Eur. Phys. J. Plus (2025) 140:1254
https://doi.org/10.1140/epjp/s13360-025-07190-3

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

Breathers and quasi-Grammians of a discrete generalized coupled dispersionless system

Zeeshan Amjad[1,a], **Bushra Haider**[2,b], **Wen-Xiu Ma**[1,3,4,5,c], **Faisal Javed**[6,d]

[1] Department of Mathematics, Zhejiang Normal University, 321004 Jinhua, PR China
[2] Department of Physics, University of the Punjab, Quaid-e-Azam Campus, 54590 Lahore, Pakistan
[3] Research Center of Astrophysics and Cosmology, Khazar University, 41 Mehseti Street, 1096 Baku, Azerbaijan
[4] Department of Mathematics and statistics, University of South Florida, Tampa FL 33620-5700, USA
[5] Material Science Innovation and Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, 2735 Mmabatho, South Africa
[6] College of Transportation, Tongji University, 201804 Shanghai, PR China

Received: 6 July 2025 / Accepted: 6 December 2025

Abstract Dispersionless hierarchies are of much consideration because of their arising in many problems of physics and applied mathematics, like quantum field theories, string theory, etc. In this paper, we consider and explore the Darboux transformation as well as the binary Darboux transformation for discrete generalized coupled dispersionless system and calculate the multi-soliton and quasi-Grammian solutions. Furthermore, we present the explicit expressions of Grammians and soliton solutions. Finally, as an explicit example, we present Grammians, discrete breather, dark and bright soliton.

1 Introduction

Dispersionless integrable systems have a great deal of interest for scientists because of their numerous applications in many areas of mathematics and physics [1–7]. Most of the dispersionless systems belong to a class of family where the systems arise as a quasiclassical limit of ordinary integrable systems containing a dispersion term. But the system under discussion here is considered as dispersionless due to the lack of dispersion term instead of quasiclassical limit. Coupled dispersionless systems have received much consideration due to integrability structure and soliton dynamics. Like the continuous coupled dispersionless system, which show many integrability aspects, the discrete generalized coupled dispersionless (dGCD) system also preserves integrability. The Darboux transformation (DT) of dGCD system is studied in [8], and soliton solutions have been presented.

In soliton theory, binary Darboux transformation (BDT) is very effective tool in generating the wide class of exact solutions. BDT is one of the well familiar techniques used to calculate Grammian and soliton solutions of integrable systems [9–15]. The general mechanism of this technique is to keep both direct and adjoint spectral problems which are associated with the given nonlinear equation invariant under BDT. In this work, we extend the concept of [8] and express the solutions which cannot be calculated through the DT. We will present the schematic way through which dGCD system reduces to semi-discrete version as well as the continuous GCD system. We will develop the skeleton of BDT and apply on dGCD system to evaluate the quasi-Grammian solutions which cannot be obtain through the DT. Also, we will show that how the BDT solutions can be transform to the DT solutions through the reduction of specific spectral parameter. Therefore, unless of applying the DT, we can also calculate the DT solutions which is the beautiful feature of BDT. At last, we will give the brief discussion on the importance of solutions.

Coupled dispersionless systems and their discrete generalizations appear in a variety of physical and geometric settings. They model nonlinear wave propagation in nearly dispersion-free media, including shallow water flows, short-pulse optical propagation and elastic wave dynamics. Such systems also arise in plasma physics, discrete mechanical lattices and the semiclassical limits of integrable field theories. In addition, coupled dispersionless equations are gauge-equivalent to geometric motions of curves and surfaces, and their discrete analogs naturally emerge in integrable lattice models, Hirota-type bilinear discretizations and discrete differential geometry. The breather and quasi-Grammian solutions obtained in this work therefore offer insight into localized oscillatory structures relevant to these physical and mathematical contexts.

[a] e-mail: z.amjadphysics@gmail.com
[b] e-mail: bushrahaider.physics@pu.edu.pk
[c] e-mail: mawx@cas.usf.edu (corresponding author)
[d] e-mail: faisaljaved.math@gmail.com

Published online: 26 December 2025

This paper is organized as follows. In Sect. 2, we give brief description about the DT of dGCD. In Sect. 3, we define BDT for the dGCD system. In Sect. 4, the dynamics of solutions like traveling of Grammian, bright and dark soliton and discrete breather are presented. Concluding remarks are written in Sect. 5.

The Lax pair of dGCD system is given by

$$E\Psi_{n,\,m} = \big(I + \eta^{-1} a[Q_{n+1,\,m} - Q_{n,\,m}]\big)\Psi_{n,\,m} \equiv \mathcal{A}_{n,\,m}\Psi_{n,\,m}, \tag{1.1}$$

$$F\Psi_{n,\,m} = \big(I + b[Q_{n,\,m+1}G - GQ_{n,\,m} + \eta G]\big)\Psi_{n,\,m} \equiv \mathcal{B}_{n,\,m}\Psi_{n,\,m}, \tag{1.2}$$

where E and F are the shift operators, i.e., $E\Psi_{n,\,m} = \Psi_{n+1,\,m}$, $F\Psi_{n,\,m} = \Psi_{n,\,m+1}$ also I=$diag(1, 1)$, G=$diag(1, -1)$ and $Q_{n,\,m}$ is 2×2 auxiliary matrix defined by

$$Q_{n,m} = \begin{pmatrix} \alpha_{n,m} & \beta_{n,m} \\ \beta_{n,m} & -\alpha_{n,m} \end{pmatrix},$$

where $\alpha_{n,m}$, $\beta_{n,m}$ are the dynamical variables depend upon variables m and n written in subscripts defined over a lattice. Also in equations (1.1), (1.2) a and b represent the lattice parameters along n and m respectively. The linear system (1.1), (1.2) fulfill the compatibility condition $\mathcal{A}_{n,\,m}\mathcal{B}_{n+1,\,m} = \mathcal{A}_{n,\,m+1}\mathcal{B}_{n,\,m}$ which give rise nonlinear dGCD system as

$$\begin{aligned} &a\big(Q_{n+1,\,m+1} - Q_{n+1,\,m} - Q_{n,\,m+1} + Q_{n,\,m}\big) + ab\big(Q_{n+1,\,m+1} - Q_{n,\,m+1}\big)\big(Q_{n,\,m+1}G - GQ_{n,\,m}\big) \\ &= ab\big(Q_{n+1,\,m+1}G - GQ_{n+1,\,m}\big)\big(Q_{n+1,\,m} - Q_{n,\,m}\big). \end{aligned} \tag{1.3}$$

The above expression (1.3) is the equation of motion which can be converted into continuum limit by setting parameters approaches to zero, i.e., a, $b \to 0$. The equation of motion in terms of dynamical variables can be written as

$$\begin{aligned} &\big(\alpha_{n+1,m+1} - \alpha_{n+1,m} - \alpha_{n,m+1} + \alpha_{n,m}\big) + \frac{b}{2}\{\big(\beta_{n+1,m+1} - \beta_{n,m+1} + \beta_{n+1,m} - \beta_{n,m}\big) \\ &\quad \times (\beta_{n,m+1} + \beta_{n+1,m})\} - \frac{b}{2}\{\big(\alpha_{n+1,m+1} - \alpha_{n+1,m}\big)\big(\alpha_{n+1,m} - \alpha_{n,m}\big) \\ &\quad - (\alpha_{n,m+1} - \alpha_{n,m})\big(\alpha_{n+1,m+1} - \alpha_{n,m+1}\big)\} = 0, \\ &\big(\beta_{n+1,m+1} - \beta_{n+1,m} - \beta_{n,m+1} + \beta_{n,m}\big) - \frac{b}{2}\{\big(\alpha_{n+1,m+1} - \alpha_{n,m}\big)\big(\beta_{n,m+1} + \beta_{n+1,m}\big) \\ &\quad - \big(\alpha_{n,m+1} - \alpha_{n+1,m}\big)\big(\beta_{n+1,m+1} + \beta_{n,m}\big)\} = 0. \end{aligned}$$

Eq. (1.3) can be converted into continuous x and t discrete form, we use $\left(\lim_{a\to 0} \frac{g_{n+1,m} - g_{n,m}}{a} = \frac{dg}{dx}\right)$, leads

$$\partial_x(Q_{m+1} - Q_m) + b\partial_x Q_{m+1}(Q_{m+1}G - GQ_m) = b(Q_{m+1}G - GQ_m)\partial_x Q_m,$$

which is the semi-discrete (t discrete) equation of motion of GCD system. Now again for continuous also in t direction we take $\left(\lim_{b\to 0} \frac{f_{n,m+1} - f_{n,m}}{b} = \frac{df}{dt}\right)$, yields

$$\partial_t\partial_x Q + [[G,\ Q],\ \partial_x Q] = 0. \tag{1.4}$$

Eq. (1.4) is known as the equation of motion for continuous GCD system.

2 Discrete Darboux transformation

DT is very powerful generating technique because it involves purely algebraic algorithms [16–22]. The onefold DT on matrix solution $\Psi_{n,m}$ is define as

$$\widetilde{\Psi}_{n,\,m} = D_{n,\,m}(\eta)\Psi_{n,\,m}, \tag{2.1}$$

where the Darboux matrix $D_{n,\,m}(\eta)$ is given by

$$D_{n,\,m}(\eta) = \eta I - \Pi_{n,\,m}, \tag{2.2}$$

where I=$diag(1, 1)$ and $\Pi_{n,\,m}$ is non-singular matrix which is yet to define. The action of DT transforms $\Psi_{n,m}$ in space Υ to $\widetilde{\Psi}_{n,m}$ in space $\widetilde{\Upsilon}$.

$$D_{n,m}(\eta) : \Upsilon \longrightarrow \widetilde{\Upsilon}.$$

The invariance of linear system (1.1), (1.2) under DT as:

$$\widetilde{G} = G,$$

$$\widetilde{Q}_{n+1,\,m} - \widetilde{Q}_{n,\,m} = Q_{n+1,\,m} - Q_{n,\,m} - \big(\Pi_{n+1,\,m} - \Pi_{n,\,m}\big). \tag{2.3}$$

The equation (2.3) can also be expressed as

$$\widetilde{Q}_{n,\,m} = Q_{n,\,m} - \Pi_{n,\,m}. \tag{2.4}$$

The conditions imposed on matrix $\Pi_{n,\,m}$ are

$$\Pi_{n+1,\,m} - \Pi_{n,\,m} = Q_{n+1,\,m} - Q_{n,\,m} - \Pi_{n+1,\,m}\big(Q_{n+1,\,m} - Q_{n,\,m}\big)\Pi^{-1}_{n,\,m}, \tag{2.5}$$

$$\begin{aligned}\Pi_{n,\,m+1} - \Pi_{n,\,m} =& b\big(Q_{n,\,m+1}G - GQ_{n,\,m}\big)\Pi_{n,\,m} - b\Pi_{n,\,m+1}\big(Q_{n,\,m+1}G - GQ_{n,\,m}\big)\\ &- b\big(\Pi_{n,\,m+1}G - G\Pi_{n,\,m}\big)\Pi_{n,\,m}.\end{aligned} \tag{2.6}$$

Now, we develop the explicit form of matrix $\Pi_{n,\,m}$ for dGCD system. For this purpose, we define N constant parameters $(\eta_1,\ \eta_2,\ \ldots,\ \eta_N)$. For each value, there exists a particular column solution $|k_i\rangle_{n,\,m}$ to the linear system (1.1), (1.2), i.e.,

$$|k_i\rangle_{n+1,\,m} = \Big[1 + \eta_i^{-1}\big(Q_{n+1,\,m} - Q_{n,\,m}\big)\Big]|k_i\rangle_{n,\,m}, \tag{2.7}$$

$$|k_i\rangle_{n,\,m+1} = \Big[1 + b\big(Q_{n,\,m+1}G - GQ_{n,\,m} + \eta_i G\big)\Big]|k_i\rangle_{n,\,m}. \tag{2.8}$$

For the diagonal matrix $\Xi = \mathrm{diag}(\eta_1,\ \eta_2,\ \ldots,\ \eta_N)$, the linear system (2.7), (2.8) can be expressed in matrix form as

$$K_{n+1,\,m} = K_{n,\,m} + \big(Q_{n+1,\,m} - Q_{n,\,m}\big)K_{n,\,m}\Xi^{-1}, \tag{2.9}$$

$$K_{n,\,m+1} = K_{n,\,m} + b\big(Q_{n,\,m+1}G - GQ_{n,\,m}\big)K_{n,\,m} + bGK_{n,\,m}\Xi. \tag{2.10}$$

If $\det K_{n,\,m} \neq 0$, we can define $\Pi_{n,\,m}$ in terms of distinct matrix solution as

$$\Pi_{n,\,m} = K_{n,\,m}\Xi K^{-1}_{n,\,m}. \tag{2.11}$$

It is straight forward to check that the choice of $\Pi_{n,m}$ satisfies the conditions (2.5) and (2.6) as appear due to covariance of Lax pair (1.1), (1.2) under DT. For $\Pi_{n,m} = K_{n,m}\Xi K^{-1}_{n,m}$, it seems appropriate here to express the solution $\widetilde{\Psi}_{n,m} = D_{n,m}\Psi_{n,m} = (\eta I - \Pi_{n,\,m})\Psi_{n,m}$ in terms of quasideterminants[1]

$$\begin{aligned}\widetilde{\Psi}_{n,\,m} \equiv& D_{n,\,m}(\eta)\Psi_{n,\,m} = \eta\Psi_{n,\,m} - K_{n,\,m}\Xi K^{-1}_{n,\,m}\Psi_{n,\,m},\\ =&\eta\Psi_{n,\,m} + \begin{vmatrix} K_{n,\,m} & \Psi_{n,\,m}\\ K_{n,\,m}\Xi & \boxed{O}\end{vmatrix} = \begin{vmatrix} K_{n,\,m} & \Psi_{n,\,m}\\ K_{n,\,m}\Xi & \boxed{\eta\Psi_{n,\,m}}\end{vmatrix}.\end{aligned} \tag{2.12}$$

Similarly, the expression (2.4) can be expressed as

$$\begin{aligned}\widetilde{Q}_{n,\,m} =& Q_{n,\,m} - K_{n,\,m}\Xi K^{-1}_{n,\,m},\\ =& Q_{n,\,m} + \begin{vmatrix} K_{n,\,m} & I\\ K_{n,\,m}\Xi & \boxed{O}\end{vmatrix}.\end{aligned} \tag{2.13}$$

The N-fold DT of $\Psi_{n,\,m}$ as (for detail see [8])

$$\Psi_{n,\,m}[N] = \begin{vmatrix} K_{n,\,m,\,1} & K_{n,\,m,\,2} & \cdots & K_{n,\,m,\,N} & \Psi_{n,\,m}\\ K_{n,\,m,\,1}\Xi_1 & K_{n,\,m,\,2}\Xi_2 & \cdots & K_{n,\,m,\,N}\Xi_N & \eta\Psi_{n,\,m}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{n,\,m,\,1}\Xi_1^N & K_{n,\,m,\,2}\Xi_2^N & \cdots & K_{n,\,m,\,N}\Xi_N^N & \boxed{\eta^N\Psi_{n,\,m}}\end{vmatrix}. \tag{2.14}$$

Similarly, $Q_{n,\,m}[N]$ can be expressed as

$$Q_{n,\,m}[N] = Q_{n,\,m} + \begin{vmatrix} K_{n,\,m,\,1} & K_{n,\,m,\,2} & \cdots & K_{n,\,m,\,N} & O\\ K_{n,\,m,\,1}\Xi_1 & K_{n,\,m,\,2}\Xi_2 & \cdots & K_{n,\,m,\,N}\Xi_N & O\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{n,\,m,\,1}\Xi_1^{N-1} & K_{n,\,m,\,2}\Xi_2^{N-1} & \cdots & K_{n,\,m,\,N}\Xi_N^{N-1} & I\\ K_{n,\,m,\,1}\Xi_1^N & K_{n,\,m,\,2}\Xi_2^N & \cdots & K_{n,\,m,\,N}\Xi_N^N & \boxed{O}\end{vmatrix}. \tag{2.15}$$

[1] We will use the notion of quasideterminants. In this paper, we will consider only quasideterminants that are expanded about an $m \times m$ matrix. The quasideterminant of $J \times J$ matrix expanded about the $m \times m$ matrix D is defined as

$$\begin{vmatrix} A & B\\ C & \boxed{D}\end{vmatrix} = D - CA^{-1}B.$$

Now, we are going to develop the DT on adjoint Lax pair. For this purpose, first we take adjoint of linear system (1.1), (1.2) as

$$E\Phi_{n,m} = \Phi_{n,m}\mathcal{A}^{\dagger}_{n,m}, \tag{2.16}$$

$$F\Phi_{n,m} = \Phi_{n,m}\mathcal{B}^{\dagger}_{n,m}, \tag{2.17}$$

where the matrix $\mathcal{A}^{\dagger}_{n,m}$ and $\mathcal{B}^{\dagger}_{n,m}$ are given as

$$\mathcal{A}^{\dagger}_{n,m} = -I - \zeta^{-1}a\left(Q^{\dagger}_{n+1,\,m} - Q^{\dagger}_{n,\,m}\right), \tag{2.18}$$

$$\mathcal{B}^{\dagger}_{n,m} = -I - b\left(Q^{\dagger}_{n,\,m+1}G^{\dagger} - G^{\dagger}Q^{\dagger}_{n,\,m} + \zeta G^{\dagger}\right). \tag{2.19}$$

Here ζ is a real (or complex) parameter and $\Phi_{n,m}$ is eigen matrix solution belongs to adjoint $\Upsilon^{\dagger}$ space. The linear system (2.16), (2.17) fulfill the compatibility condition $\mathcal{A}^{\dagger}_{n,\,m}\mathcal{B}^{\dagger}_{n+1,\,m} = \mathcal{A}^{\dagger}_{n,\,m+1}\mathcal{B}^{\dagger}_{n,\,m}$ which gives the equation of motion for adjoint dGCD system as

$$\begin{aligned} &a\left(Q^{\dagger}_{n+1,\,m+1} - Q^{\dagger}_{n+1,\,m} - Q^{\dagger}_{n,\,m+1} + Q^{\dagger}_{n,\,m}\right) + ab\left(Q^{\dagger}_{n+1,\,m+1} - Q^{\dagger}_{n,\,m+1}\right)\left(Q^{\dagger}_{n,\,m+1}G^{\dagger} - G^{\dagger}Q^{\dagger}_{n,\,m}\right) \\ =&ab\left(Q^{\dagger}_{n+1,\,m+1}G^{\dagger} - G^{\dagger}Q^{\dagger}_{n+1,\,m}\right)\left(Q^{\dagger}_{n+1,\,m} - Q^{\dagger}_{n,\,m}\right). \end{aligned} \tag{2.20}$$

The onefold DT on $\Phi_{n,m}$ is defined as

$$\widetilde{\Phi}_{n,m} \equiv D_{n,m}(\zeta)\Phi_{n,m} = -(\zeta I - \Omega_{n,m})\Phi_{n,m} = -(\zeta I - H_{n,m}\digamma H_{n,m})\Phi_{n,m}, \tag{2.21}$$

where $\digamma$ is the eigen valued matrix which have only the diagonal entries $(\zeta_1, \ldots, \zeta_n)$. Also the matrix $H_{n,m}$ is the particular matrix solution of the linear system (2.16), (2.17). Now, the N-fold DT on matrix solution $\Phi_{n,m}$ can be expressed as

$$\Psi_{n,\,m}[N] = \begin{vmatrix} H_{n,\,m,\,1} & H_{n,\,m,\,2} & \cdots & H_{n,\,m,\,N} & \Phi_{n,\,m} \\ H_{n,\,m,\,1}\digamma_1 & H_{n,\,m,\,2}\digamma_2 & \cdots & H_{n,\,m,\,N}\digamma_N & \zeta\Phi_{n,\,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{n,\,m,\,1}\digamma_1^N & H_{n,\,m,\,2}\digamma_2^N & \cdots & H_{n,\,m,\,N}\digamma_N^N & \boxed{\zeta^N\Phi_{n,\,m}} \end{vmatrix}. \tag{2.22}$$

Similarly, $Q^{\dagger}_{n,\,m}[N]$ can be expressed as

$$Q^{\dagger}_{n,\,m}[N] = Q^{\dagger}_{n,\,m} + \begin{vmatrix} H_{n,\,m,\,1} & H_{n,\,m,\,2} & \cdots & H_{n,\,m,\,N} & O \\ H_{n,\,m,\,1}\digamma_1 & H_{n,\,m,\,2}\digamma_2 & \cdots & H_{n,\,m,\,N}\digamma_N & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{n,\,m,\,1}\digamma_1^{N-1} & H_{n,\,m,\,2}\digamma_2^{N-1} & \cdots & H^{N-1}_{n,\,m,\,N}\digamma_N^{N-1} & I \\ H_{n,\,m,\,1}\digamma_1^{N} & H_{n,\,m,\,2}\digamma_2^{N} & \cdots & H^{N}_{n,\,m,\,N}\digamma_N^{N} & \boxed{O} \end{vmatrix}. \tag{2.23}$$

The N-fold solutions (2.22) and (2.23) reduce to those of the continuous system by taking the continuum limit as $a,\ b \to 0$ presented in [15].

3 Standard binary Darboux transformation

Now consider hat space $\hat{\Upsilon}$ for BDT so the corresponding solutions $\hat{\Psi}_{n,m} \in \hat{\Upsilon}$. The linear system for hat space is given by

$$E\widehat{\Psi}_{n,\,m} \equiv \widehat{\mathcal{A}}_{n,\,m}\widehat{\Psi}_{n,\,m} = \left(I + \eta^{-1}a\left(\widehat{Q}_{n+1,\,m} - \widehat{Q}_{n,\,m}\right)\right)\widehat{\Psi}_{n,\,m}, \tag{3.1}$$

$$F\widehat{\Psi}_{n,\,m+} \equiv \widehat{\mathcal{B}}_{n,\,m}\widehat{\Psi}_{n,\,m} = \left(I + b\left(\widehat{Q}_{n,\,m+1}\widehat{G} - \widehat{G}\widehat{Q}_{n,\,m} + \eta\widehat{G}\right)\right)\widehat{\Psi}_{n,\,m}. \tag{3.2}$$

The linear system (3.1), (3.2) fulfill the compatibility condition $\widehat{\mathcal{A}}_{n,\,m}\widehat{\mathcal{B}}_{n+1,\,m} = \widehat{\mathcal{A}}_{n,\,m+1}\widehat{\mathcal{B}}_{n,\,m}$ which leads the equation of motion for hat space

$$\begin{aligned} &a\left(\widehat{Q}_{n+1,\,m+1} - \widehat{Q}_{n+1,\,m} - \widehat{Q}_{n,\,m+1} + \widehat{Q}_{n,\,m}\right) + ab\left(\widehat{Q}_{n+1,\,m+1} - \widehat{Q}_{n,\,m+1}\right)\left(\widehat{Q}_{n,\,m+1}\widehat{G} - \widehat{G}\widehat{Q}_{n,\,m}\right) \\ =&ab\left(\widehat{Q}_{n+1,\,m+1}\widehat{G} - \widehat{G}\widehat{Q}_{n+1,\,m}\right)\left(\widehat{Q}_{n+1,\,m} - \widehat{Q}_{n,\,m}\right). \end{aligned} \tag{3.3}$$

The particular matrix solutions for direct as well as adjoint spaces are $K_{n,m}$, $H_{n,m}$ respectively. So the corresponding solutions for $\widehat{\Upsilon}$ are $\hat{K}_{n,m} \in \hat{\Upsilon}$ and $\hat{\Phi}_{n,m} \in \hat{\Upsilon}^{\dagger}$. Assuming $i(\hat{K}_{n,m}) \in \widetilde{\Upsilon}^{\dagger}$, one can write as

$$D^{(-1)\dagger}_{n,m}(\eta) : \Upsilon^{\dagger} \to \widetilde{\Upsilon}^{\dagger}. \tag{3.4}$$

Since $\Phi_{n,m} \in \Upsilon^{\dagger}$, we have

$$i(\widehat{K}_{n,m}) = D_{n,m}^{(-1)\dagger}(\eta)\Phi_{n,m}. \tag{3.5}$$

Also from $D_{n,m}^{\dagger}(\eta)(i(K_{n,m})) = 0$, we get $i(K_{n,m}) = K_{n,m}^{(-1)\dagger}$, similarly $i(\widehat{K}_{n,m}) = \widehat{K}_{n,m}^{(-1)\dagger}$. So, we can write

$$\widehat{K}_{n,m}^{(-1)\dagger} = D_{n,m}^{(-1)\dagger}(\eta)\Phi_{n,m},$$

and

$$\widehat{K}_{n,m} = (D_{n,m}^{(-1)\dagger}(\eta)\Phi_{n,m})^{(-1)\dagger}, \tag{3.6}$$

where $D_n(\equiv) = \eta I - K_{n,m}\Xi K_{n,m}^{-1}$ and using in (3.6), we obtain

$$\begin{aligned}\widehat{K}_{n,m} &= ((\eta I - K_{n,m}\Xi K_{n,m}^{-1})^{(-1)\dagger}\Phi_{n,m})^{(-1)\dagger},\\ &= (\eta I - K_{n,m}\Xi K_{n,m}^{-1})\Phi_{n,m}^{(-1)\dagger},\\ &= K_{n,m}(\eta I - \Xi)K_{n,m}^{-1}\Phi_{n}^{(-1)\dagger},\\ &= K_{n,m}(\eta I - \Xi)(\Phi_{n,m}^{\dagger}K_{n,m})^{-1},\\ &= K_{n,m}\Delta_{n,m}(K_{n,m}, \Phi_{n,m})^{-1},\end{aligned} \tag{3.7}$$

where the eigen potential $\Delta_{n,m}$ is given by

$$\Delta_{n,m}(K_{n,m}, \Phi_{n,m}) = (\Phi_{n,m}^{\dagger}K_{n,m})(\eta I - \Xi)^{-1}. \tag{3.8}$$

Similarly for adjoint space

$$\hat{H}_{n,m} = H_{n,m}\Delta_{n,m}(\Psi_{n,m}, H_{n,m})^{(-1)\dagger}, \tag{3.9}$$

where

$$\Delta_n(\Psi_{n,m}, H_{n,m}) = -(\eta I - F^{\dagger})^{-1}(H_{n,m}^{\dagger}\Psi_{n,m}). \tag{3.10}$$

By expressing equations (3.8), (3.10) in the form of matrix, the condition on $\Delta_{n,m}$ given by

$$F^{\dagger}\Delta_{n,m}(K_{n,m}, H_{n,m}) - \Delta_{n,m}(K_{n,m}, H_{n,m})\Xi = H_{n,m}^{\dagger}K_{n,m}, \tag{3.11}$$

where $\Delta_{n,m}$ is eigen potential matrix given by

$$\Delta_{n,m}(K_{n,m}, H_{n,m})_{ij} = \frac{\langle H_{n,m}|K_{n,m}\rangle_{(i\ j)}}{F^{\dagger} - \Xi}. \tag{3.12}$$

Now, we are able to define Darboux matrix in hat space

$$\widehat{D}_{n,m}(\eta) \equiv (\eta I - \widehat{\Pi}_{n,m}) = (\eta I - \widehat{K}_{n,m}F^{\dagger}\widehat{K}_{n,m}^{-1}), \tag{3.13}$$

where

$$\widehat{D}_{n,m}(\eta)\widehat{\Psi}_{n,m} = \widetilde{\Psi}_{n,m}. \tag{3.14}$$

We may write the following Darboux map as

$$\begin{aligned}D_{n,m}(\eta) &: \Upsilon \to \widetilde{\Upsilon},\\ \widehat{D}_{n,m}(\eta) &: \hat{\Upsilon} \to \widetilde{\Upsilon},\\ D_{n,m}(\zeta) &: \Upsilon^{\dagger} \to \widetilde{\Upsilon}^{\dagger}.\end{aligned} \tag{3.15}$$

Now, we can write the definition of standard BDT as

$$\widehat{D}_{n,m}(\eta)\widehat{\Psi}_{n,m} = D_{n,m}(\eta)\Psi_{n,m}, \tag{3.16}$$

which implies

$$\widehat{\Psi}_{n,m} = \widehat{D}_{n,m}^{-1}(\eta)D_{n,m}(\eta)\Psi_{n,m}. \tag{3.17}$$

Substituting the values of $\widehat{D}_{n,m}^{-1}(\eta)$ and $D_{n,m}(\eta)$, we obtain

$$\begin{aligned}\widehat{\Psi}_{n,m} &= (\eta I - \widehat{K}_{n,m}F^{\dagger}\widehat{K}_{n,m}^{-1})^{-1}(\eta I - K_{n,m}\Xi K_{n,m}^{-1})\Psi_{n,m},\\ &= \widehat{K}_{n,m}(\eta I - F^{\dagger})^{-1}\widehat{K}_{n,m}^{-1}K_{n,m}(\eta I - \Xi)K_{n,m}^{-1}\Psi_{n,m}.\end{aligned}$$

Using (3.7), we get

$$\begin{aligned}\widehat{\Psi}_{n,m} =&K_{n,m}^{-1}\Delta_{n,m}(K_{n,m},H_{n,m})(\eta I-F^{\dagger})^{-1}\\ &\times \Delta_{n,m}(K_{n,m},H_{n,m})K_{n,m}^{-1}K_{n,m}(\eta I-\Xi)K_{n,m}^{-1}\Psi_{n,m},\\ =&K_{n,m}^{-1}\Delta_{n,m}(K_{n,m},H_{n,m})(\eta I-F^{\dagger})^{-1}\\ &\times (\eta\Delta_{n,m}(K_{n,m},H_{n,m})-\Delta_{n,m}(K_{n,m},H_{n,m})\Xi)K_{n,m}^{-1}\Psi_{n,m}.\end{aligned} \tag{3.18}$$

Substituting (3.11) in (3.18), we get

$$\begin{aligned}\widehat{\Psi}_{n,m} =&K_{n,m}^{-1}\Delta_{n,m}(K_{n,m},\ H_{n,m})(\eta I-F^{\dagger})^{-1}\\ &\times (\eta\Delta_{n,m}(K_{n,m},\ H_{n,m})-F^{\dagger}\Delta_{n,m}(K_{n,m},\ H_{n,m})+H_{n,m}^{\dagger}K_{n,m})K_{n,m}^{-1}\Psi_{n,m},\\ =&K_{n,m}^{-1}\Delta_{n,m}(K_{n,m},\ H_{n,m})(\eta I-F^{\dagger})^{-1}\\ &\times (\eta\Delta_{n,m}(K_{n,m},\ H_{n,m})K_{n,m}^{-1}-F^{\dagger}\Delta_{n,m}(K_{n,m},\ H_{n,m})K_{n,m}^{-1}+H_{n,m}^{\dagger})\Psi_{n,m},\\ =&(\eta I-F^{\dagger})^{-1}(\eta I-F^{\dagger})\left(I+\frac{K_{n,m}^{-1}\Delta_{n,m}(K_{n,m},\ H_{n,m})H_{n,m}^{\dagger}}{\eta I-F^{\dagger}}\right)\Psi_{n,m},\\ =&\Psi_{n,m}+K_{n,m}^{-1}\Delta_{n,m}(K_{n,m},\ H_{n,m})(\eta I-F^{\dagger})^{-1}H_{n,m}^{\dagger}\Psi_{n,m}.\end{aligned}$$

By using (3.10), above expression will become

$$\widehat{\Psi}_{n,m} = \Psi_{n,m}-K_{n,m}\Delta_{n,m}(K_{n,m},H_{n,m})^{-1}\Delta_{n,m}(\Psi_{n,m},H_{n,m}). \tag{3.19}$$

In terms of quasideterminants

$$\widehat{\Psi}_{n,m}=\begin{vmatrix}\Delta_{n,m}(K_{n,m},H_{n,m}) & \Delta_{n,m}(\Psi_{n,m},H_{n,m})\\ K_{n,m} & \boxed{\Psi_{n,m}}\end{vmatrix}. \tag{3.20}$$

The expression (3.20) called quasi-Grammian solution of dGCD system. Same as, for adjoint space

$$\begin{aligned}\widehat{\Phi}_{n,m} =&\Phi_{n,m}-H_{n,m}\Delta_{n,m}(K_{n,m},H_{n,m})^{(-1)\dagger}\Delta_{n,m}(K_{n,m},\Phi_{n,m})^{\dagger}\\ =&\begin{vmatrix}\Delta_{n,m}(K_{n,m},H_{n,m})^{\dagger} & \Delta_{n,m}(K_{n,m},\Phi_{n,m})^{\dagger}\\ H_{n,m} & \boxed{\Phi_{n,m}}\end{vmatrix}.\end{aligned} \tag{3.21}$$

Now using the definition of BDT on matrix solution $Q_{n,m}$

$$\begin{aligned}\widehat{Q}_{n,m}-\widehat{\Pi}_{n,m} =&Q_{n,m}-\Pi_{n,m},\\ \widehat{Q}_{n,m} =&Q_{n,m}+\widehat{K}_{n,m}F^{\dagger}\widehat{K}_{n,m}^{-1}-K_{n,m}\Xi K_{n,m}^{-1},\\ =&Q_{n,m}+K_{n,m}\Delta_{n,m}(K_{n,m},\ H_{n,m})^{-1}\big(F^{\dagger}\Delta_{n,m}(K_{n,m},\ H_{n,m})\big)K_{n,m}^{-1}-K_{n,m}\Xi K_{n,m}^{-1},\end{aligned}$$

using the relation (3.11), we get

$$\begin{aligned}\widehat{Q}_{n,m} =&Q_{n,m}+K_{n,m}\Delta_{n,m}(K_{n,m},H_{n,m})^{-1}H_{n,m}^{\dagger},\\ =&Q_{n,m}-\begin{vmatrix}\Delta_{n,m}(K_{n,m},H_{n,m}) & H_{n,m}^{\dagger}\\ K_{n,m} & \boxed{O}\end{vmatrix}.\end{aligned} \tag{3.22}$$

Now, for $b\to 0$, we obtain BDT for t continuous semi-discrete GCD system as

$$\begin{aligned}\widehat{Q}_{n} =&Q_{n}+K_{n}\Delta_{n}(K_{n},\ H_{n})^{-1}H_{n}^{\dagger},\\ =&Q_{n}-\begin{vmatrix}\Delta_{n}(K_{n},\ H_{n}) & H_{n}^{\dagger}\\ K_{n} & \boxed{O}\end{vmatrix}.\end{aligned}$$

Also, for $a\to 0$, BDT for x continuous semi-discrete GCD system as

$$\begin{aligned}\widehat{Q}_{m} =&Q_{m}+K_{m}\Delta_{m}(K_{m},\ H_{m})^{-1}H_{m}^{\dagger},\\ =&Q_{m}-\begin{vmatrix}\Delta_{m}(K_{m},\ H_{m}) & H_{m}^{\dagger}\\ K_{m} & \boxed{O}\end{vmatrix}.\end{aligned}$$

The continuous GCD system can be obtained by $a,\ b\to 0$, as

$$\widehat{Q} = Q - \begin{vmatrix} \Delta(K,\ H) & H^{\dagger} \\ K & \boxed{O} \end{vmatrix},$$

which is presented in [15]. Through iteration of BDT, one can write N-th iteration of $Q_{n,m}[N+1]$ as

$$\widehat{Q}_{n,m}[N+1] = Q_{n,m} - \begin{vmatrix} \Delta_{n,m}(K_{n,m,1}, H_{n,m,1}) & \cdots & \Delta_{n,m}(K_{n,m,N}, H_{n,m,1}) & H^{\dagger}_{n,m,1} \\ \vdots & \cdots & \vdots & \vdots \\ \Delta_{n,m}(K_{n,m,1}, H_{n,m,N}) & \cdots & \Delta_{n,m}(K_{n,m,N}, H_{n,m,N}) & H^{\dagger}_{n,m,N} \\ K_{n,m,1} & \cdots & K_{n,m,N} & \boxed{O} \end{vmatrix}. \quad (3.23)$$

These are the quasi-Grammian solutions of dGCD system. The key feature of this method is that it could significantly enhance our approach toward nonlinear waves because it gives the different solutions from the DT. By using BDT, one can obtain Grammian, breather and also the soliton solutions.

4 Explicit soliton solutions

In this section, we consider dGCD system based upon Lie group $SU(2)$ and calculate the explicit solutions by using BDT. For the Lie group $SU(2)$, the matrix fields $Q_{n,m}$ and $G_{n,m}$ belong to Lie algebra $su(2)$ so, we have

$$Q^{\dagger}_{n,m} = -\,Q_{n,m},\quad G^{\dagger}_{n,m} = -G_{n,m},$$
$$\mathrm{Tr}Q_{n,m} = 0,\ \ \mathrm{Tr}G_{n,m} = 0.$$

The 2×2 matrix $Q_{n,m}$ is given by

$$Q_{n,m} = \begin{pmatrix} \alpha_{n,m} & \beta_{n,m} \\ \beta_{n,m} & -\alpha_{n,m} \end{pmatrix},$$

where $\alpha_{n,m}$, $\beta_{n,m}$ are the scalar functions. Taking seed solution, i.e., $\alpha_{n+1,\,m} - \alpha_{n,\,m} = q \neq 0$, $\alpha_{n,\,m+1} - \alpha_{n,\,m} = 0$, $\beta_{n,\,m} = 0$, where q is constant. The explicit solution of linear system (1.1), (1.2) can be written as

$$\Psi_{n,m} = \begin{pmatrix} X_{n,m} & 0 \\ 0 & Y_{n,m} \end{pmatrix}, \quad (4.1)$$

where the solutions $X_{n,\,m}$, $Y_{n,\,m}$ are computed as

$$X_{n,\,m}(\eta) = \left(1 + i\eta^{-1}q\right)^n \left(1 - b\frac{i\eta}{2}\right)^m + i\left(1 - i\eta^{-1}q\right)^n \left(1 + b\frac{i\eta}{2}\right)^m, \quad (4.2)$$

$$Y_{n,\,m}(\eta) = \left(1 + i\eta^{-1}q\right)^n \left(1 - b\frac{i\eta}{2}\right)^m - i\left(1 - i\eta^{-1}q\right)^n \left(1 + b\frac{i\eta}{2}\right)^m. \quad (4.3)$$

The distinct matrix solution $K_{n,m}$ can be expressed as

$$K_{n,m} = (\Psi_{n,m}(\eta)|1\rangle, \Psi_{n,m}(\bar{\eta})|2\rangle) = \begin{pmatrix} X_{n,m}(\eta) & X_{n,m}(\bar{\eta}) \\ Y_{n,m}(\eta) & -Y_{n,m}(\bar{\eta}) \end{pmatrix}. \quad (4.4)$$

Also, for adjoint space $H_{n,m}$ can be written as

$$H_{n,m} = (\Phi_{n,m}(\zeta)|1\rangle, \Phi_{n,m}(\bar{\zeta})|2\rangle) = \begin{pmatrix} E_{n,m}(\zeta) & E_{n,m}(\bar{\zeta}) \\ F_{n,m}(\zeta) & -F_{n,m}(\bar{\zeta}) \end{pmatrix}, \quad (4.5)$$

where

$$E_{n,\,m}(\zeta) = \left(1 + i\zeta^{-1}q\right)^n \left(1 - b\frac{i\zeta}{2}\right)^m + i\left(1 - i\zeta^{-1}q\right)^n \left(1 + b\frac{i\zeta}{2}\right)^m, \quad (4.6)$$

$$F_{n,\,m}(\zeta) = \left(1 + i\zeta^{-1}q\right)^n \left(1 - b\frac{i\zeta}{2}\right)^m - i\left(1 - i\zeta^{-1}q\right)^n \left(1 + b\frac{i\zeta}{2}\right)^m. \quad (4.7)$$

Now using (4.4), (4.5) in the definition of $\Delta_{n,m}(K_{n,m},\ H_{n,m})$, we may write

$$\Delta_{n,m}(K_{n,m}, H_{n,m}) = \begin{pmatrix} \frac{2(A_{n,m}+\bar{A}_{n,m})}{\zeta_1-\eta_1} & \frac{2i(\bar{B}_{n,m}-B_{n,m})}{\zeta_1+\bar{\eta}_1} \\ -\frac{2i(\bar{B}_{n,m}-B_{n,m})}{\zeta_1+\eta_1} & -\frac{2(A_{n,m}+\bar{A}_{n,m})}{\zeta_1-\eta_1} \end{pmatrix}, \quad (4.8)$$

where

$$A_{n,m} =(1+i\zeta^{-1}q)^n(1-i\eta^{-1}q)^n\left(1-\frac{ib\zeta}{2}\right)^m\left(1+\frac{ib\eta}{2}\right)^m, \tag{4.9}$$

$$B_{n,m} =(1+i\zeta^{-1}q)^n(1+i\eta^{-1}q)^n\left(1-\frac{ib\zeta}{2}\right)^m\left(1-\frac{ib\eta}{2}\right)^m. \tag{4.10}$$

Now, we take

$$\begin{aligned}\widehat{\Pi}_{n,m} &=K_{n,m}\Delta_{n,m}(K_{n,m},H_{n,m})^{-1}H^{\dagger}_{n,m} = \begin{pmatrix}\widehat{\Pi}_{n,m,11} & \widehat{\Pi}_{n,m,12}\\ \widehat{\Pi}_{n,m,21} & \widehat{\Pi}_{n,m,22}\end{pmatrix},\\ &=\frac{1}{L_{n,m}}\begin{pmatrix} \frac{i(A+\bar{A})}{\zeta-\eta}X(\eta)E(\zeta)-\frac{(\bar{B}-B)}{\zeta+\eta}X(\eta)E(\zeta) & -\frac{i(A+\bar{A})}{\zeta-\eta}X(\eta)F(\zeta)-\frac{(\bar{B}-B)}{\zeta+\eta}X(\eta)F(\zeta)\\ +\frac{(\bar{B}-B)}{\zeta+\eta}X(\eta)E(\zeta)-\frac{i(A+\bar{A})}{\zeta-\eta}X(\eta)E(\zeta) & -\frac{(\bar{B}-B)}{\zeta+\eta}X(\eta)F(\zeta)-\frac{i(A+\bar{A})}{\zeta-\eta}X(\eta)F(\zeta)\\ \frac{i(A+\bar{A})}{\zeta-\eta}Y(\eta)E(\zeta)-\frac{(\bar{B}-B)}{\zeta+\eta}Y(\eta)E(\zeta) & -\frac{i(A+\bar{A})}{\zeta-\eta}Y(\eta)F(\zeta)-\frac{(\bar{B}-B)}{\zeta+\eta}Y(\eta)F(\zeta)\\ -\frac{(\bar{B}-B)}{\zeta+\eta}F(\eta)X(\zeta)+\frac{i(A+\bar{A})}{\zeta-\eta}F(\eta)X(\zeta) & +\frac{(\bar{B}-B)}{\zeta+\eta}Y(\eta)F(\zeta)+\frac{i(A+\bar{A})}{\zeta-\eta}Y(\eta)F(\zeta)\end{pmatrix},\end{aligned} \tag{4.11}$$

where

$$L_{n,m} = \frac{-4(A_{n,m}+\bar{A}_{n,m})^2}{(\zeta-\eta)^2}-\frac{4(\bar{B}_{n,m}-B_{n,m})^2}{(\zeta+\eta)^2}. \tag{4.12}$$

By using (3.22), we can write the solutions as

$$\widehat{\alpha}_{n,m} =\alpha_{n,m}+\widehat{\Pi}_{n,m,11}, \tag{4.13}$$

$$\widehat{\beta}_n =\beta_{n,m}+\widehat{\Pi}_{n,m,12}. \tag{4.14}$$

Figures 1 and 2 show the curved type motion of discrete soliton solutions known as Grammian solutions whose wave fronts are curved and amplitude is varying along the wave front which is clearly shown by surface and contour plot. In solution (4.11), q, b are the lattice parameters which defines the amplitude of the wave. Figure 1 shows the dynamics of Grammian toward the negative m whereas Fig. 2 presents the motion of Grammian toward negative m direction.

4.1 Reduction I

Now, we transform the BDT solutions to classical DT solutions by substituting $\zeta = -\eta$, we get $A_{n,m}+\bar{A}_{n,m} = X(\eta)Y(\eta)$, and $B_{n,m}-\bar{B}_{n,m} = 0$, also $L_{n,m} = -\frac{X^2(\eta)Y^2(\eta)}{\eta^2}$; therefore, we may write the above expressions (4.11) as

$$\alpha_{n,m}[1] = \eta\frac{(1+i\eta^{-1}q)^n\left(1-\frac{ib\eta}{2}\right)^m+i(1-i\eta^{-1}q)^n\left(1+\frac{ib\eta}{2}\right)^m)}{(1+i\eta^{-1}q)^n\left(1-\frac{ib\eta}{2}\right)^m-i(1-i\eta^{-1}q)^n\left(1+\frac{ib\eta}{2}\right)^m}, \tag{4.15}$$

$$\beta_{n,m}[1] = \eta\frac{(1+i\eta^{-1}q)^n\left(1-\frac{ib\eta}{2}\right)^m-i(1-i\eta^{-1}q)^n\left(1+\frac{ib\eta}{2}\right)^m}{(1+i\eta^{-1}q)^n\left(1-\frac{ib\eta}{2}\right)^m+i(1-i\eta^{-1}q)^n\left(1+\frac{ib\eta}{2}\right)^m}. \tag{4.16}$$

Expressions (4.15) and (4.16) display the bright and dark solitons depicted in Fig. 3. Bright solitons are positive-amplitude and localized solutions of nonlinear wave equations that retain their shape throughout the propagation. They are often associated with focusing nonlinearity. Because of localized optical pulse that does not lose energy while traveling over a far off distances, bright solitons are used for high-speed communication systems, whereas dark solitons have negative-amplitude and associated with defocusing nonlinearity. Due to defocusing nonlinearity, darks solitons are studied in optical fibers as pulse propagation. For controlling light pulses, dark solitons can be helpful as optical switching and soliton-based communication systems.

4.2 Reduction II

For another reduction put $\zeta = \eta$, Eq. (4.11) gives

$$\alpha_{n,m} = \beta_{n,m} = \eta\frac{\begin{aligned}&\left\{(1+i\eta^{-1}q)^n\left(1-\frac{ib\eta}{2}\right)^m+i(1-i\eta^{-1}q)^n\left(1+\frac{ib\eta}{2}\right)^m\right\}\times\\ &\left\{(1+i\eta^{-1}q)^n\left(1-\frac{ib\eta}{2}\right)^m-i(1-i\eta^{-1}q)^n\left(1+\frac{ib\eta}{2}\right)^m\right\}\end{aligned}}{(1-i\eta^{-1}q)^{2n}\left(1+\frac{ib\eta}{2}\right)^m-(1+i\eta^{-1}q)^{2n}\left(1-\frac{ib\eta}{2}\right)^{2m}}. \tag{4.17}$$

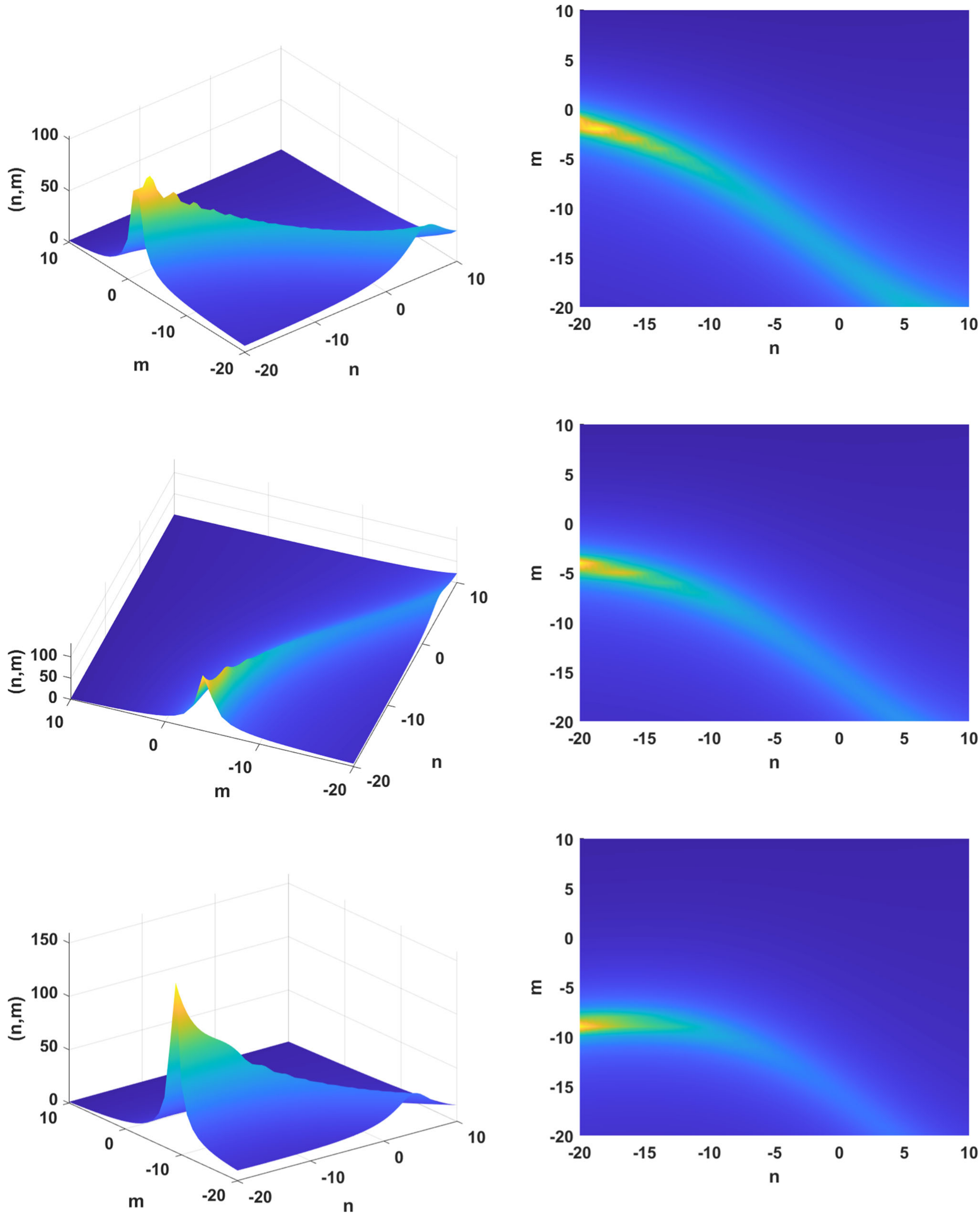

Fig. 1 Grammian $\widehat{\alpha}_{n,m}$: for numerical values $\eta = 0.8 - 0.54\mathrm{i}$, $\zeta = 4.5 + 0.25\mathrm{i}$, $q = 3$, $b = 2$

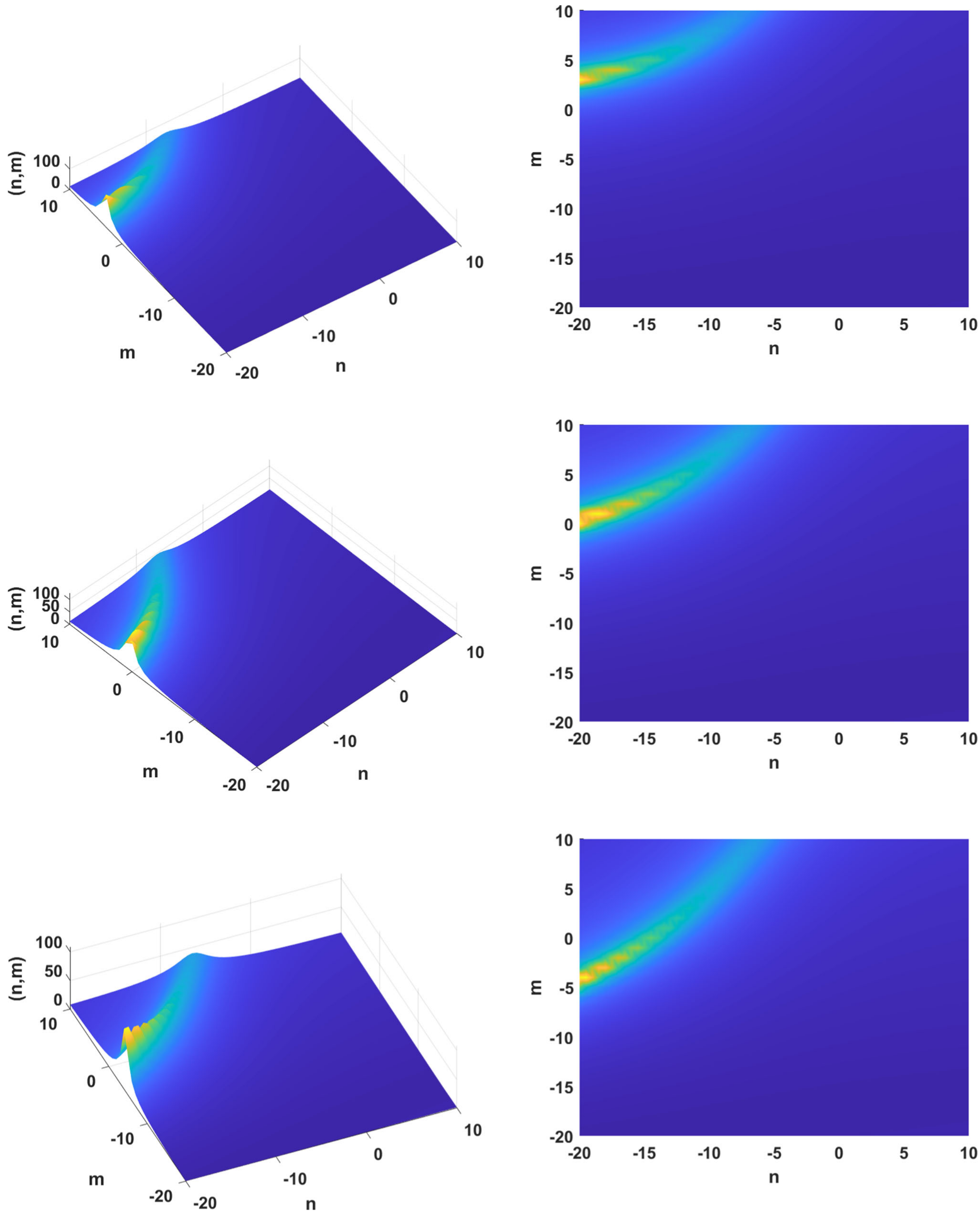

Fig. 2 Grammian $\widehat{\beta}_{n,m}$: for numerical values $\eta = 0.8 + 0.54\text{i}$, $\zeta = 4.1 - 0.45\text{i}$, $q = 3$, $b = 2$

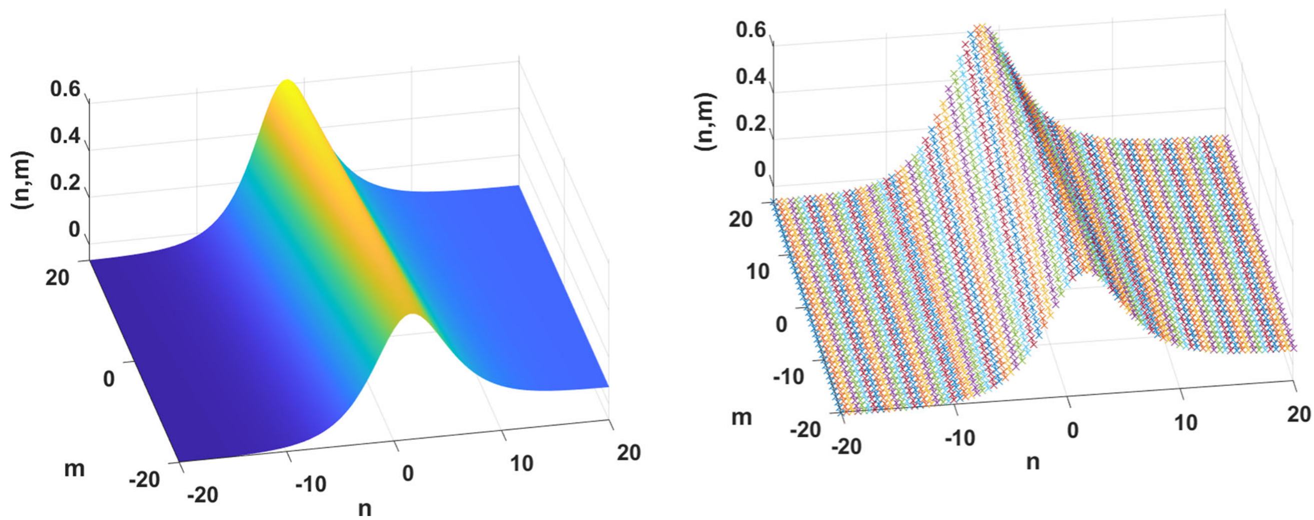

Fig. 3 Bright soliton: for numerical values $\eta = 0.7 + 0.54\text{i}$, $q = 3$, $b = 2$

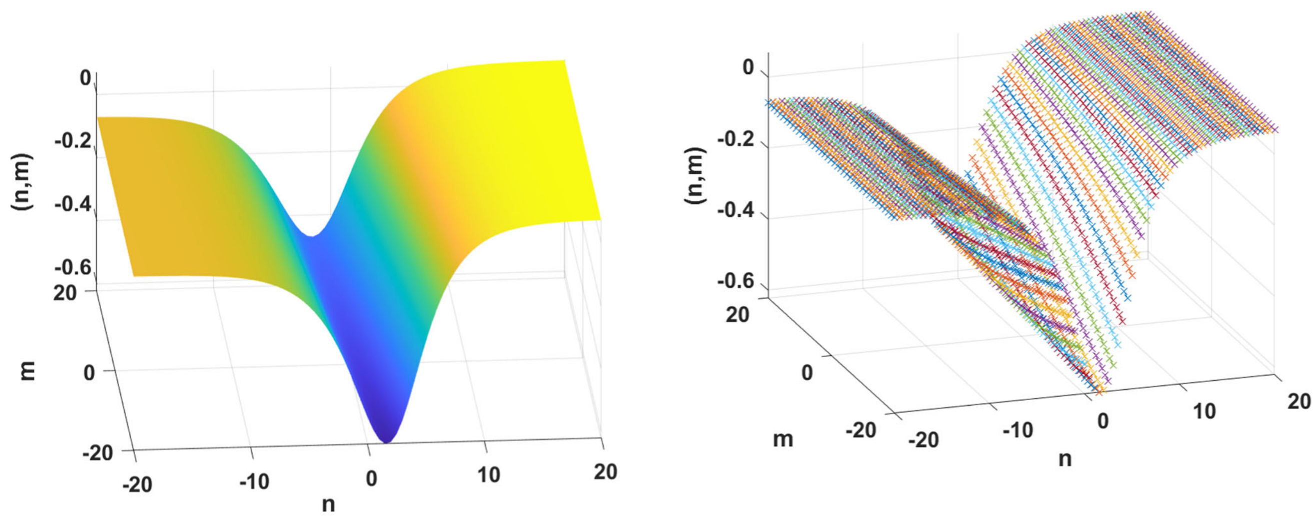

Fig. 4 Dark soliton: for numerical values $\eta = 0.8 + 0.53\text{i}$, $q = 3$, $b = 2$

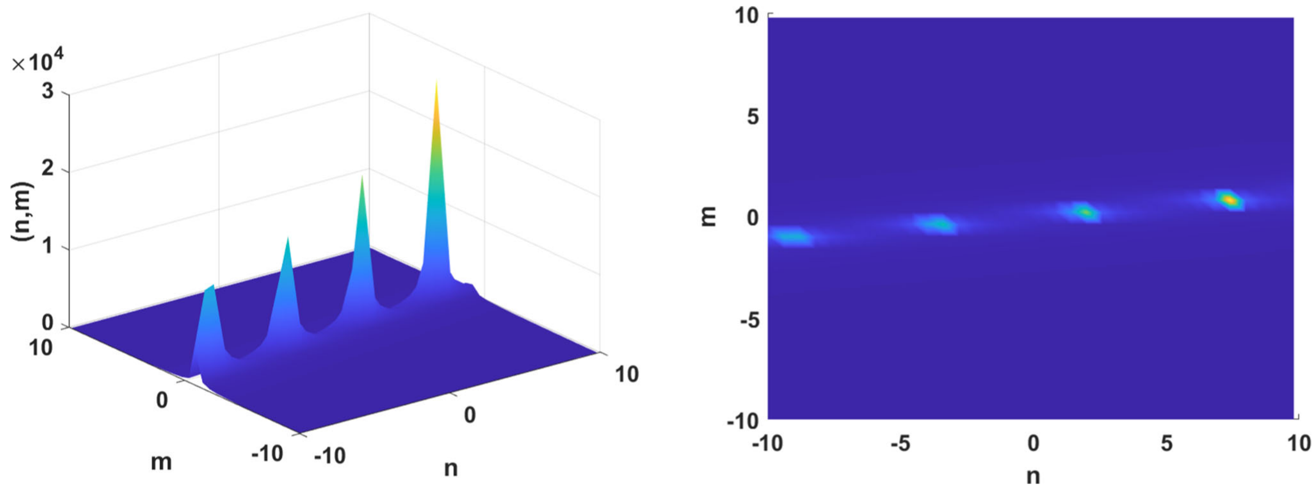

Fig. 5 Discrete breather: for numerical values $\eta = 5.9 + 0.54\mathrm{i}$, $q = 0.9$, $b = 0.6$

The expression (4.17) represents the breather solution shown in Fig. 5. In optical fiber, breather solutions are particularly important to get understanding about the controlling light and pulse dynamics where the temporal evolution of light pulses may be controlled to attain desirable signal characteristics.

5 Conclusion

This study presents a comprehensive analysis of the dGCD system. We present the Lax pair of dGCD system and also elaborate how this model reduces to the semi-discrete versions as well as continuous system under the continuum limits. The DT was developed and applied to obtain the discrete soliton solutions for both direct and adjoint spectral problems. Further, through the application of BDT, we have calculated the explicit expressions of quasi-Grammian solutions for the system. Furthermore, we present explicit expressions of the quasi-Grammians. Finally, we reduced the BDT solutions to elementary DT solutions. The dynamics of Grammian, dark and bright soliton, also breather solutions of the dGCD system are shown through surface and contours plots.

In the study of propagation of ultra-short pulses for an optical fiber, breather solutions play an important role. Also, one important characteristic that both dark and bright solitons have in common is their robustness, which is crucial for guaranteeing their usefulness in optical communications. These solutions can also continue to move at the same speed and form extended distances.

It would be interesting to investigate the multi-component of the dGCD system by applying the DT and BDT to derive more general classes of solutions. In future work, we would like to investigate BDT for $\mathcal{PT}$-symmetric integrable systems to calculate symmetry breaking and symmetry preserving soliton and breather solutions. Furthermore, employing physics-informed neural networks can provide improved insights into the systems behavior and support the analysis of more complex scenarios. We shall address this problem in a separate paper.

Acknowledgements Z. Amjad would like to thanks Department of Mathematics, Zhejiang Normal University, Jinhua, China, for providing the research facilities. The authors would like to thank the reviewers for their deep analysis and kind suggestions. The work was supported in part by NSFC under the grants 12271488, 11975145 and 11972291 and the Ministry of Science and Technology of China under the grants G2021016032L and G2023016011L.

Data Availability Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. R. Carroll, Y. Kodama, Solution of the dispersionless Hirota equations. J. Phys. A **28**, 6373 (1995)
2. M. Dunajski, Interpolating Dispersionless Integrable System. J. Phys. A **41**, 315202 (2008)
3. E.V. Ferapontov, B. Kruglikov, Dispersionless integrable systems in 3D and Einstein-Weyl geometry. J. Diff. Geom. **97**, 215–254 (2014)
4. B. Kruglikov, O. Morozov, Integrable dispersionless PDEs in 4D, their symmetry pseudogroups and deformations. Lett. Math. Phys. **105**, 1703–1723 (2015)
5. K. Takasaki, T. Takebe, Quasi-classical limit of Toda hierarchy and Winfinity symmetries. Lett. Math. Phys. **28**, 165–176 (1993)
6. K. Takasaki, T. Takebe, Integrable hierarchies and dispersionless limit. Rev. Math. Phys. **7**, 743–808 (1995)

7. K. Takasaki, Dispersionless Toda hierarchy and two-dimensional string theory. Commun. Math. Phys. **170**, 101–116 (1995)
8. H.W.A. Riaz, M. Hassan, A discrete generalized coupled dispersionless integrable system and its multisoliton solutions. J. Math. Anal. Appl. **458**, 1639–1652 (2018)
9. Z. Amjad, B. Haider, Binary Darboux transformations of the supersymmetric Heisenberg magnet model. Theoret. Math. Phys. **199**, 784–797 (2019)
10. W.X. Ma, Binary Darboux transformation for general matrix mKdV equations and reduced counterparts. Choas Solitons Fractals **146**, 110824 (2021)
11. W.X. Ma, Binary Darboux transformation of vector nonlocal reverse-time integrable NLS equations. Chaos, Solitons Fractals **180**, 114539 (2024)
12. Z. Amjad, B. Haider, Binary Darboux transformation of time-discrete generalized lattice Heisenberg magnet model. Chaos Solitons Fractals **130**, 109404 (2020)
13. Z. Amjad, W.X. Ma, R.M. Zulqarnain, Double breather and peak solitons of a semi discrete short pulse equation. Reviews Math. Phys. **37**, 2550002 (2025)
14. Z. Amjad, Breather and soliton solutions of semi-discrete negative order AKNS equation. Eur. Phys. J. Plus **137**, 1036 (2022)
15. B. Haider, M. Hassan, Quasi-Grammian solutions of the generalized coupled dispersionless integrable system. SIGMA **8**, 084 (2012)
16. V.B. Matveev, M.A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991)
17. J. Cieslinski, An algebraic method to construct the Darboux matrix. J. Math. Phys. **36**, 5670 (1995)
18. J.J.C. Nimmo, Darboux transformations for discrete systems. Chaos, Solitons Fractals **11**, 115–120 (2000)
19. Z. Amjad, B. Haider, W.X. Ma, Integrable Discretization and Multi-soliton Solutions of Negative Order AKNS Equation. Qual. Theory Dyn. Syst **23**, 280 (2024)
20. J.J.C. Nimmo, C.R. Gilson, Y. Ohta, Applications of Darboux transformations to the self-dual Yang-Mills equations. Theoret. and Math. Phys. **122**, 239–246 (2000)

21. C. Rogers, W. K. Schief, *Bäcklund and Darboux transformations: geometry and modern applications in soliton theory*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, (2002)
22. C. Gu, H. Hu, Z. Zhou, *Darboux Transformations in Integrable Systems, Theory and their Applications to Geometry* (Springer, Berlin, 2005)