(\Gamma,\Pi) = \begin{pmatrix} \frac{\cosh(\eta+\zeta)}{-\lambda+\mu} & \frac{\sinh(\eta-\bar{\zeta})}{-\zeta+\mu} \\ \frac{\sinh(-\bar{\eta}+\zeta)}{-\bar{\lambda}+\bar{\mu}} & \frac{\cosh(\bar{\eta}+\bar{\zeta})}{-\bar{\bar{\lambda}}+\bar{\mu}} \end{pmatrix}. \tag{6.6}$$

Now, consider

$$\begin{aligned} \hat{N} &= \Gamma\Delta^{-1}(\Gamma,\Pi)\Pi^{\dagger} = \frac{1}{2K}\begin{pmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{pmatrix}, \\ &= \frac{1}{2K}\begin{pmatrix} \begin{array}{c} \left(\frac{\cosh(\bar{\eta}+\bar{\zeta})}{\bar{\mu}-\bar{\bar{\lambda}}}\right)e^{(\eta+\zeta)} \\ -\left(\frac{\sinh(\eta-\bar{\zeta})}{\mu-\bar{\bar{\lambda}}}\right)e^{(-\bar{\eta}+\zeta)} \\ -\left(\frac{\sinh(-\bar{\eta}+\zeta)}{\bar{\mu}-\lambda}\right)e^{(\eta-\bar{\zeta})} \\ +\left(\frac{\cosh(\eta+\zeta)}{\mu-\lambda}\right)e^{(-\bar{\eta}-\bar{\zeta})} \end{array} & \begin{array}{c} -\left(\frac{\cosh(\bar{\bar{\eta}}+\bar{\zeta})}{\bar{\mu}-\bar{\bar{\lambda}}}\right)e^{(-\eta+\zeta)} \\ -\left(\frac{\sinh(\eta-\bar{\zeta})}{\mu-\bar{\bar{\lambda}}}\right)e^{(\bar{\eta}+\zeta)} \\ +\left(\frac{\cosh(\eta+\zeta)}{\mu-\lambda}\right)e^{(\bar{\eta}-\bar{\zeta})} \\ -\left(\frac{\sinh(-\bar{\eta}+\zeta)}{\bar{\mu}-\lambda}\right)e^{(-\eta-\bar{\zeta})} \end{array} \\ \\ \begin{array}{c} -\left(\frac{\cosh(\bar{\eta}+\bar{\zeta})}{\bar{\mu}-\bar{\bar{\lambda}}}\right)e^{(\eta-\zeta)} \\ +\left(\frac{\sinh(\eta-\bar{\zeta})}{\mu-\bar{\bar{\lambda}}}\right)e^{(-\bar{\eta}-\zeta)} \\ +\left(\frac{\cosh(\eta+\zeta)}{\mu-\lambda}\right)e^{(-\bar{\eta}+\bar{\zeta})} \\ -\left(\frac{\sinh(-\bar{\eta}+\zeta)}{\bar{\mu}-\lambda}\right)e^{(\eta+\bar{\zeta})} \end{array} & \begin{array}{c} \left(\frac{\cosh(\bar{\eta}+\bar{\zeta})}{\bar{\mu}-\bar{\bar{\lambda}}}\right)e^{(-\eta-\zeta)} \\ +\left(\frac{\sinh(\eta-\bar{\zeta})}{\mu-\bar{\bar{\lambda}}}\right)e^{(\bar{\eta}-\zeta)} \\ -\left(\frac{\sinh(\bar{\bar{\eta}}+\zeta)}{\bar{\mu}-\lambda}\right)e^{(-\eta+\bar{\zeta})} \\ +\left(\frac{\cosh(\eta+\zeta)}{\mu-\lambda}\right)e^{(\bar{\eta}+\bar{\zeta})} \end{array} \end{pmatrix}, \end{aligned} \tag{6.7}$$

where

$$K = \frac{\cosh(\eta+\zeta)\cosh(\bar{\eta}+\bar{\zeta})}{(\mu-\lambda)(\bar{\mu}-\bar{\lambda})} - \frac{\sinh(\eta-\bar{\zeta})\sinh(-\bar{\eta}+\bar{\zeta})}{(\mu-\bar{\lambda})(\bar{\mu}-\lambda)}. \tag{6.8}$$

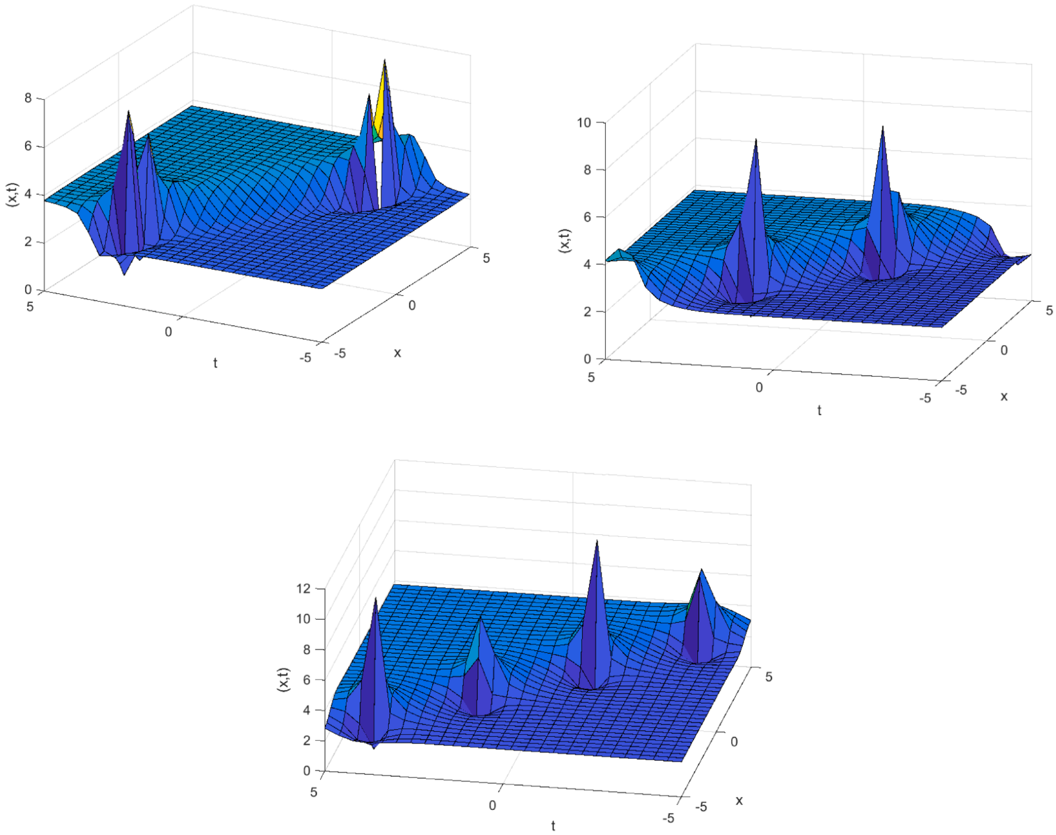

Fig. 7. Dynamics of nonzero background $x[1]$.

$$\hat{x} = X - \hat{N}_{11}, \tag{6.9}$$

$$\hat{u} = \hat{N}_{12}. \tag{6.10}$$

Fig. 1 represents the interaction of breather and grammian while the Fig. 2 shows the interaction of kink and grammian. So, by using BDT we are able to calculate the solutions which cannot be derive with the help of elementary DT. It means BDT has a key role in developing the better understanding about the nonlinear systems. Also, we can reduce the expressions of BDT into elementary DT by some particular substitutions.

6.1. Reduction

Now, substitute $\overline{\mu} = \lambda$, $\mu = \bar{\lambda}$, we get the bright and dark breather respectively shown in Fig. 3
Now, further substituting $\lambda = \bar{\lambda}$, we reduce our solutions to elementary DT as

$$x[1] = X - \frac{1}{2\lambda}\tanh(2\zeta),$$
$$u[1] = \frac{1}{2\lambda}\sec\text{h}(2\zeta), \tag{6.11}$$

Bright and dark solitons and also the loop soliton is presented in Figs. 4 and 5 for vanishing background ($u = 0$) seed solution. Now, for nonvanishing background seed solution ($u = x = 3$), the dynamics of Eq. (6.11) presents in Figs. 6 and 7 for $\lambda = 0.4 + 0.3i, 0.4 + 0.35i$ and $0.4 + 0.39i$.

So, we have obtained explicit expressions for BDT which shows the interaction of grammian with breather and also the interaction of kink with grammian. Further, we reduce these solutions to elementary DT by some suitable substitution and derive the loop and bright, dark soliton solutions.

7. Conclusion

The aim of paper is to propose BDT for SP equation. The general idea of BDT is to keep two sets of eigenfunctions and adjoint eigenfunctions associated with nonlinear equation invariant under BDT. We developed skeleton of BDT and applied on SP equation to calculate quasi-grammian solutions. The explicit expressions for quasi-grammian solutions were derived. Finally, we reduced the BDT solutions to elementary DT solutions. The grammians, dark and bright breather and soliton solutions, also loop soliton for SP equation were plotted for zero seed solution. Also, for the non-zero seed solution periodic rogue solutions are depicted.

In order to study the propagation of ultra-short pulses in an optical fibre, breather solutions may plays an important role. Also, one important characteristic that both dark and bright solitons have in common is their robustness, which is crucial for guaranteeing their usefulness in optical communications. These solutions can also continue to move at the same speed and form extended distances.

Recently, with the development of numerical algorithms and also the physical knowledge embedded into deep neural networks, physics-informed neural network (PINN) has been used to solve many nonlinear complex problems and got significant results with the small amount of data [42–44].

The present work can be extended to many directions, for example one can study discrete and semi-discrete versions of many integrable systems and study their multi-soliton, breather, rogue and hump wave solutions. It could also be interesting that one can apply PINN technique on continuous and discrete integrable systems.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no conflict of interest.

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