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Investigation of a new similarity solution for the (2+1)-dimensional Schwarzian Korteweg–de Vries equation

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We study the (2+1)-dimensional Schwarzian Korteweg–de Vries equation (SKdV). The explored solutions describe new Lump soliton colliding with visible soliton, an interaction between multi-soliton waves with one soliton, multi peaks of waves moving in a curved path, two hyperbolic waves moving together without interaction and some of periodic waves. We examine the commutative product between multi unknown Lie infinitesimals for the (2+1)-dimensional (SKdV) equation, and this study result in some new Lie vectors. The commutative product generates a system of nonlinear ODEs which had been solved manually. Through two

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stages of Lie symmetry reduction, SKdV equation is reduced to non-solvable nonlinear ODEs using various combinations of optimal Lie vectors. Using the Riccati–Bernoulli sub-ODE and Integration methods, we investigate new analytical solutions for these ODEs. Back substituting for the original variables generates new solutions for SKdV. Some selected solutions are illustrated through three-dimensional plots.

Keywords: Lie symmetries; invariant solutions; commutative product; (2+1)-dimensional Schwarzian Korteweg–de Vries equation.

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1. Introduction

The differential equations play important role in medical, fluid, signal processing and engineering applications.[1–6] Here, we consider the Schwarzian Korteweg–de Vries (SKdV) equation in (2+1) dimensions as follows:

$$u_t + \frac{1}{4}u_{xxz} - \frac{u_x u_{xz}}{2u} - \frac{u_{xx}u_z}{4u} + \frac{u_x^2 u_z}{2u^2} - \frac{u_x}{8}\partial_x^{-1}\left(\frac{u_x^2}{u^2}\right)_z = 0. \tag{1}$$

In Ref. 7, Toda and Yu derived Eq. (1) from a well-known higher-dimensional manner to the Lax pair for the SKdV equation. Here, ∂_x^{-1} is the integration w.r.t. (x).

Following the Möbius transformations in Refs. 8 and 9, we will transfer the (2+1) SKdV to nonlinear system of PDEs as follows:

$$\begin{gathered} 4w^2v_x - wvw_x + w^2w_{xxz} - ww_{xx}w_z - 3ww_xw_{xz} + 3w_x^2w_z - 4w^4w_z = 0, \\ w_t - v_x = 0. \end{gathered} \tag{2}$$

The Korteweg–de Vries (KdV) equation.[10] has an important role in the physical and optical fields. So, many researchers are interested in KdV equations and its extensions and originated from them. Furthermore, we can say the SKdV equation is the most essential equation.

In Ref. 9, the authors used Miura transformation and the connection between KdV and SKdV equation to explore some of the solitary waves. Ramìrez *et al.*[8] applied the classical Lie method to obtain reductions with optimal system of vectors to reduce the SKdV equation. Through Lax Pair of (1), the authors in Ref. 11 derived the Darboux transformation and then they generated multi soliton solution for (1). The extended simplest equation method, the sech-tanh method, the Adomian decomposition method and the cubic spline scheme were applied in Ref. 12 to explore analytical, numerical solutions for (1). Li[13] applied the projective equation method which satisfies the Riccati equation to generate rouge wave-type and cross-like fractal soliton solutions. Some of the Isochronous Solutions were presented in Ref. 14 for the modified version of (1). Some numerical solutions were obtained in Ref. 15 using Adomian and B-spline methods. Lax pairs equations with Darboux transformation were applied in Ref. 16 to obtain some explicit solutions. Miscellaneous solutions were obtained in Ref. 17 as rational soliton, hyperbolic function and bright soliton solutions.

In this paper, we consider the SKdV equation and based on Lie algebra,[18–24] system (2) has four unknown vectors as determinate in Ref. 25, so, we optimize Lie vectors containing arbitrary functions of time through a commutative product for various values of functions. Through one or two stages of reduction, some ODEs that have no quadrature are solved using two methods: the Riccati–Bernoulli sub-ODE.[26] and the integrating factors as in Ref. 19.

The rest of the paper is organized as follows. In Sec. 2, we investigate the unknown subalgebras for system (2) We explore new solutions for SKdV system in Sec. 3 using the new Lie vectors that we obtained in the previous section. In Sec. 4, we analyze our solutions based on the physical meaning. In Sec. 5, we end the work with the conclusion.

2. Investigation of Lie Symmetry Generators of (2)

The system (1) admits four Lie vectors. Each of these vectors has arbitrary functions in time. Through the commutative product between the resulted vectors, we investigate new infinitesimals for SKdV system.

$$\begin{cases} X_1 = f_1(t)\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial z} - (f_1'(t)w)\dfrac{\partial}{\partial v}, \\ X_2 = f_2(t)\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial t} - (f_2'(t)w)\dfrac{\partial}{\partial v}, \\ X_3 = \left(f_3(t) - \dfrac{x}{2}\right)\dfrac{\partial}{\partial x} + z\dfrac{\partial}{\partial z} + \dfrac{w}{2}\dfrac{\partial}{\partial w} - (f_3'(t)w)\dfrac{\partial}{\partial v}, \\ X_4 = \left(f_4(t) + \dfrac{x}{2}\right)\dfrac{\partial}{\partial x} + t\dfrac{\partial}{\partial t} - \dfrac{w}{2}\dfrac{\partial}{\partial w} - \left(v + f_4'(t)w\right)\dfrac{\partial}{\partial v}. \end{cases} \tag{3}$$

There are an infinite number of possibilities for these vectors due to the existence of the arbitrary functions $f_i(t), i = 1\ldots 4$. We derive optimal values for these functions first by evaluating the commutative product of these infinitesimals as listed in Table 1.

Simplifying Table 1 by the commutative product properties generates a nonlinear system of ODEs as follows:

$$\begin{cases} \dfrac{f_1}{2} + f_1' - tf_1' = 0, \\ f_4' - tf_2' - \dfrac{f_2}{2} = 0, \\ \dfrac{f_3}{2} - tf_3' + \dfrac{f_2}{2} - f_3' + \dfrac{f_4}{2} = 0. \end{cases} \tag{4}$$

Solving this system of differential equations manually and the assumption of some values results in

$$f_1 = \sqrt{t-1}, \quad f_2 = \frac{1}{\sqrt{t}}, \quad f_3 = \sqrt{t+1} + \sqrt{t} + c_1, \quad f_4 = c_1. \tag{5}$$

Table 1. Commutator table.

	X_1	X_2	X_3	X_4
X_1	0	$-f_1'\frac{\partial}{\partial x} + f_1''w\frac{\partial}{\partial v}$	$-\frac{f_1}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial z} + \frac{f_1'}{2}w\frac{\partial}{\partial v}$	$\left(\frac{f_1}{2} - tf_1'\right)\frac{\partial}{\partial x} + \left(f_1'w + tf_1''w - \frac{f_1'}{2}w\right)\frac{\partial}{\partial v}$
X_2	$f_1'\frac{\partial}{\partial x} - f_1''w\frac{\partial}{\partial v}$	0	$(-\frac{f_1}{2} + f_3')\frac{\partial}{\partial x} + (-f_3''w + \frac{f_2'}{2})\frac{\partial}{\partial v}$	$\left(\frac{f_2}{2} + f_4' - tf_2'\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ $+\left(-f_4''w + f_2'w + tf_2''w - \frac{f_2'}{2}w\right)\frac{\partial}{\partial v}$
X_3	$\frac{f_1}{2}\frac{\partial}{\partial x} - \frac{\partial}{\partial z} - \frac{f_1'}{2}w\frac{\partial}{\partial v}$	$-\left(-\frac{f_1}{2} + f_3'\right)\frac{\partial}{\partial x} - (-f_3''w + \frac{f_2'}{2})\frac{\partial}{\partial v}$	0	$(f_3 + f_4 - tf_3')\frac{\partial}{\partial x} + (f_3'w + tf_3''w - \frac{f_3'}{2}w)\frac{\partial}{\partial v}$
X_4	$-\left(\frac{f_1}{2} - tf_1'\right)\frac{\partial}{\partial x}$ $-\left(f_1'w + tf_1''w - \frac{f_1'}{2}w\right)\frac{\partial}{\partial v}$	$-\left(\frac{f_2}{2} + f_4' - tf_2'\right)\frac{\partial}{\partial x} - \frac{\partial}{\partial t}$ $-\left(-f_4''w + f_2'w + tf_2''w - \frac{f_1'}{2}w\right)\frac{\partial}{\partial v}$	$-(f_3 + f_4 - tf_3')\frac{\partial}{\partial x}$ $-(f_3'w + tf_3''w - \frac{f_3'}{2}w)\frac{\partial}{\partial v}$	0

Table 2. Commutator table after optimization.

	X_1	X_2	X_3	X_4
X_1	0	$-zX_1 - tX_2 + X_3 + X_4$	X_1	$-zX_1 - tX_2 + X_3 + X_4$
X_2	$zX_1 + tX_2 - X_3 - X_4$	0	$-zX_1 - tX_2 + X_3 + X_4$	X_2
X_3	$-X_1$	$zX_1 + tX_2 - X_3 - X_4$	0	$-zX_1 - tX_2 + X_3 + X_4$
X_4	$zX_1 + tX_2 - X_3 - X_4$	$-X_2$	$zX_1 + tX_2 - X_3 - X_4$	0

Substituting from (5) in (3), we explore the four unknown Lie infinitesimals then simplifies Table 1 to an optimized form described in Table 2.

$$\begin{cases} X_1 = \sqrt{t-1}\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial z} - \left(\dfrac{w}{2\sqrt{t-1}}\right)\dfrac{\partial}{\partial v}, \\ X_2 = \dfrac{1}{\sqrt{t}}\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial t} + \left(\dfrac{1}{2t^{\frac{3}{2}}}w\right)\dfrac{\partial}{\partial v}, \\ X_3 = \left(\sqrt{t+1} + \sqrt{t} + c_1 - \dfrac{x}{2}\right)\dfrac{\partial}{\partial x} + z\dfrac{\partial}{\partial z} + \dfrac{w}{2}\dfrac{\partial}{\partial w} - \left(\dfrac{w}{2\sqrt{t+1}} + \dfrac{w}{2\sqrt{t}}\right)\dfrac{\partial}{\partial v}, \\ X_4 = \left(c_1 + \dfrac{x}{2}\right)\dfrac{\partial}{\partial x} + t\dfrac{\partial}{\partial t} - \dfrac{w}{2}\dfrac{\partial}{\partial w} - (v)\dfrac{\partial}{\partial v}. \end{cases} \tag{6}$$

3. Reduction of the Independent Variables in SKdV System

3.1. *Subalgebra X_1 in Eq. (6)*

$$X_1{=}\sqrt{t-1}\frac{\partial}{\partial x} + \frac{\partial}{\partial z} - \left(\frac{w}{2\sqrt{t-1}}\right)\frac{\partial}{\partial v}. \tag{7}$$

So, the characteristic equation will be

$$\frac{dx}{\sqrt{t-1}} = \frac{dz}{1} = \frac{dv}{-\left(\frac{w}{2\sqrt{t-1}}\right)}. \tag{8}$$

Solving this equation leads to

$$w(x,z,t) = F(r,s), v(x,z,t) = \frac{(2(t-1)K(r,s) - xw(x,z,t))}{2(t-1)}, \tag{9}$$

where $F(r,s)$ and $K(r,s)$ are new independent variables and

$$r = t, \quad s = \frac{z\sqrt{t-1} - x}{\sqrt{t-1}}. \tag{10}$$

Using the new similarity and the dependent variables, system (2) will be reduced to

$$\sqrt{r-1}F_r + \frac{1}{2}\frac{F}{\sqrt{r-1}} + K_s = 0, \tag{11}$$

$$\begin{aligned} -\frac{F^2 F_{sss}\sqrt{r-1}}{4} + \left(F F_{ss} F_s - \frac{3F_s^3}{4} + F^4(r-1)F_s + \frac{F_s^3}{2}\right) \\ \sqrt{r-1} + F(FF_s - KF_s)(r-1) = 0. \end{aligned} \tag{12}$$

Separating the function $K(r,s)$ from Eq. (12) and differentiating it w.r.t s once, then substituting from (11) into (12) generates nonlinear fourth-order PDE as follows:

$$\begin{aligned} &F_{ssss}F_sF^3 - 12F_s^3F^4r + 12F_s^3F^4 + 4F_{rs}F_sF^3r - 4F_{ss}F_rF^3r \\ &- 3F_{sss}F_s^2F^2 - F_{sss}F_{ss}F^3 - 3F_s^5 + 6F_s^3F_{ss}F - 2F_s^2F^3 \\ &- 4F_{rs}F_sF^3 + 4F_{ss}F_rF^3 = 0. \end{aligned} \tag{13}$$

So, Eq. (13) admits two unknown vectors as follows:

$$\begin{aligned} V_1 &= (-1+r)\frac{\partial}{\partial r} + g_1(r)\frac{\partial}{\partial s} - \frac{F}{2}\frac{\partial}{\partial F}, \\ V_2 &= \frac{1}{\sqrt{(-1+r)}}\frac{\partial}{\partial r} + \left(\frac{-s}{2(-1+r)^{3/2}} + g_2(r)\right)\frac{\partial}{\partial s}. \end{aligned} \tag{14}$$

Through the commutative product between V_1, V_2, we get

$$\begin{aligned} V_1 \cdot V_2 = [V_1, V_2] = V_1(V_2) - V_2(V_1) = \frac{-3}{2\sqrt{(-1+r)}}\frac{\partial}{\partial r} + (-1+r) \\ \times \left(\frac{3s}{4(-1+r)^{5/2}} - \frac{g_1}{2(-1+r)^{5/2}} - \frac{g_1'}{(-1+r)^{3/2}} + g_2'\right)\frac{\partial}{\partial s} = \frac{-3}{2}V_2. \end{aligned} \tag{15}$$

Comparing the two sides in (15) results in

$$(-1+r)g_2' + \frac{3}{2}g_2 = \frac{g_1}{2(-1+r)^{3/2}} + \frac{g_1'}{(-1+r)^{1/2}}. \tag{16}$$

By solving Eq. (16), we get

$$g_2(r) = \frac{1}{(-1+r)^{3/2}}\left[g_1(r) + \int \frac{g_1(r)}{2(-1+r)}dr + C_1\right]. \tag{17}$$

Here, we will introduce some choices of g_1 as follows:

Case I

Assume $g_1 = r$, then select V_1 to reduce Eq. (13) as follows:

$$\theta_{\eta\eta\eta\eta}\theta^3\theta_\eta - \theta^3\theta_{\eta\eta}\theta_{\eta\eta\eta} - 6\theta\theta_\eta^3\theta_{\eta\eta} + 3\theta_\eta^5 + 12\theta^4\theta_\eta^3 + 4\theta^3\theta_\eta^2\theta_{\eta\eta\eta} - 2\theta_{\eta\eta}\theta^4 = 0, \tag{18}$$

where

$$\eta = -r - \ln(-1+r) + s, \theta(\eta) = \sqrt{-1+r}F(r,s) \tag{19}$$

has no analytical solution. So, we solve Eq. (18) using the Riccati–Bernoulli sub-ODE method as follows: Suppose the solution of Eq. (18) be in the form solution of Riccati equation

$$\theta'(\eta) = a\theta(\eta)^2 + b\theta(\eta) + c. \tag{20}$$

By getting the second, third and fourth derivative of θ and substituting from (20) into (18) and set the coefficients of $\theta(\eta)$ to be zero, we obtain an algebraic system in a, b, c. Solving this system, we obtain a set of solutions as follows:

$$a = \frac{\pm 2I}{\sqrt{7}}, \quad b = c = 0.$$

So

$$\theta(\eta) = \left(\frac{\mp 2I}{\sqrt{7}}(\eta + C)\right)^{-1}. \tag{21}$$

Back substituting using (19) and (21), we get

$$F(r,s) = \frac{(\frac{\mp 2I}{\sqrt{7}}(-r - \ln(-1+r) + s + C))^{-1}}{\sqrt{-1+r}}. \tag{22}$$

Using Eq. (11) we get the function $K(r,s)$; as follows:

$$K(r,s) = \frac{\frac{\mp\sqrt{7}I}{4}(+(3C - 5r + 3s)\ln(-1+r)^2 - 3(-r+s+C)\left(C - \frac{7r}{3} + s\right)\ln(-1+r) + g(r,s))}{(-r - \ln(-1+r) + s + C)^3(-1+r)}, \tag{23}$$

where

$$\begin{aligned} g(r,s) = &-\ln(-1+r)^3 - 3r^3 + (7C + 7s)r^2 - 5(s+C)^2 r + C^3 + 3C^2 s \\ &+ 3Cs^2 + s^3 + 4. \end{aligned} \tag{24}$$

Using the transformations in (10) and (9), we will get a new solution for system (2) as follows:

$$w(x,z,t) = \frac{(\frac{\mp 2I}{\sqrt{7}}(-t - \ln(-1+t) + \frac{z\sqrt{t-1}-x}{\sqrt{t-1}} + C))^{-1}}{\sqrt{-1+t}}, \tag{25}$$

$$v(x,z,t) = \frac{\frac{\mp\sqrt{7}I}{4}\left(+\left(3C - 5t + 3\frac{z\sqrt{t-1}-x}{\sqrt{t-1}}\right)\ln(-1+t)^2 - 3\left(-t + \frac{z\sqrt{t-1}-x}{\sqrt{t-1}} + C\right)\left(C - \frac{7t}{3} + \frac{z\sqrt{t-1}-x}{\sqrt{t-1}}\right)\ln(-1+t) + G(x,z,t)\right)}{\left(-t - \ln(-1+t) + \frac{z\sqrt{t-1}-x}{\sqrt{t-1}} + C\right)^3(-1+t)}, \tag{26}$$

where

$$\begin{aligned} G(x,z,t) = &-\ln(-1+t)^3 - 3t^3 + \left(7C + 7\frac{z\sqrt{t-1}-x}{\sqrt{t-1}}\right)t^2 - 5\left(\frac{z\sqrt{t-1}-x}{\sqrt{t-1}} + C\right)^2 t \\ &+ C^3 + 3C^2\frac{z\sqrt{t-1}-x}{\sqrt{t-1}} + 3C\left(\frac{z\sqrt{t-1}-x}{\sqrt{t-1}}\right)^2 + \left(\frac{z\sqrt{t-1}-x}{\sqrt{t-1}}\right)^3 + 4. \end{aligned} \tag{27}$$

Case II. Assuming $g_1 = 1$ and solving Eq. (16),
we get $g_2 = \frac{1}{(-1+r)^{3/2}} + \frac{\ln(r-1)}{2(-1+r)^{3/2}}$, then using vector V_2 to reduce Eq. (13), we get

$$-\theta_{\eta\eta\eta}\theta^3\theta_\eta + \theta^3\theta_{\eta\eta}\theta_{\eta\eta\eta} + 3\theta_{\eta\eta\eta}\theta^2\theta_\eta^2 - 6\theta\theta_\eta^3\theta_{\eta\eta} + 3\theta_\eta^5 + 12\theta^4\theta_\eta^3 = 0, \tag{28}$$

where

$$\eta = s\sqrt{r-1} - \ln(r-1)\sqrt{r-1}\theta, (\eta) = F(r,s). \tag{29}$$

This equation has no analytical solution, so we try to solve it using the integrating factors as follows.

- **Integration method**

  First, multiply Eq. (28) by $\frac{1}{\theta^3\theta_\eta{}^2}$ then integrate it once with respect to η and set the integration constant equal to zero to obtain

$$\theta_{\eta\eta\eta} = -\frac{\theta_\eta(-12\theta^4 + 3\theta_\eta^2 - 6\theta\theta_{\eta\eta})}{2\theta^2}. \tag{30}$$

  For more solutions, we multiply Eq. (28) by $\frac{(\theta_\eta - 2\theta^2)(\theta_\eta + 2\theta^2)}{\theta^5\theta_\eta{}^2}$ then integrate it once to obtain

$$\theta_{\eta\eta\eta} = -\frac{\theta_\eta(48\theta^8 + 80\theta_{\eta\eta}\theta^5 - 88\theta^4\theta_\eta^2 - 4\theta^2\theta_{\eta\eta}^2 - 4\theta\theta_{\eta\eta}\theta_\eta^2 + 3\theta_\eta^4)}{4\theta^2(4\theta^4 - \theta_\eta^2)}. \tag{31}$$

  Equating Eq. (30) to Eq. (31) results in

$$4\theta^4 + 3\theta_\eta^2 - 2\theta\theta_{\eta\eta} = 0, \tag{32}$$

  which has an analytical solution

$$\theta(\eta) = \frac{4c_1}{c_1^2c_2^2 + 2c_1^2c_2\eta + c_1^2\eta^2 - 16}. \tag{33}$$

  Using Eq. (33) and back substituting using the invariant transformations in (29), (9), (10), we get

$$F(r,s) = \frac{4c_1}{\begin{aligned} c_1^2c_2^2 + 2c_1^2c_2(s\sqrt{r-1} - \ln(r-1)\sqrt{r-1}) \\ + c_1^2(s\sqrt{r-1} - \ln(r-1)\sqrt{r-1})^2 - 16 \end{aligned}}, \tag{34}$$

$$w(x,z,t)=\frac{4c_1}{c_1^2c_2^2+2c_1^2c_2\left(\frac{z\sqrt{t-1}-x}{\sqrt{t-1}}\sqrt{t-1}-\ln(t-1)\sqrt{t-1}\right)+zc_1^2\left(\frac{z\sqrt{t-1}-x}{\sqrt{t-1}}\sqrt{t-1}-\ln(t-1)\sqrt{t-1}\right)^2-16}, \tag{35}$$

$$v(x,z,t)=\frac{\left(2(t-1)K(r,s)-x\left(\frac{4c_1}{c_1^2c_2^2+2c_1^2c_2\left(\frac{z\sqrt{t-1}-x}{\sqrt{t-1}}\sqrt{t-1}-\ln(t-1)\sqrt{t-1}\right)+zc_1^2\left(\frac{z\sqrt{t-1}-x}{\sqrt{t-1}}\sqrt{t-1}-\ln(t-1)\sqrt{t-1}\right)^2-16}\right)\right)}{2(t-1)}. \tag{36}$$

3.2. *Subalgebra X_2 in Eq. (6)*

$$X_2=\frac{1}{\sqrt{t}}\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\left(\frac{1}{2t^{\frac{3}{2}}}w\right)\frac{\partial}{\partial v}. \tag{37}$$

So, the characteristic equation will be

$$\frac{dx}{\frac{1}{\sqrt{t}}}=\frac{dt}{1}=\frac{dw}{\frac{1}{2t^{\frac{3}{2}}}w}. \tag{38}$$

Solving this equation leads to

$$w(x,z,t)=F(r,s),v(x,z,t)=\frac{\sqrt{t}K(r,s)-w(x,z,t))}{\sqrt{t}}, \tag{39}$$

where $F(r,s)$ and $K(r,s)$ are new independent variables and

$$r=-2\sqrt{t}+x, s=z. \tag{40}$$

Using the new similarity and the dependent variables, system (2) will be reduced to

$$K_r=0, \tag{41}$$

$$-4F^4F_s+F^2F_{rrs}+4F^2K_r-4FKF_r-FF_{rr}F_s-3FF_rF_{rs}+3F_r^2F_s=0. \tag{42}$$

This system has an analytical solution as follows:

$$F(r,s)=\frac{(-3c_1f_1(s)+c_2)^{1/3}}{(-3c_1r+c_2)^{1/3}},\quad K(r,s)=\frac{d}{ds}f_1(s), \tag{43}$$

where $f_1(s)$ is an arbitrary function in the variable s.

Back substitution process will explore new solutions for system (2) as follows:

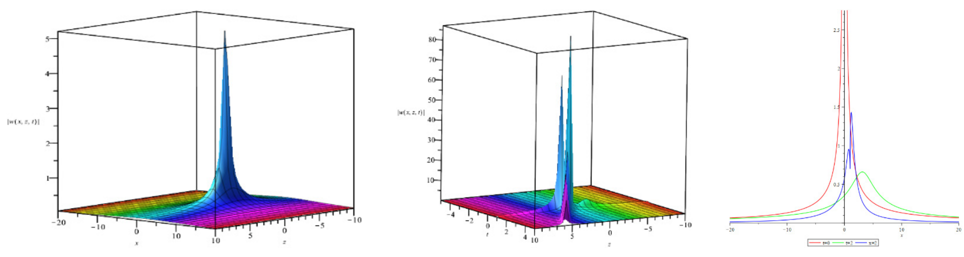
$$w(x,z,t) = \frac{(-3c_1 f_1(z) + c_2)^{1/3}}{(-3c_1(-2\sqrt{t} + x) + c_2)^{1/3}}, \tag{44}$$

$$v(x,z,t) = \frac{d}{dz} f_1(z) - \frac{(-3c_1 f_1(z) + c_2)^{1/3}}{\sqrt{t}(-3c_1(-2\sqrt{t} + x) + c_2)^{1/3}}. \tag{45}$$

4. Results and Discussion

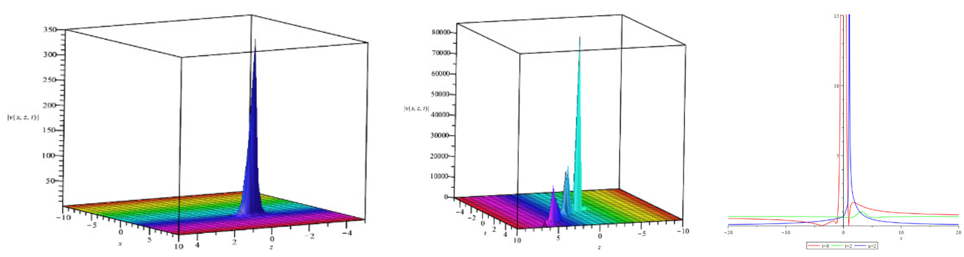
Now, we analyze our solutions from the physical meaning view. Starting with solution (24) in $(x - z)$ domain we notice that the resulted wave has a Lump wave shape but in $(t - z)$ domain, which presents an interaction between two waves as depicted in Fig. 1.

In Fig. 2, we plot solution (26), which presents in $(x - z)$ domain a Lump solution but in $(t - z)$ domain it will present multi peaks of waves move in curved path. Moreover, in Fig. 3, we plot $w(x,z,t)$ in the third row in Eq. (34) in two different domains. First in $(x - z)$ domain, which show two Lump solutions together



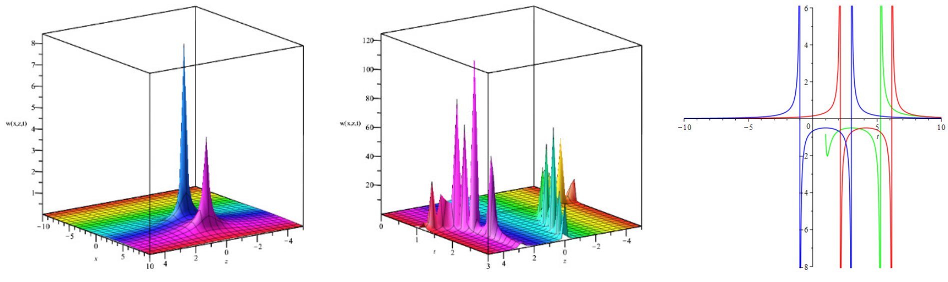
(a) (b) (c)

Fig. 1. (Color online) Three-dimensional plots for $w(x,z,t)$ for $C = 1$. (a) $t = 0$, (b) $x = 2$ and (c) $t = 0, t = 2, x = 2$.



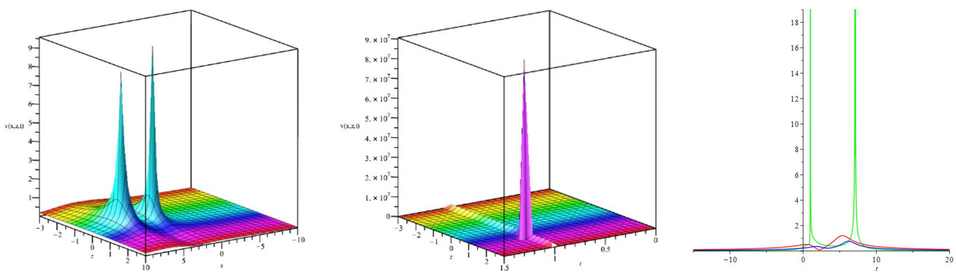
(a) (b) (c)

Fig. 2. (Color online) Three-dimensional plots for $v(x,z,t)$ for $C = 1$. (a) $t = 0$, (b) $x = 2$ and (c) $t = 0, t = 2, x = 2$.



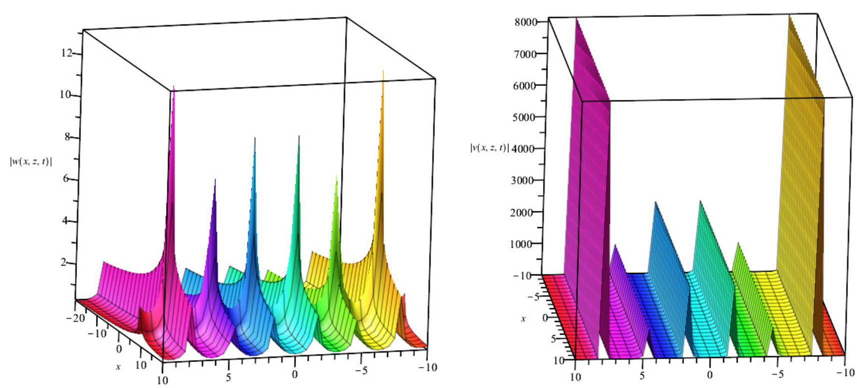
(a) (b) (c)

Fig. 3. (Color online) Three-dimensional plots for $w(x,z,t)$ in Eq. (34) for $c_1 = 2, c_2 = 1$. (a) $t = 0$, (b) $x = 2$ and (c) $t = 0, t = 0.9, x = 1$.



(a) (b) (c)

Fig. 4. (Color online) Three-dimensional plots for $v(x,z,t)$ in Eq. (35) for $c_1 = 2, c_2 = 1$. (a) $t = 0$, (b) $x = 0.1$ and (c) $t = 0, t = 0.5, x = 1$.



(a) (b)

Fig. 5. (Color online) Three-dimensional plots for $w(x,z,t), v(x,z,t)$ in Eqs. (44) and (45) for $c_1 = 1, c_2 = 1$. (a) $t = 20$, and (b) $t = 10$.

but in $(x - z)$ domain $(t - z)$ domain, we find two curved obliques of some wave's peaks.

Figure 4 presents $v(x, z, t)$ in Eq. (36), which also show two closer Lump solutions different than Fig. 3(a) in $x - z$ domain and in $t - z$ domain. Figure 4(b) represents interaction between one soliton solution moves in time axis and multi soliton waves move in curved oblique path. In solutions (44) and (45), we select $f_1(z) = 1 + \tan(z)^2$ and plot the functions for $w(x, z, t)$ and $v(x, z, t)$, which represent periodic interacted solutions and multi soliton solutions, respectively.

5. Conclusion

We study the nonlinear time-dependent Schwarzian Korteweg–de Vries equation in (2+1) dimensions and by using some logarithmic transformations, we transfer the (2+1)-dimensional SKDV equation to system of PDEs. The transformed system admits infinite numbers of infinitesimals. We determined optimal functions forms via a commutative product. Based on our method, we investigate new Lie vectors. Through two methods the Riccati-Bernoulli sub-ODE and integrating factor methods and using the new Lie vectors, we generate new solitary wave solutions. The summary of our results is as follows.

- The investigated Lie vectors are different and new compared with Ref. 8.
- Based on the new Lie vectors, we reduced system (2) to some ODEs and solved these equations using the direct, integration and Riccati–Bernoulli sub-ODE methods.
- We can analyze our solutions as follows:

  (I) Figures 1(a) and 2(a) imply one Lump soliton solution.
  (II) Figures 1(b) and 4(b) represent an interaction between multi-soliton waves with one soliton solution together and this means the physical properties do not change after the interaction.
  (III) Figure 2(b) presents multi peaks of waves move in curved path.
  (IV) Figures 3(a) and 4(a) represent two Lump soliton solutions.
  (V) Figure 4(b) represents two hyperbolic waves.
  (VI) Figures 5(a) and 5(b) represent periodic and multi-soliton solutions that their amplitude decreases with increasing the time and the wave peaks moves to lift direction.

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References

1. A. Adiga, D. Dubhashi, B. Lewis, M. Marathe and A. Vullikanti, *J. Indian Inst. Sci.* **100**(4) (2020) 793–807.
2. M. A. Attia and R. A. Shanab, *Aerosp. Sci. Technol.* **111** (2021) 106552.
3. T. Botmart, Z. Sabir, M. Ali and R. Sadat, *Informatics Med. Unlocked* **32** (2022) 101028.
4. S. de Oliveira Gonçalves *et al.*, *Sci. Total Environ.* **825** (2022) 153988.
5. M. Karimirad, C. Michailides and A. Nematbakhsh, *Offshore Mechanics: Structural and Fluid Dynamics for Recent Applications* (John Wiley & Sons, 2018).
6. A. Korobkin, E. I. Părău and J.-M. Vanden-Broeck, *The Mathematical Challenges and Modelling of Hydroelasticity* (The Royal Society Publishing, 2011), pp. 2803–2812.
7. K. Toda and S.-J. Yu, *J. Math. Phys.* **41**(7) (2000) 4747.
8. J. Ramirez, M. S. Bruzón, C. Muriel and M. L. Gandarias, *J. Phys. A: Math. Gen.* **36**(5) (2003) 1467.
9. H. Ma and Y. Bai, *Int. J. Nonlinear Sci.* **17**(1) (2014) 41.
10. B. A. Dubrovin, *Funct. Anal. Appl.* **9**(3) (1975) 215.
11. X. Li and M. Zhang, *Mod. Phys. Lett. B* **34**(25) (2020) 2050270.
12. R. A. Attia, S. H. Alfalqi, J. F. Alzaidi, M. A. Khater and D. Lu, *Complexity* **2020** (2020) 1.
13. Z. Li, *Therm. Sci.* **22**(4) (2018) 1781.
14. F. Calogero and M. Mariani, *Phys. At. Nucl.* **68**(10) (2005) 1646.
15. M. Şenol, *Mod. Phys. Lett. B* **38**(32) (2024) 2450475.
16. X. Li and M. Zhang, *Mod. Phys. Lett. B* **34**(25) (2020) 2050270.
17. F. Shehzad, A. Seadawy, S. Ahmed and S. Rizvi, *Mod. Phys. Lett. B* **38**(26) (2024) 2450255.
18. R. Saleh, M. Kassem and S. Mabrouk, *Math. Methods Appl. Sci.* **40**(16) (2017) 5851.
19. R. Sadat and M. Kassem, *Math. Comput. Appl.* **23**(1) (2018) 15.
20. G. Baumann, *Symmetry Analysis of Differential Equations with Mathematica®* (Springer Science & Business Media, 2013).
21. G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations*, Vol. 13 (Springer Science & Business Media, 2012).
22. V. Jadaun and S. Kumar, *Int. J. Geom. Methods Mod. Phys.* **15**(8) (2018) 1850125.
23. R. Sadat and M. M. Kassem, *Int. J. Appl. Comput. Math.* **5** (2019) 28.
24. R. Sadat, M. Kassem and W.-X. Ma, *Mod. Phys. Lett. B* **33**(20) (2019) 1950229.
25. M. A. Abdulwahhab, *Comput. Math. Appl.* **75**(12) (2018) 4300.
26. X.-F. Yang, Z.-C. Deng and Y. Wei, *Adv. Differ. Equ.* **2015**(1) (2015) 1.