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Integrable Discretization and Multi-soliton Solutions of Negative Order AKNS Equation

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Abstract

In this paper, we propose the discrete and two semi-discrete versions of negative order AKNS equation through the discretization of associated Lax pair of continuous negative order AKNS equation. Discrete and semi-discrete multi-soliton solutions are computed by using Darboux transformation and are presented in the form of quasideterminants. The dynamics of one soliton and interaction of two soliton solutions for negative order AKNS equation are presented in the end.

Keywords Discrete integrable systems · Darboux transformation · Negative order AKNS · Soliton solution

1 Introduction

Many integrable systems, such as Korteweg-de Vries (KdV), modified KdV (mKdV), and so on admit their own hierarchy

$$u_t = T^m k_0(u) = k_m, \quad m = \dots, \ -1, \ 0, \ 1, \dots \tag{1.1}$$

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where T is referred as a recursion operator, whereas m with positive values correspond to positive flow and m with negative values referred to as negative flow of hierarchy. For example, the sine-Gordon equation is the well-studied equation (which has potential applications in Josephson transmission line [1, 2], ultrashort pulse propagation in a resonant medium [3]) that can be derived as negative order of the mKdV hierarchy [4]. The coupled integrable dispersionless equations are derived from the first negative flow of the nonlinear Schrödinger (NLS) hierarchy [5]. Numerous integrable systems such as Degasperis-Procesi equation [6], Camassa-Holm equation [7–9] and the short pulse equation [10, 11] are referred to as the negative order equations by reciprocal transformations. The importance to study the negative order equations is that one can construct infinitely many symmetries for nonisospectral Ablowitz-Ladik hierarchy [12]. Another fact is that it results in determining the time dependence in the phases of solitons. Therefore, the negative order flow equations give many interesting new results not only from the integrability point of view but also provide interesting information dynamically (both physically and mathematically).

The study of discrete integrable systems are of particular importance in various fields not only as a physical model but also for numerical analysis. Discrete integrable systems are the systems which have their own independent variables defined over lattice point rather than the systems with their independent variables having continuous values have received much attention by the researchers working in the field of mathematical and applied sciences.

Many techniques such as B$\ddot{a}$cklund transformation, Inverse scattering transform, Hirota method, Darboux transformation, binary Darboux transformation etc., have been employed to obtain exact solutions of many nonlinear partial differential equations [13–23]. Among all of the above techniques, Darboux transformation is a very effective tool for soliton generation of nonlinear evolution equations. Also, the soliton solutions can be expressed in terms of quasi-Grammian, quasi-Wronskian, quasi-determinants and Wronskian [21–27]. Integrable discrete analogues of negative order AKNS were considered in [30].

The present work is devoted for the study of Darboux transformation (DT) of the discrete negative order AKNS (nAKNS) equation. Like the continuous nAKNS equation which exhibits numerous aspects of integrability, discrete nAKNS equation also preserves the integrability. The DT of continuous nAKNS equation is presented in [28]. The breathers of nAKNS equation and its multi-component generalizations have been investigated and soliton solutions have been obtained [29–32].

In this paper, we discuss discrete (difference-difference) nAKNS equation. We present the Lax pair of the discrete nAKNS (dnAKNS) equation, whereas both the space and time variables are defined by the discrete steps on the lattice. The DT is applied on discrete Lax pair, the solutions of the matrix field equations are calculated in terms of quasi-determinants. By iterating the DT, we get the K-soliton solutions. Further, the explicit one and two soliton solutions are calculated for dnAKNS equation.

2 Lax Pair

In this section, we present the discrete generalization of nAKNS equation. We know that the Lax pair of continuous nAKNS equation can be written as [28]

$$\partial_x \Phi \equiv A\Phi = \big(-2\lambda V^{(0)} + V\big)\Phi, \tag{2.1}$$

$$\partial_t \Phi \equiv B\Phi = -\frac{1}{2\lambda}\big(V^{(0)} + \partial_t W\big)\Phi, \tag{2.2}$$

where the matrices W, V, $V^{(0)}$ are defined as

$$W = \begin{pmatrix} w & -u \\ -v & -w \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -u \\ v & 0 \end{pmatrix}, V^{(0)} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3}$$

In Eq. (2.3), $u(x,t)$ and $v(x,t)$ are the scalar functions and $w(x,t) = \partial_x^{-1}(uv)$. The compatibility condition ($\partial_x\partial_t\Phi = \partial_t\partial_x\Phi$) for the Lax pair (2.1), (2.2) yields the zero-curvature condition i.e., $A_t - B_x + [A, B] = 0$, gives the equation of motion

$$\partial_x\partial_t W = [V, \partial_t W] + \left[V, V^{(0)}\right]. \tag{2.4}$$

The Eq. (2.4) is known as the matrix nAKNS equation. Now, we derive the discretize version of the nAKNS equation by discretization of linear system given by (2.1) and Eq. (2.2) for both space and time. The Lax pair of difference-difference nAKNS equation is given by

$$\mathfrak{T}_n \Phi_{n,m} \equiv \Phi_{n+1,m} = A_{n,m}\Phi_{n,m}, \tag{2.5}$$

$$\mathfrak{T}_m \Phi_{n,m} \equiv \Phi_{n,m+1} = B_{n,m}\Phi_{n,m}, \tag{2.6}$$

with the matrices $A_{n,m}$ and $B_{n,m}$ given as

$$A_{n,m} = I + a\left(V_{n,m} - 2\lambda^{-1}V^{(0)}\right), \tag{2.7}$$

$$B_{n,m} = I - \frac{\lambda}{2}\left(bV^{(0)} + W_{n,m+1} - W_{n,m}\right), \tag{2.8}$$

where $\Phi_{n,m} \equiv \Phi_{n,m}(\lambda)$ is an $N \times N$ eigen matrix which depends upon variables m and n written in subscripts defined over a lattice. Also note that a and b in Eqs. (2.7) and (2.8) represent the lattice parameters along n and m respectively. The consistency condition ($\mathfrak{T}_n\mathfrak{T}_m\Phi_{n,m} = \mathfrak{T}_m\mathfrak{T}_n\Phi_{n,m}$) of difference-difference Lax pair (2.5), (2.6) leads to the discrete zero-curvature condition i.e., $A_{n,m+1}B_{n,m} - B_{n+1,m}A_{n,m} = 0$, which is equivalent to dnAKNS equation of motion:

$$\begin{aligned} &W_{n+1,m+1} - W_{n+1,m} - W_{n,m+1} + W_{n,m} - abV_{n,m+1}V^{(0)} + abV^{(0)}V_{n,m} \\ &\quad = +aV_{n,m+1}\left(W_{n,m+1} - W_{n,m}\right) - a\left(W_{n+1,m+1} - W_{n+1,m}\right)V_{n,m}. \end{aligned} \tag{2.9}$$

The Eq. (2.9) is the discrete version of the matrix nAKNS equation.

In order to calculate the semi-discrete (difference-differential) version of nAKNS equation, taking t continuous $\left(\lim_{b\to 0} \frac{g_{n,m+1}-g_{n,m}}{b} = \frac{dg}{dt}\right)$ and x is discrete and is given by

$$\frac{d}{dt}\left(W_{n+1}-W_n\right) = aV_nV^{(0)} - aV^{(0)}V_n + aV_n\frac{d}{dt}W_n - a\frac{d}{dt}(W_{n+1})V_n. \tag{2.10}$$

Similarly, the linear system (2.5), (2.6) can be converted into semi-discrete version as

$$\Phi_{n+1} = A_n\Phi_n = \left(I + a\left(V_n - 2\lambda^{-1}V^{(0)}\right)\right)\Phi_n, \tag{2.11}$$

$$\frac{d}{dt}\Phi_n = B_n\Phi_n = -\frac{\lambda}{2}\left(V^{(0)} + \partial_t W_n\right)\Phi_n, \tag{2.12}$$

where the equation of motion can be calculated by $\frac{d}{dt}A_n + A_nB_n - B_{n+1}A_n = 0$, which is equivalent to Eq. (2.9). Now if we take lattice parameter $a \to 0$, we get Eq. (2.4) from Eq. (2.9). Also, we get Eqs. (2.1), (2.2) from Eqs. (2.11), (2.12).

We can easily derive the semi-discrete version of nAKNS equation which is continuous in x and discrete in t by taking the continuum limit $\left(\lim_{a\to 0} \frac{g_{n+1,m}-g_{n,m}}{a} = \frac{dg}{dx}\right)$, i.e.,

$$\begin{aligned}\frac{d}{dx}\left(W_{m+1}-W_m\right) &= bV_{m+1}V^{(0)} - bV^{(0)}V_m + V_{m+1}\left(W_{m+1}-W_m\right)\\ &\quad - \left(W_{m+1}-W_m\right)V_m.\end{aligned} \tag{2.13}$$

Similarly, the Lax pair (2.5), (2.6) can be converted into semi-discrete version as

$$\frac{d}{dx}\Phi_m = A_m\Phi_m = \left(V_m - 2\lambda^{-1}V^{(0)}\right)\Phi_m, \tag{2.14}$$

$$\Phi_{m+1} = B_m\Phi_m = \left(I - \frac{\lambda}{2}\left(bV^{(0)} + W_{m+1} - W_m\right)\right)\Phi_m, \tag{2.15}$$

where the equation of motion can be obtained by $\frac{d}{dx}B_m + B_mA_m - A_{m+1}B_m = 0$, which is equivalent to equation (2.13). Further, if we take the lattice parameter $b \to 0$, we obtain Eq. (2.4) from Eq. (2.13). Also, we can obtain Eqs. (2.1), (2.2) from Eq. (2.14), (2.15).

3 Discrete Darboux Transformation

Now, we define DT on discrete Lax pair given by Eqs. (2.5), (2.6) to calculate the soliton solutions in terms of quasi-determinants. The Darboux matrix $D_{n,m}(\lambda)$ acts on the seed solution $\Phi_{n,m}$ and gives the transformed solution $\Phi_{n,m}[1]$. The one-fold DT on the solution $\Phi_{n,m}$ is defined as

$$\Phi_{n,m}[1] = D_{n,m}(\lambda)\Phi_{n,m}, \tag{3.1}$$

where the Darboux matrix is defined as

$$D_{n,m}(\lambda) = \lambda^{-1}I - Q_{n,m}. \tag{3.2}$$

Note that in Eq. (3.2) I is the $N \times N$ identity matrix and $Q_{n,m}$ is another $N \times N$ matrix which is yet to be evaluated. From the covariance of Lax pair (2.5), (2.6) under DT, we conclude

$$V_{n,m}[1] = V_{n,m} + 2Q_{n+1,m}V^{(0)} - 2V^{(0)}Q_{n,m}, \tag{3.3}$$
$$V^{(0)}[1] = V^{(0)}, \tag{3.4}$$
$$W_{n,+m+1}[1] - W_{n,m}[1] = W_{n,m+1} - W_{n,m} - 2Q_{n,m+1} + 2Q_{n,m}. \tag{3.5}$$

Equation (3.5) can be re-written as

$$W_{n,m}[1] = W_{n,m} - 2Q_{n,m}. \tag{3.6}$$

Similarly the covariance of equations (2.5) and (2.6) under DT shows that the following conditions on $Q_{n,m}$ should be satisfied at every point of lattice

$$\begin{aligned} Q_{n+1,m} - Q_{n,m} = a\left(V_{n,m}Q_{n,m} - Q_{n+1,m}V_{n,m}\right) \\ +2a\left(Q_{n+1,m}V^{(0)} - V^{(0)}Q_{n,m}\right)Q_{n,m}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} Q_{n,m+1} - Q_{n,m} = \frac{1}{2}\left\{ \begin{matrix} bV^{(0)}Q_{n,m} - bQ_{n,m+1}V^{(0)} + \left(W_{n,m+1} - W_{n,m}\right)Q_{n,m} \\ -Q_{n,m+1}\left(W_{n,m+1} - W_{n,m}\right) \end{matrix} \right\} \\ Q_{n,m}^{-1}. \end{aligned} \tag{3.8}$$

Now, we develop the matrix $Q_{n,m}$ for the dnAKNS equation. Let us define N constant parameters (real/complex) $\lambda_1, \lambda_2, \ldots, \lambda_N$. For each value of parameter, there exists a particular column vector solution $|\sigma_i\rangle_{n,m}$ for the linear system (2.5), (2.6) i.e.,

$$|\sigma\rangle_{n+1,m} = \left(1 + a\left(V_{n,m} - 2\lambda^{-1}V^{(0)}\right)\right)|\sigma\rangle_{n,m}, \tag{3.9}$$

$$|\sigma\rangle_{n,m+1} = \left(1 - \frac{\lambda}{2}\left(bV^{(0)} + W_{n,m+1} - W_{n,m}\right)\right)|\sigma\rangle_{n,m}. \tag{3.10}$$

We now need to define $Q_{n,m}$ so that Eqs. (3.7) and (3.8) are satisfied. Therefore, we choose

$$Q_{n,m} = \Omega_{n,m}\Lambda^{-1}\Omega_{n,m}^{-1}. \tag{3.11}$$

By taking $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, the Lax pair (3.9), (3.10) can be written in the form of matrix as

$$\Omega_{n+1,m} = \Omega_{n,m} + a\left(V_{n,m}\Omega_{n,m} - 2V^{(0)}\Omega_{n,m}\Lambda^{-1}\right), \tag{3.12}$$

$$\Omega_{n,m+1} = \Omega_{n,m} - \frac{1}{2}\left(bV^{(0)}\Omega_{n,m} + W_{n,m+1}\Omega_{n,m} - W_{n,m}\Omega_{n,m}\right)\Lambda. \quad (3.13)$$

Now, we will show when we operate difference operator Δ_n and Δ_m on $Q_{n,m}$ given by Eq. (3.11), we get Eqs. (3.7) and (3.8), as

$$\begin{aligned}\Delta_n Q_{n,m} &= Q_{n+1,m} - Q_{n,m},\\ &= \Omega_{n+1,m}\Lambda^{-1}\Omega^{-1}_{n+1,m} - \Omega_{n,m}\Lambda^{-1}\Omega^{-1}_{n,m} + \Omega_{n+1,m}\Lambda^{-1}\Omega^{-1}_{n,m}\Omega_{n,m}\Omega^{-1}_{n,m}\\ &\quad -\Omega_{n+1,m}\Lambda^{-1}\Omega^{-1}_{n+1,m}\Omega_{n+1,m}\Omega^{-1}_{n,m},\\ &= \Omega_{n+1,m}\Lambda^{-1}\Omega^{-1}_{n+1,m}\left(I - \Omega_{n+1,m}\Omega^{-1}_{n,m}\right) + \left(\Omega_{n+1,m} - \Omega_{n,m}\right)\Lambda^{-1}\Omega^{-1}_{n,m},\\ &= Q_{n+1,m}\left(I - \left(\Omega_{n,m} + aV_{n,m}\Omega_{n,m} - 2aV^{(0)}\Omega_{n,m}\Lambda^{-1}\right)\Omega^{-1}_{n,m}\right)\\ &\quad + \left(\Omega_{n,m} + aV_{n,m}\Omega_{n,m} - 2aV^{(0)}\Omega_{n,m}\Lambda^{-1} - \Omega_{n,m}\right)\Lambda^{-1}\Omega^{-1}_{n,m},\\ &= a\left(V_{n,m}Q_{n,m} - Q_{n+1,m}V_{n,m}\right) + 2a\left(Q_{n+1,m}V^{(0)} - V^{(0)}Q_{n,m}\right)Q_{n,m},\end{aligned}$$

which is similar to Eq. (3.7). Similarly, by operating Δ_m on $Q_{n,m}$, we get

$$\begin{aligned}\Delta_m Q_{n,m} &= Q_{n,m+1} - Q_{n,m},\\ &= \Omega_{n,m+1}\Lambda^{-1}\Omega^{-1}_{n,m+1} - \Omega_{n,m}\Lambda^{-1}\Omega^{-1}_{n,m} + \Omega_{n,m+1}\Lambda^{-1}\Omega^{-1}_{n,m}\Omega_{n,m}\Omega^{-1}_{n,m}\\ &\quad -\Omega_{n,m+1}\Lambda^{-1}\Omega^{-1}_{n,m+1}\Omega_{n,m+1}\Omega^{-1}_{n,m},\\ &= \Omega_{n,m+1}\Lambda^{-1}\Omega^{-1}_{n,m+1}\left(I - \Omega_{n,m+1}\Omega^{-1}_{n,m}\right) + \left(\Omega_{n,m+1} - \Omega_{n,m}\right)\Lambda^{-1}\Omega^{-1}_{n,m},\\ &= Q_{n,m+1}\left(I - \left\{\Omega_{n,m} - \frac{b}{2}V^{(0)}\Omega_{n,m}\Lambda - \frac{1}{2}\left(W_{n,m+1} - W_{n,m}\right)\Omega_{n,m}\Lambda\right\}\Omega^{-1}_{n,m}\right)\\ &\quad + \left(\Omega_{n,m} - \frac{b}{2}V^{(0)}\Omega_{n,m}\Lambda - \frac{1}{2}\left(W_{n,m+1} - W_{n,m}\right)\Omega_{n,m}\Lambda - \Omega_{n,m}\right)\Lambda^{-1}\Omega^{-1}_{n,m},\\ &= \frac{1}{2}\left\{\begin{array}{c} bV^{(0)}Q_{n,m} - bQ_{n,m+1}V^{(0)} + \left(W_{n,m+1} - W_{n,m}\right)Q_{n,m}\\ -Q_{n,m+1}\left(W_{n,m+1} - W_{n,m}\right)\end{array}\right\}Q^{-1}_{n,m},\end{aligned}$$

which is same as Eq. (3.8). Therefore, we can write the DT on matrix solution $\Phi_{n,m}$ and also the matrix solutions $V_{n,m}$, $W_{n,m}$ as

$$\Phi_{n,m}[1] = \left(\lambda^{-1}I - \Omega_{n,m}\Lambda^{-1}\Omega^{-1}_{n,m}\right)\Phi_{n,m}, \quad (3.14)$$

$$V_{n,m}[1] = V_{n,m} + 2\Omega_{n+1,m}\Lambda^{-1}\Omega^{-1}_{n+1,m}V^{(0)} - 2V^{(0)}\Omega_{n,m}\Lambda^{-1}\Omega^{-1}_{n,m}, \quad (3.15)$$

$$W_{n,m}[1] = W_{n,m} - 2\Omega_{n,m}\Lambda^{-1}\Omega^{-1}_{n,m}. \quad (3.16)$$

Now, we express the solutions in terms of quasideterminants (for detail see [33, 34]). The one-fold Darboux matrix transformed solution (3.14) can be written as

$$\Phi_{n,m}[1] \equiv D_{n,m}(\lambda)\Phi_{n,m} = \left(\lambda I - \Omega_{n,m}\Lambda^{-1}\Omega_{n,m}^{-1}\right)\Phi_{n,m},$$
$$= \begin{vmatrix} \Omega_{n,m} & \Phi_{n,m} \\ \Omega_{n,m}\Lambda^{-1} & \boxed{\lambda^{-1}I} \end{vmatrix}. \tag{3.17}$$

The result (3.17) can be extended by repeating iterations of Darboux matrix and can be generalized to K-fold DT. For this, we define the particular matrix solution $\Omega^k_{n,m}$ of the Lax pair for $\Lambda = \Lambda_k$ where $k = 1, 2, \ldots, K$. So, the K-fold DT is given by

$$\Phi^{[K]}_{n,m} = \begin{vmatrix} \Omega_{n,m,1} & \Omega_{n,m,2} & \cdots & \Omega_{n,m,K} & \Phi_{n,m} \\ \Omega_{n,m,1}\Lambda_1^{-1} & \Omega_{n,m,2}\Lambda_2^{-1} & \cdots & \Omega_{n,m,K}\Lambda_K^{-1} & \lambda^{-1}\Phi_{n,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Omega_{n,m,1}\Lambda_1^{-(K-1)} & \Omega_{n,m,2}\Lambda_2^{-(K-1)} & \cdots & \Omega_{n,m,K}\Lambda_K^{-(K-1)} & \lambda^{-(K-1)}\Phi_{n,m} \\ \Omega_{n,m,1}\Lambda_1^{-K} & \Omega_{n,m,2}\Lambda_2^{-K} & \cdots & \Omega_{n,m,K}\Lambda_K^{-K} & \boxed{\lambda^{-K}\Phi_{n,m}} \end{vmatrix}. \tag{3.18}$$

The expression (3.15) can also be expressed as

$$V_{n,m}[1] = V_{n,m} - 2[V^{(0)}, Q_{n,m}],$$
$$= V_{n,m} + 2[V^{(0)}, (O - \Omega_{n,m}\Lambda^{-1}\Omega_{n,m}^{-1})],$$
$$= V_{n,m} + 2\left[V^{(0)}, \begin{vmatrix} \Omega_{n,m} & I \\ \Omega_{n,m}\Lambda^{-1} & \boxed{O} \end{vmatrix}\right]. \tag{3.19}$$

Similarly, expression (3.16) can be written as

$$W_{n,m}[1] = W_{n,m} + 2(O - \Omega_{n,m}\Lambda^{-1}\Omega_{n,m}^{-1}),$$
$$= W_{n,m} + 2\begin{vmatrix} \Omega_{n,m} & I \\ \Omega_{n,m}\Lambda^{-1} & \boxed{O} \end{vmatrix}. \tag{3.20}$$

The K-times DT on $V_{n,m}$, $V^{(0)}$and $W_{n,m}$ can be written as

$$V^{[K]}_{n,m} = V_{n,m} + 2\left[V_0, \begin{vmatrix} \Omega_{n,m,1} & \Omega_{n,m,2} & \cdots & \Omega_{n,m,K} & O \\ \Omega_{n,m,1}\Lambda_1^{-1} & \Omega_{n,m,2}\Lambda_2^{-1} & \cdots & \Omega_{n,m,K}\Lambda_K^{-1} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Omega_{n,m,1}\Lambda_1^{-(K-1)} & \Omega_{n,m,2}\Lambda_2^{-(K-1)} & \cdots & \Omega_{n,m,K}\Lambda_K^{-(K-1)} & I \\ \Omega_{n,m,1}\Lambda_1^{-K} & \Omega_{n,m,2}\Lambda_2^{-K} & \cdots & \Omega_{n,m,K}\Lambda_K^{-K} & \boxed{O} \end{vmatrix}\right], \tag{3.21}$$

$$V^{(0)[K]} = V^{(0)}, \tag{3.22}$$

$$W^{[K]}_{n,m} = W_{n,m} + 2 \begin{vmatrix} \Omega_{n,m,1} & \Omega_{n,m,2} & \cdots & \Omega_{n,m,K} & O \\ \Omega_{n,m,1}\Lambda_1^{-1} & \Omega_{n,m,2}\Lambda_2^{-1} & \cdots & \Omega_{n,m,K}\Lambda_K^{-1} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Omega_{n,m,1}\Lambda_1^{-(K-1)} & \Omega_{n,m,2}\Lambda_2^{-(K-1)} & \cdots & \Omega_{n,m,K}\Lambda_K^{-(K-1)} & I \\ \Omega_{n,m,1}\Lambda_1^{-K} & \Omega_{n,m,2}\Lambda_2^{-K} & \cdots & \Omega_{n,m,K}\Lambda_K^{-K} & \boxed{O} \end{vmatrix}. \tag{3.23}$$

The expressions given by equations (3.18), (3.21), (3.22) and (3.23) are the Kth solutions of discrete nAKNS equation and these results can be easily derived by induction.

3.1 Semi-Discrete Darboux transformation

3.1.1 Continuous in t-Variable

The one-fold DT on the solution Φ_n is defined as

$$\Phi_n[1] = D_n(\lambda)\Phi_n, \tag{3.24}$$

where the Darboux matrix is defined as

$$D_n(\lambda) = \lambda^{-1} I_1 - Q_n. \tag{3.25}$$

Note that in Eq. (3.25) I_1 is the $N \times N$ identity matrix and Q_n is a $N \times N$ matrix. From the covariance of Lax pair (2.11), (2.12) under DT, we conclude

$$V_n[1] = V_n + 2Q_{n+1}V^{(0)} - 2V^{(0)}Q_n, \tag{3.26}$$

$$V^{(0)}[1] = V^{(0)}, \tag{3.27}$$

$$W_n[1] = W_n - 2Q_n. \tag{3.28}$$

Similarly the covariance of equations (2.11) and (2.12) under DT shows that the following conditions on Q_n should be satisfied at every lattice point

$$Q_{n+1} - Q_n = a\,(V_n Q_n - Q_{n+1}V_n) + 2a\left(Q_{n+1}V^{(0)} - V^{(0)}Q_n\right)Q_n, \tag{3.29}$$

$$\frac{d}{dt}Q_n = \frac{1}{2}\left([V^{(0)}, Q_n] + [\partial_t W_n, Q_n]\right). \tag{3.30}$$

By putting $a \to 0$, the solutions reduce to continuous nAKNS solutions calculated in [28].

3.1.2 Continuous in x-Variable

The one-fold DT on the solution Φ_m is defined as

$$\Phi_m[1] = D_m(\lambda)\Phi_m, \tag{3.31}$$

where the Darboux matrix is defined as

$$D_m(\lambda) = \lambda^{-1} I_2 - Q_m. \tag{3.32}$$

Note that in Eq. (3.32), I_2 is the $N \times N$ identity matrix and Q_m is a $N \times N$ matrix. From the covariance of Lax pair (2.14), (2.15) under DT, we conclude

$$V_m[1] = V_m + 2[Q_m, V^0], \tag{3.33}$$
$$V^{(0)}[1] = V^{(0)}, \tag{3.34}$$
$$W_m[1] = W_m - 2Q_m. \tag{3.35}$$

Similarly the covariance of equations (2.14) and (2.15) under DT shows that the following conditions on Q_m should be satisfied at every lattice point

$$Q_{m+1} - Q_m = \frac{1}{2}\left\{ \begin{array}{c} bV^{(0)}Q_m - bQ_{m+1}V^{(0)} + (W_{m+1} - W_m)\,Q_m \\ -Q_{m+1}\,(W_{m+1} - W_m) \end{array} \right\} Q_m^{-1}, \tag{3.36}$$

$$\frac{d}{dt} Q_m = [V_m, Q_m] + 2[Q_m, V^{(0)}]Q_m. \tag{3.37}$$

By putting $b \to 0$, the solutions reduce to continuous nAKNS solutions presented in [28].

4 Explicit Solutions

In this section, we calculate soliton solutions from a trivial (seed) solution by solving the Lax pair (2.5), (2.6) of dnAKNS equation. For this, we re-write the matrix $Q^{(K)}_{n,m}$ from (3.23) in a more appropriate form as follows

$$Q^{(K)}_{n,m} = \begin{vmatrix} \Omega_{n,m} & \mathfrak{I}^{(K)} \\ \widetilde{\Omega}_{n,m} & \boxed{O} \end{vmatrix}, \tag{4.1}$$

where $I^{(K)}$, $\widetilde{\Omega}_{n,m}$ and $\Omega_{n,m}$ are $NK \times N$, $N \times NK$ and $NK \times NK$ matrices respectively. These matrices are given by

$$\begin{aligned} \mathfrak{I}^{(K)} &= \left(I \;\; O \cdots O\right)^T, \\ \widetilde{\Omega}_{n,m} &= \left(\Omega_{n,m,1}\Lambda_1^{-K} \;\; \Omega_{n,m,2}\Lambda_2^{-K} \cdots \Omega_{n,m,K}\Lambda_K^{-K}\right), \\ \Omega_{n,m} &= \begin{pmatrix} \Omega_{n,m,1} & \Omega_{n,m,2} & \cdots & \Omega_{n,m,K} \\ \Omega_{n,m,1}\Lambda_1^{-1} & \Omega_{n,m,2}\Lambda_2^{-1} & \cdots & \Omega_{n,m,K}\Lambda_K^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{n,m,1}\Lambda_1^{-(K-1)} & \Omega_{n,m,2}\Lambda_2^{-(K-1)} & \cdots & \Omega_{n,m,K}\Lambda_K^{-(K-1)} \end{pmatrix}. \end{aligned} \tag{4.2}$$

The elements of $Q^{(K)}_{n,m}$ can be written as

$$Q^{(K)}_{n,m\ ij} = \left(\begin{vmatrix} \Omega_{n,m} & \mathfrak{I}^{(K)} \\ \widetilde{\Omega}_{n,m} & \boxed{O} \end{vmatrix} \right)_{ij} = \begin{vmatrix} \Omega_{n,m} & \mathfrak{I}^{(K)}_j \\ (\widetilde{\Omega}_{n,m})_i & \boxed{0} \end{vmatrix},$$
$$= (-1)^{i+j} \frac{\det(\Omega_{n,m})_{ij}}{\det(\Omega_{n,m})}, \quad i, j = 1, 2, \ldots, K, \tag{4.3}$$

where $(\widetilde{\Omega}_{n,m})_i$ and $\mathfrak{I}^{(K)}_j$ represent the i-th row and j-th column of the matrices $\widetilde{\Omega}_{n,m}$ and $\mathfrak{I}^{(K)}$ respectively. Let us now consider the simplest case for $N = 2$. The matrix $Q^{(K)}_{n,m}$ can be written as

$$Q^{(K)}_{n,m} \equiv \begin{pmatrix} Q^{(K)}_{n,m\ 11} & Q^{(K)}_{n,m\ 12} \\ Q^{(K)}_{n,m\ 21} & Q^{(K)}_{n,m\ 22} \end{pmatrix} = \begin{vmatrix} \Omega_{n,m} & \mathfrak{I}^{(K)} \\ \widetilde{\Omega}_{n,m} & \boxed{O_2} \end{vmatrix},$$

where

$$Q^{(K)}_{n,m\ ij} = \begin{vmatrix} \Omega_{n,m} & \mathfrak{I}^{(K)}_j \\ (\widetilde{\Omega}_{n,m})_i & \boxed{0} \end{vmatrix} = (-1)^{i+j} \frac{\det(\Omega_{n,m})_{ij}}{\det(\Omega_{n,m})}, \quad i, j = 1, 2. \tag{4.4}$$

For one soliton $K = 1$, we have

$$\mathfrak{I}^{(1)} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\Omega_{n,m,\ 1} = \begin{pmatrix} \pi^{(1)}_{n,m,\ 11} & \pi^{(2)}_{n,m,\ 12} \\ \pi^{(1)}_{n,m,\ 21} & \pi^{(2)}_{n,m,\ 22} \end{pmatrix}, \qquad \Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \bar{\lambda}_1 \end{pmatrix},$$
$$\widetilde{\Omega}_{n,m} = \Omega_{n,m,\ 1}\Lambda_1^{-1},$$
$$= \begin{pmatrix} \lambda_1^{-1}\pi^{(1)}_{n,m,\ 11} & \bar{\lambda}_1^{-1}\pi^{(2)}_{n,m\ 12} \\ \lambda_1^{-1}\pi^{(1)}_{n,m\ 21} & \bar{\lambda}_1^{-1}\pi^{(2)}_{n,m,\ 22} \end{pmatrix}. \tag{4.5}$$

Substituting Eq. (4.5) in Eq. (4.4), the matrix element $Q^{(1)}_{n,m,\ 12}$ of the matrix $Q^{(1)}_{n,m}$ can be obtained as

$$Q^{(1)}_{n,m\ 12} = \begin{vmatrix} \Omega_{n,m} & \mathfrak{I}^{(1)}_2 \\ (\widetilde{\Omega}_{n,m})_1 & \boxed{O_2} \end{vmatrix} = \begin{vmatrix} \pi^{(1)}_{n,m,\ 11} & \pi^{(2)}_{n,m,\ 12} & 0 \\ \pi^{(1)}_{n,m,\ 21} & \pi^{(2)}_{n,m,\ 22} & 1 \\ \lambda_1^{-1}\pi^{(1)}_{n,m,\ 11} & \bar{\lambda}_1^{-1}\pi^{(2)}_{n,m\ 12} & \boxed{0} \end{vmatrix},$$
$$= -\frac{\det\begin{pmatrix} \pi^{(1)}_{n,m,\ 11} & \pi^{(2)}_{n,m,\ 12} \\ \lambda_1^{-1}\pi^{(1)}_{n,m,\ 11} & \bar{\lambda}_1^{-1}\pi^{(2)}_{n,m\ 12} \end{pmatrix}}{\det\begin{pmatrix} \pi^{(1)}_{n,m,\ 11} & \pi^{(2)}_{n,m,\ 12} \\ \pi^{(1)}_{n,m,\ 21} & \pi^{(2)}_{n,m,\ 22} \end{pmatrix}}. \tag{4.6}$$

Similarly, we have

$$Q^{(1)}_{n,m,\,21} = \begin{vmatrix} \Omega_{n,m} & \mathfrak{I}_1^{(1)} \\ (\widetilde{\Omega}_{n,m})_2 & \boxed{O_2} \end{vmatrix} = \begin{vmatrix} \pi^{(1)}_{n,m,\,11} & \pi^{(2)}_{n,m,\,12} & 1 \\ \pi^{(1)}_{n,m,\,21} & \pi^{(2)}_{n,m,\,22} & 0 \\ \lambda_1^{-1}\pi^{(1)}_{n,m\,21} & \bar{\lambda}_1^{-1}\pi^{(2)}_{n,m,\,22} & \boxed{0} \end{vmatrix},$$

$$= -\frac{\det\begin{pmatrix} \lambda_1^{-1}\pi^{(1)}_{n,m\,21} & \bar{\lambda}_1^{-1}\pi^{(2)}_{n,m,\,22} \\ \pi^{(1)}_{n,m,\,21} & \pi^{(2)}_{n,m,\,22} \end{pmatrix}}{\det\begin{pmatrix} \pi^{(1)}_{n,m,\,11} & \pi^{(2)}_{n,m,\,12} \\ \pi^{(1)}_{n,m,\,21} & \pi^{(2)}_{n,m,\,22} \end{pmatrix}}. \tag{4.7}$$

Similarly, the other two terms can also be obtained. In order to get an explicit expression of the soliton solution, let us take $u,\ v = 0$ as the seed solution, so that the dnAKNS Lax pair (2.5), (2.6) reduces to

$$\Phi_{n+1,m} = I - 2a\lambda^{-1}V^{(0)}\Phi_{n,m}, \tag{4.8}$$

$$\Phi_{n,m+1} = I - \frac{\lambda}{2}bV^{(0)}\Phi_{n,m}. \tag{4.9}$$

Therefore, Eqs. (4.8), (4.9) give the matrix solution $\Phi_{n,m}$ of the Lax pair (2.5), (2.6) having the following form

$$\begin{pmatrix} X_{n+1,m} \\ Y_{n+1,m} \end{pmatrix} = \begin{pmatrix} 1 + a\lambda^{-1} & \\ & 1 - a\lambda^{-1} \end{pmatrix} \begin{pmatrix} X_{n,m} \\ Y_{n,m} \end{pmatrix}. \tag{4.10}$$

Also

$$\begin{pmatrix} X_{n,m+1} \\ Y_{n,m+1} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\lambda b}{4} & \\ & 1 - \frac{\lambda b}{4} \end{pmatrix} \begin{pmatrix} X_{n,m} \\ Y_{n,m} \end{pmatrix}. \tag{4.11}$$

The solution of Lax pair (4.10), (4.11) yields

$$X_{n,m} = \left(1 + a\lambda^{-1}\right)^n \left(1 + \frac{\lambda b}{4}\right)^m, \tag{4.12}$$

$$Y_{n,m} = \left(1 - a\lambda^{-1}\right)^n \left(1 - \frac{\lambda b}{4}\right)^m. \tag{4.13}$$

Now substituting the choice $\pi^{(1)}_{n,m,\,11} = \pi^{(1)}_{n,m,\,22} = X_{n,m}$, $\pi^{(2)}_{n,m,\,12} = -Y_{n,m}$, $\pi^{(2)}_{n,m,\,21} = Y_{n,m}$ and $\lambda_1 = \lambda$, $\bar{\lambda}_1 = -\lambda$ in Eqs. (4.6) and (4.7), we get

$$Q^{(1)}_{n,m\,12} = -\frac{2\lambda^{-1}X_{n,m}Y_{n,m}}{X^2_{n,m} + Y^2_{n,m}}, \tag{4.14}$$

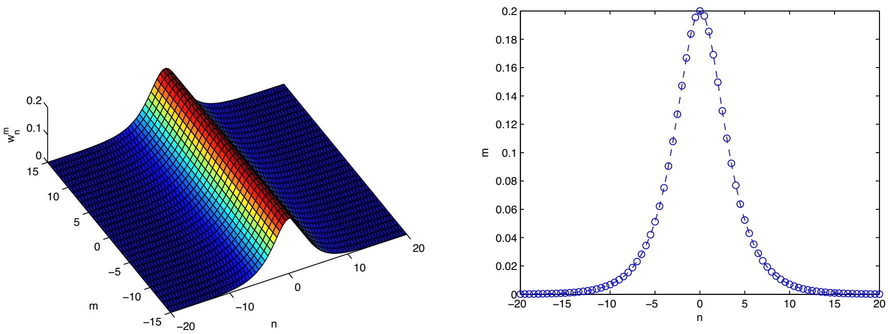


Fig. 1 Discrete one-soliton: for numerical values $\lambda = 5, a = 1, b = 2$

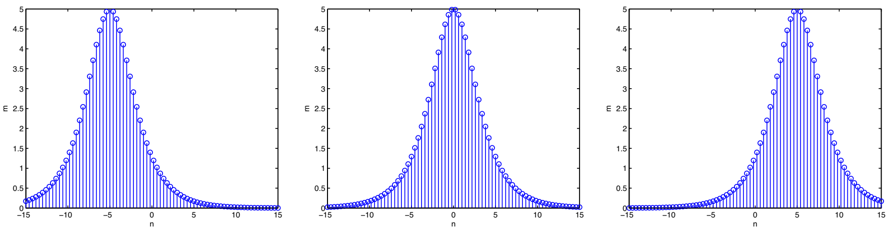


Fig. 2 Discrete travelling one-soliton: for numerical values $\lambda = 5, a = 1, b = 2$.

$$Q^{(1)}_{n,m,\,21} = -\frac{2\lambda^{-1} X_{n,m} Y_{n,m}}{X^2_{n,m} + Y^2_{n,m}}. \tag{4.15}$$

Expressions (4.14), (4.15) are identical and represent the one soliton solution which is shown in Figs. 1 and 2.

For two soliton solution, substitute $K = 2$ in Eq. (4.4). The corresponding matrices are given below

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{pmatrix},$$

$$\mathfrak{I}^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\widetilde{\Omega}_{n,m} = \left(\Omega_{n,m,\,1}\Lambda_1^{-2} \vdots\, \Omega_{n,m,\,2}\Lambda_2^{-2} \right),$$

$$= \begin{pmatrix} \lambda_1^{-2} X_{n,m,\,1} & -\lambda_1^{-2} Y_{n,m,\,1} & \vdots & \lambda_2^{-2} X_{n,m,\,2} & -\lambda_1^{-2} Y_{n,m,\,2} \\ \lambda_1^{-2} Y_{n,m,\,1} & \lambda_1^{-2} X_{n,m,\,1} & \vdots & \lambda_2^{-2} Y_{n,m,\,2} & \lambda_2^{-2} X_{n,m,\,2} \end{pmatrix},$$

$$\Omega_{n,m} = \begin{pmatrix} \Omega_{n,m,\,1} & \Omega_{n,m,\,2} \\ \Omega_{n,m,\,1}\Lambda_1^{-1} & \Omega_{n,m,\,2}\Lambda_2^{-1} \end{pmatrix}$$

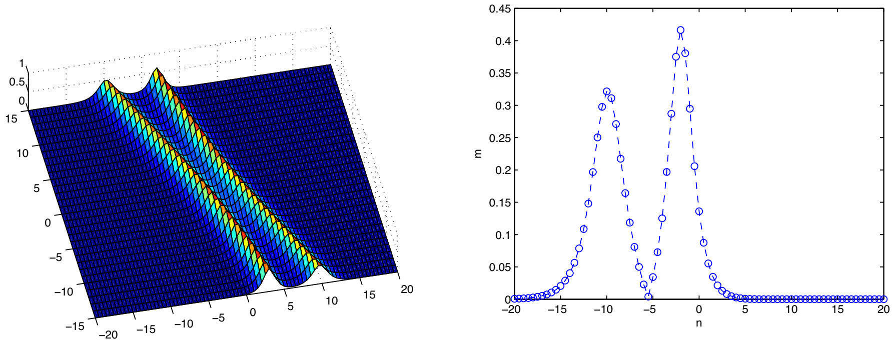


Fig. 3 Discrete two-soliton: for numerical values $\lambda = 3.1$, $\eta = 2.6$, $a = 1$, $b = 2$.

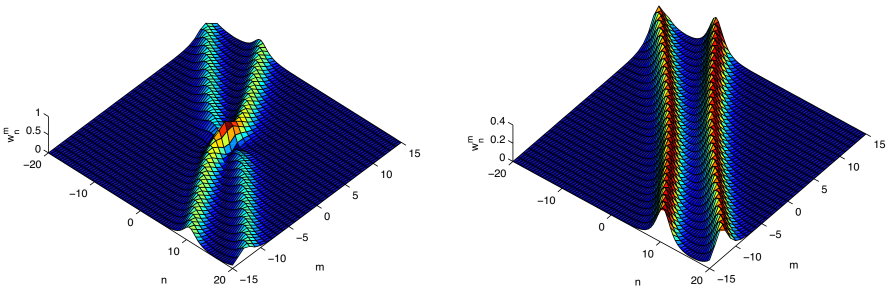


Fig. 4 Discrete two-soliton: left figure shows the interaction and right shows after the interaction, for numerical values $\lambda = 3.2 + 0.8i$, $\eta = 2.6 + 1.02i$, $a = 1$, $b = 2$.

$$= \begin{pmatrix} X_{n,m,\,1} & -Y_{n,m,\,1} & X_{n,m,\,2} & -Y_{n,m,\,2} \\ Y_{n,m,\,1} & X_{n,m,\,1} & Y_{n,m,\,2} & X_{n,m,\,2} \\ \lambda_1^{-1}X_{n,m,\,1} & \lambda_1^{-1}Y_{n,m,\,1} & \lambda_2^{-1}X_{n,m,\,2} & \lambda_2^{-1}Y_{n,m,\,2} \\ \lambda_1^{-1}Y_{n,m,\,1} & -\lambda_1^{-1}X_{n,m,\,1} & \lambda_2^{-1}Y_{n,m,\,2} & -\lambda_2^{-1}X_{n,m,\,2} \end{pmatrix}.$$

The matrix $Q^{(2)}_{n,m\ 21}$ can be calculated as

$$Q^{(2)}_{n,m\ 21} = -\frac{\det\begin{pmatrix} X_{n,m,\,1} & -Y_{n,m,\,1} & X_{n,m,\,2} & -Y_{n,m,\,2} \\ Y_{n,m,\,1} & X_{n,m,\,1} & Y_{n,m,\,2} & X_{n,m,\,2} \\ \lambda_1^{-2}Y_{n,m,\,1} & \lambda_1^{-2}X_{n,m,\,1} & \lambda_2^{-2}Y_{n,m,\,2} & \lambda_2^{-2}X_{n,m,\,2} \\ \lambda_1^{-1}Y_{n,m,\,1} & -\lambda_1^{-1}X_{n,m,\,1} & \lambda_2^{-1}Y_{n,m,\,2} & -\lambda_2^{-1}X_{n,m,\,2} \end{pmatrix}}{\det\begin{pmatrix} X_{n,m,\,1} & -Y_{n,m,\,1} & X_{n,m,\,2} & -Y_{n,m,\,2} \\ Y_{n,m,\,1} & X_{n,m,\,1} & Y_{n,m,\,2} & X_{n,m,\,2} \\ \lambda_1^{-1}X_{n,m,\,1} & \lambda_1^{-1}Y_{n,m,\,1} & \lambda_2^{-1}X_{n,m,\,2} & \lambda_2^{-1}Y_{n,m,\,2} \\ \lambda_1^{-1}Y_{n,m,\,1} & -\lambda_1^{-1}X_{n,m,\,1} & \lambda_2^{-1}Y_{n,m,\,2} & -\lambda_2^{-1}X_{n,m,\,2} \end{pmatrix}}. \quad (4.16)$$

The two soliton and also the interaction of two solitons are depicted in Figs. 3, 4 and 5.

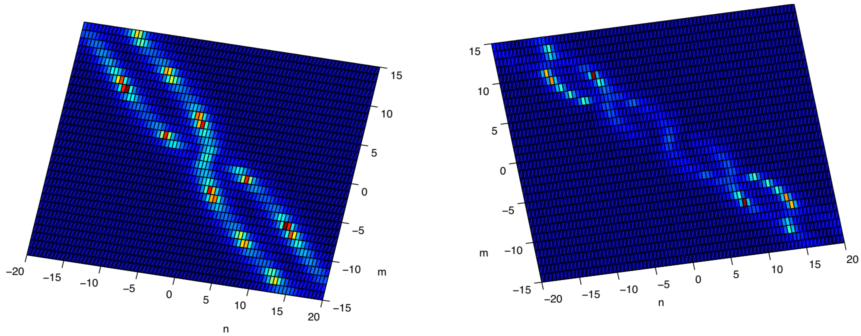


Fig. 5 Discrete two-soliton: left figure presents the interaction with same velocities and right presents the interaction with different velocities, for numerical values $\lambda = 3.1 + 0.6i$, $\eta = 2.6 - 1.02i$, $a = 1$, $b = 2$.

4.1 Continuous in t-Variable

In this case, the explicit solution of semi-discrete nAKNS (sdnAKNS) Lax pair (2.11), (2.12) for the seed solution reduces to

$$\Phi_{n+1} = \left(I - 2a\lambda^{-1}V^{(0)}\right)\Phi_n, \tag{4.17}$$

$$\frac{d}{dt}\Phi_n = -\frac{\lambda}{2}V^{(0)}\Phi_n. \tag{4.18}$$

Therefore, Eqs. (4.17), (4.18) give the matrix solution Φ_n of the Lax pair (2.11), (2.12) having the following form

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + a\lambda^{-1} & \\ & 1 - a\lambda^{-1} \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}. \tag{4.19}$$

Also

$$\begin{pmatrix} \frac{\partial}{\partial t} X_n \\ \frac{\partial}{\partial t} Y_n \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{4} & \\ & -\frac{\lambda}{4} \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}. \tag{4.20}$$

The solution of Lax pair (4.19), (4.20) yields

$$X_n = \left(1 + a\lambda^{-1}\right)^n \exp\left(\frac{\lambda}{4}t\right), \tag{4.21}$$

$$Y_n = \left(1 - a\lambda^{-1}\right)^n \exp\left(\frac{-\lambda}{4}t\right). \tag{4.22}$$

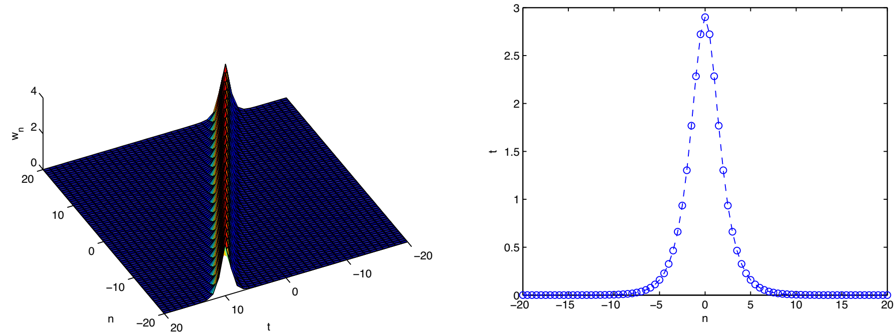


Fig. 6 Semi-Discrete one-soliton: for numerical values $\lambda = 2.9, a = 1$.

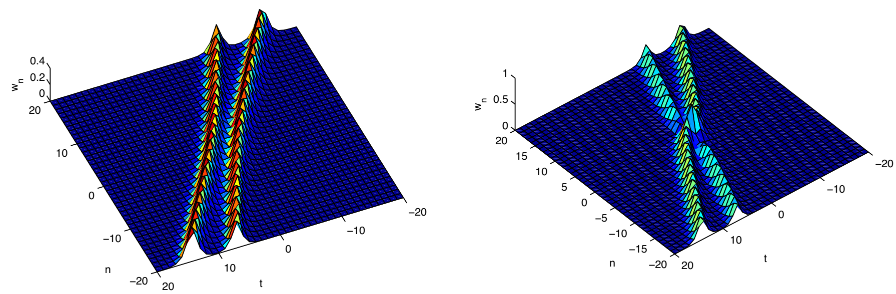


Fig. 7 Semi-Discrete two soliton: left figure shows before interaction and right shows interaction, for numerical values $\lambda = 1.4, \eta = 1.9, a = 1$.

Now substituting the choice $\pi^{(1)}_{n,\,11} = \pi^{(1)}_{n,\,22} = X_n, \pi^{(2)}_{n,\,12} = -Y_n, \pi^{(2)}_{n,\,21} = Y_n$ and $\lambda_1 = \lambda, \bar{\lambda}_1 = -\lambda$ in Eqs. (4.6) and (4.7), we get

$$Q^{(1)}_{n\ 12} = -\frac{2\lambda^{-1} X_n Y_n}{X_n^2 + Y_n^2}, \tag{4.23}$$

$$Q^{(1)}_{n,\,21} = -\frac{2\lambda^{-1} X_n Y_n}{X_n^2 + Y_n^2}. \tag{4.24}$$

Expressions (4.23), (4.24) are identical and represent the one soliton solution which is shown in Fig. 6.

Similarly, the two soliton solution can be calculated by using the relation (4.16) which is depicted in Fig. 7.

4.2 Continuous in x-Variable

In this case, the explicit solution of sdnAKNS Lax pair (2.14), (2.15) for the seed solution reduces to

$$\Phi_{m+1} = \left(I - \frac{b\lambda V^{(0)}}{2}\right)\Phi_m, \tag{4.25}$$

$$\frac{d}{dx}\Phi_m = -\frac{2V^0}{\lambda}\Phi_m. \tag{4.26}$$

Therefore, the Eqs. (4.25), (4.26) give the matrix solution Φ_m of the Lax pair (2.14), (2.15) which has the following expression

$$\begin{pmatrix} X_{m+1} \\ Y_{m+1} \end{pmatrix} = \begin{pmatrix} 1+\frac{b\lambda}{4} & \\ & 1-\frac{b\lambda}{4} \end{pmatrix}\begin{pmatrix} X_m \\ Y_m \end{pmatrix}. \tag{4.27}$$

Also

$$\begin{pmatrix} \frac{\partial}{\partial x} X_m \\ \frac{\partial}{\partial x} Y_m \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & \\ & -\lambda^{-1} \end{pmatrix}\begin{pmatrix} X_m \\ Y_m \end{pmatrix}. \tag{4.28}$$

The solution of Lax pair (4.27), (4.28) yields

$$X_m = \left(1+\frac{b\lambda}{4}\right)^m \exp\left(\lambda^{-1}x\right), \tag{4.29}$$

$$Y_m = \left(1-\frac{b\lambda}{4}\right)^m \exp\left(-\lambda^{-1}x\right). \tag{4.30}$$

Now, substituting the choice $\pi^{(1)}_{m,\,11} = \pi^{(1)}_{m,\,22} = X_m, \pi^{(2)}_{m,\,12} = -Y_m, \pi^{(2)}_{m,\,21} = Y_m$ and $\lambda_1 = \lambda, \bar{\lambda}_1 = -\lambda$ in Eqs. (4.6) and (4.7), we get

$$Q^{(1)}_{m,\,12} = -\frac{2\lambda^{-1}X_mY_m}{X_m^2 + Y_m^2}, \tag{4.31}$$

$$Q^{(1)}_{m,\,21} = -\frac{2\lambda^{-1}X_mY_m}{X_m^2 + Y_m^2}. \tag{4.32}$$

Expressions (4.31), (4.32) are identical and represent the one soliton solution which is presented in Fig. 8.

Similarly, the two soliton solution can be calculated by using the relation (4.16) which is depicted in Fig. 9.

In the continuum limit $b \to 0$, Eqs. (4.29), (4.30) becomes

$$X = exp\left(\lambda^{-1}x + \frac{\lambda}{4}t\right), \tag{4.33}$$

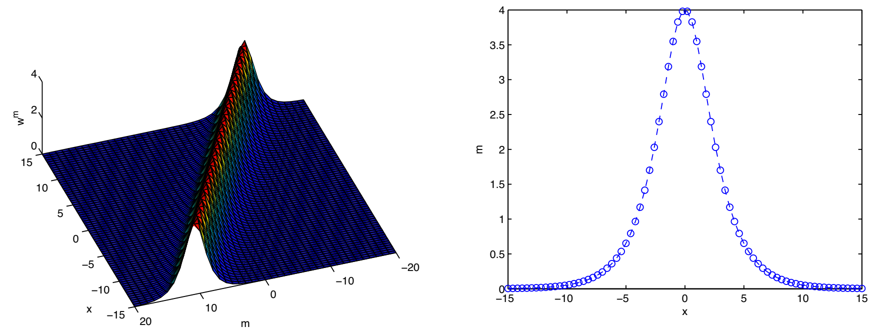


Fig. 8 Semi-Discrete one-soliton: for numerical values $\lambda = 4$, $b = 3$.

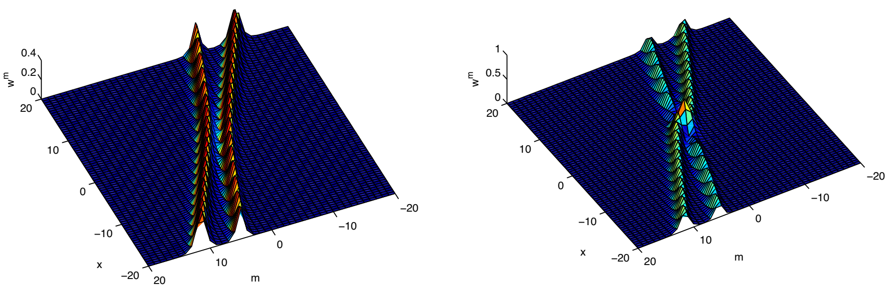


Fig. 9 Discrete two soliton: left figure showing before interaction and right presenting interaction, for numerical values $\lambda = 1.4$, $\eta = 1.9$, $b = 1$.

$$Y = exp\left(-\lambda^{-1}x - \frac{\lambda}{4}t\right), \tag{4.34}$$

which leads the solutions (4.31), (4.32) as

$$Q_{12}^{(1)} = -\frac{2\lambda^{-1}XY}{X^2 + Y^2}, \tag{4.35}$$

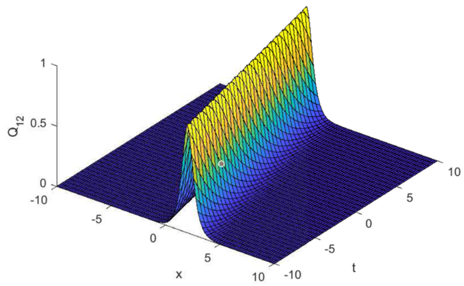
$$Q_{21}^{(1)} = -\frac{2\lambda^{-1}XY}{X^2 + Y^2}, \tag{4.36}$$

represents the one soliton solution for vanishing background $(u, v = 0)$ as a seed solution presented in Fig. 10.

Now, for nonvanishing background seed solution $(u = v = 3)$ result (4.35), (4.36) gives rogue type solutions shown in Fig. 11.

The results obtained can be used to derive conserved quantities and other useful properties. It has been shown that, each soliton able to resume its original shape after collision which shows that the collision between solitons is elastic.

Fig. 10 One soliton: for numerical values $\lambda = 0.9 + 0.0002i$.



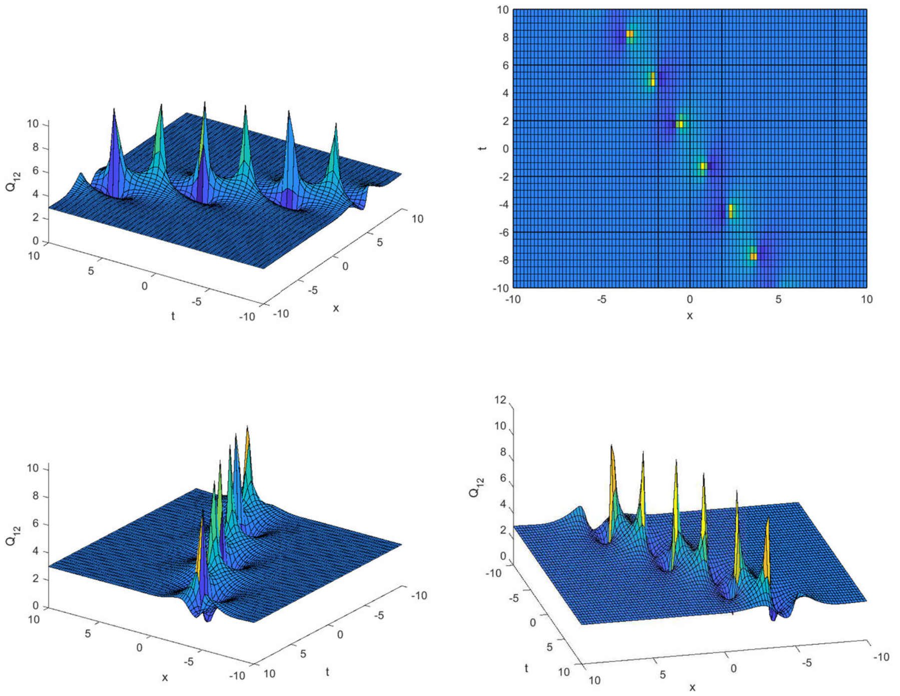


Fig. 11 Rogue solution: for numerical values $\lambda = 0.9 + 1.0002i$.

5 Conclusion

In this paper, we developed the discrete version of nAKNS and two sdnAKNS equations. We also presented that how this model reduces to the continuous equation under continuum limits. Darboux transformation was developed and applied to obtain the discrete soliton solutions. The one and two solitons (interactions) for the dnAKNS and sdnAKNS equation were plotted. Also, we discussed the nonvanishing background seed solution case and calculate the rogue type solutions. This work can also be extended in a variety of interesting directions, for example, one can study discrete and semi-discrete versions of multi-component nAKNS equation and study their multi-soliton solutions. It would also be interesting to study discrete rogue, breather and lump wave solutions for the discrete and two semi-discrete versions of negative order AKNS equation.

Data availability No data was used for the research described in the article.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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