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Lie symmetry analysis and invariant solutions for (2+1) dimensional Bogoyavlensky-Konopelchenko equation with variable-coefficient in wave propagation

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ABSTRACT

This work aims to present nonlinear models that arise in ocean engineering. There are many models of ocean waves that are present in nature. In shallow water, the linearization of the equations requires critical conditions on wave capacity than it make in deep water, and the strong nonlinear belongings are spotted. We use Lie symmetry analysis to obtain different types of soliton solutions like one, two, and three-soliton solutions in a (2 + 1) dimensional variable-coefficient Bogoyavlensky Konopelchenko (VCBK) equation that describes the interaction of a Riemann wave reproducing along the y-axis and a long wave reproducing along the x-axis in engineering and science. We use the Lie symmetry analysis then the integrating factor method to obtain new solutions of the VCBK equation. To demonstrate the physical meaning of the solutions obtained by the presented techniques, the graphical performance has been demonstrated with some values. The presented equation has fewer dimensions and is reduced to ordinary differential equations using the Lie symmetry technique.

1. Introduction

The Bogoyavlensky Konopelchenko models have many valuable applications in physical systems, physiology, mathematical science problems, engineering areas and the chemical reaction of species in the porous catalyst particle. The historical Bogoyavlensky Konopelchenko models has great significance and importance due to the singularity at the origin. Investigation of the nonlinear partial differential equations (NLPDEs) has involved the attention of the author's sets because they represent pertinent nonlinear events in several Engineering and Science [1–10]. The study of the symmetry approach and other correlating properties of the NLPDEs is constantly expanding its dimensions in various fields of research and technology. The primary role of the symmetry approach is to reduce the dimensions of NLPDEs to an ordinary differential equation. The Lie symmetry approach is the widest approach for the authors [11–15] in mathematical Engineering and sciences. The main target, of the present study deals with an investigation of the (2 + 1) dimensional Bogoyavlensky Konopelchenko (BK) equation [16–19];

$$w_t + \alpha w_{xxx} + \beta w_{xxy} + 6\alpha w w_x + 4\beta w w_y + 4\beta w_x \partial_x^{-1} w_y = 0. \quad (1)$$

where ∂_x^{-1} is the integral concerning x and α and β are arbitrary constants. Substituting $w(x,y,t) = v_x$ to get the following simple equation;

$$v_{xt} + \alpha v_{xxxx} + \beta v_{xxxy} + 6\alpha v_x v_{xx} + 4\beta v_x v_{xy} + 4\beta v_y v_{xx} = 0. \quad (2)$$

Many kinds of research such as in [16] depicted that $v(x,y,t)$ describes the interaction of a Riemann wave propagating along the y-axis and x-axis. From our fast review, there are many presented solutions for (2). Authors in [16,17], apply the Lie symmetry analysis to present some generators through the prolongation theorem and the geometric approach, respectively, to reduce (2) to ODEs and generate exact solutions. Also, the authors in [24], apply Lie point symmetry to (2) and investigate its conservation

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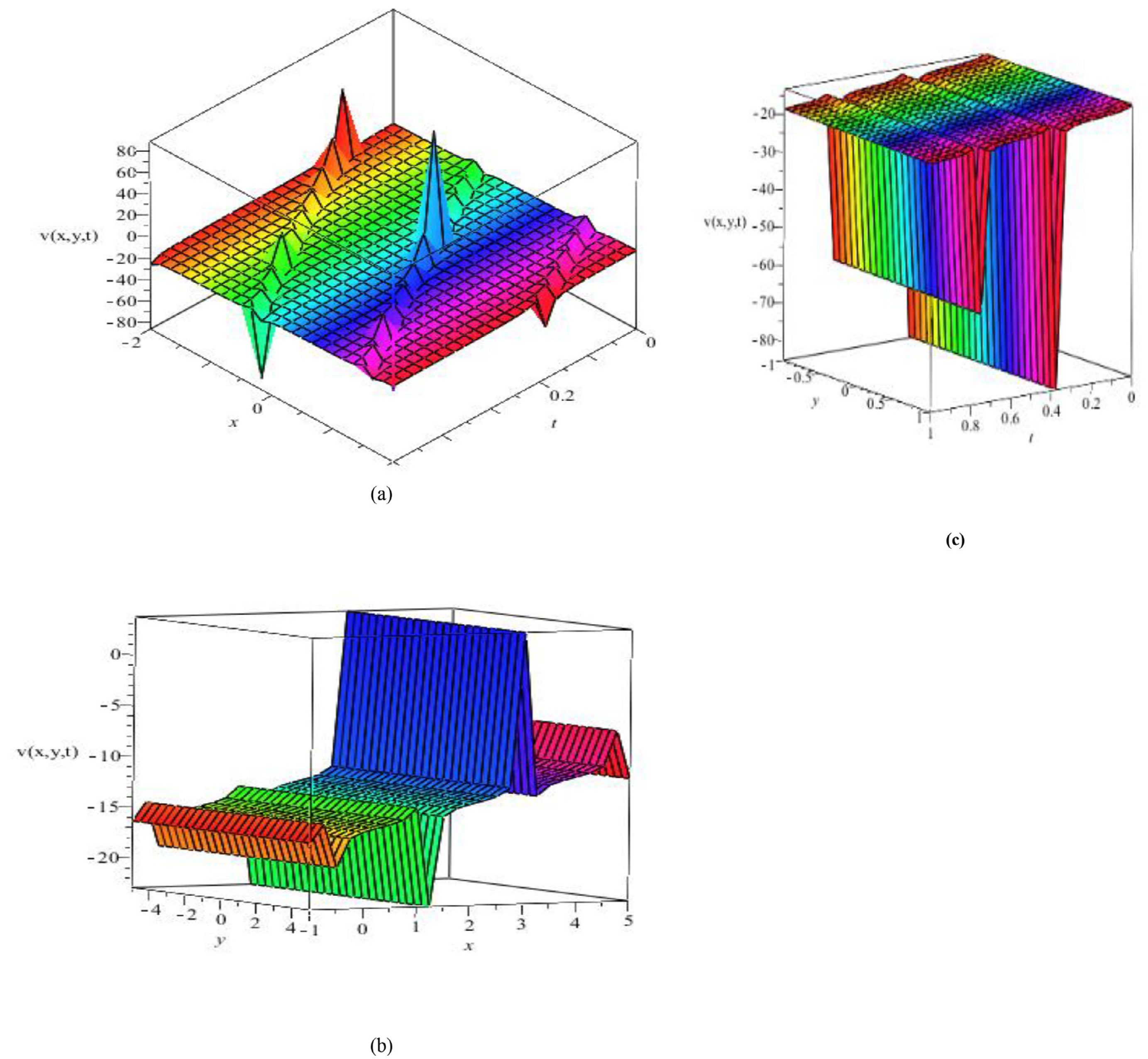

Fig. 1. Three dimensional plots for $v(x, y, t)$ at $c = 1$, $c_2 = 0$, $c_4 = 1$, (a) $t = 0$, (b) $y = 0$ and (c) $x = 1$.

laws. Some Lump solutions and interacted soliton solutions had been obtained using the Hirota bilinear method in [20–25]. On the other hand, some authors studied the VCBK equation [24];

$$v_{xt} + \alpha(t)v_{xxxx} + \beta(t)v_{xxxy} + \gamma(t)v_x v_{xx} + \delta(t)v_x v_{xy} + \theta(t)v_y v_{xx} = 0. \quad (3)$$

Where $\alpha(t)$, $\beta(t), \gamma(t), \delta(t)$ and $\theta(t)$ are real functions in time and $\gamma(t) = 6\alpha(t)$ and $\delta(t) = \theta(t) = 4\beta(t)$.

In [1], they determine the parallel Bell polynomial and Hirota technique to research one and two solitons for (1.3). The collaboration between one soliton, either two soliton arrangements on the off chance that the functions are steady or variable is explored in [12] utilizing the summed up brought together technique. Many investigations produce different arrangements of (1.3) utilizing various techniques as the reverse dispersing strategy with the guide of Lax sets administrators and the $(\frac{G'}{G})$ expansion method [13,14].

In this article, we consider the (VCBK) equation and in view of Lie approach [23-20,14,15,2,5], Eq. (1.3) have four obscure vectors as determinate in [1], in this way, we improve Lie vectors containing self-assertive elements of time through a commutative item for different upsides of capacities and furthermore apply similar system for steady coefficients condition (1.2). Through a couple of phases of decreases, a few ODEs that had no quadrature are settled utilizing the coordinating components as in [18].

The organizing of this paper is cleared as follows: In Section 2, we demonstrate three events for Eq. (3) using Lie symmetry approach by demonstrating a various values for $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ and $\theta(t)$. Abbreviate the NLPDEs to find the resulted ODEs are demonstrated using Lie symmetry then new solitons and other solutions for the VCBK equation are demonstrated in Section 3. In Section 4, the aim of the work is demonstrated at the end of the paper.

Table 1
Commutator table.

	$\boldsymbol{X}_1$	$\boldsymbol{X}_2$	$\boldsymbol{X}_3$	$\boldsymbol{X}_4$
$\boldsymbol{X}_1$	0	$-f_1'\ e^{-t}\frac{\partial}{\partial x}+a_1\frac{\partial}{\partial \nu}$	$a_2\frac{\partial}{\partial \nu}$	$a_3\frac{\partial}{\partial x}+\frac{1}{3}\frac{\partial}{\partial y}+a_4\frac{\partial}{\partial \nu}$
$\boldsymbol{X}_2$	$f_1'\ e^{-t}\frac{\partial}{\partial x}-a_1\frac{\partial}{\partial \nu}$	0	$(e^{-t}f_5'+\frac{1}{4}f_3)\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+a_5\frac{\partial}{\partial \nu}$	$(\frac{1}{3}f_3+e^{-t}f_7'-f_3')\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+a_6\frac{\partial}{\partial \nu}$
$\boldsymbol{X}_3$	$-a_2\frac{\partial}{\partial \nu}$	$(-e^{-t}f_5'-\frac{1}{4}f_3)\frac{\partial}{\partial x}-\frac{\partial}{\partial y}-a_5\frac{\partial}{\partial \nu}$	0	$(\frac{1}{3}f_5-f_5')\frac{\partial}{\partial x}-\frac{2}{3}\frac{\partial}{\partial y}+a_7\frac{\partial}{\partial \nu}$
$\boldsymbol{X}_4$	$-a_3\frac{\partial}{\partial x}-\frac{1}{3}\frac{\partial}{\partial y}-a_4\frac{\partial}{\partial \nu}$	$-(\frac{1}{3}f_3+e^{-t}f_7'-f_3')\frac{\partial}{\partial x}-\frac{\partial}{\partial t}-a_6\frac{\partial}{\partial \nu}$	$(-\frac{1}{3}f_5+f_5')\frac{\partial}{\partial x}+\frac{2}{3}\frac{\partial}{\partial y}-a_7\frac{\partial}{\partial \nu}$	0

2. Lie symmetry approach of VCBK equation

The objective of this part is to present three cases of Lie infinitesimals for the **VCBK** equation using various values of the real function $(\alpha,\ \delta, \beta, \nu, \theta)$. By using various values of the real function, the four vectors $X_1,\ X_2, X_3, X_4$ for the Bogoyavlensky-Konopelchenko equation can be established as

2.1. Case I

Suppose $\alpha(t)=\delta(t)=\frac{g(t)}{2},\ \beta(t)=\frac{g(t)}{4}$ and $\gamma(t)=\theta(t)=g(t)$. Eq. (3) admits the following Lie infinitesimals;

$$\begin{cases} X_1=f_1(t)\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\left(\frac{1}{g(t)}yf_1'(t)+f_2(t)\right)\frac{\partial}{\partial \nu}, \\ X_2=f_3(t)\frac{\partial}{\partial x}+\frac{1}{g(t)}\frac{\partial}{\partial t}+\left(\frac{1}{g(t)}yf_3'(t)+f_4(t)\right)\frac{\partial}{\partial \nu}, \\ X_3=f_5(t)\frac{\partial}{\partial x}+g(t)\frac{\partial}{\partial y}+\left(\frac{1}{4g(t)}yf_5'(t)+f_6(t)+\frac{1}{4}x-\frac{3}{8}y\right)\frac{\partial}{\partial \nu}, \\ X_4=\left(\frac{1}{3}x+f_7(t)\right)\frac{\partial}{\partial x}+\frac{1}{3}y\frac{\partial}{\partial y}+\frac{\partial}{\partial t}+\left(\frac{1}{4g(t)}yf_7'(t)+f_8(t)-\frac{1}{3}\nu\right)\frac{\partial}{\partial \nu} \end{cases} \tag{4}$$

where $g(t)=e^t$, there is an infinite number of possibilities for these vectors due to the existence of the arbitrary functions $f_i(t),\ i=1\ldots8$. We derive optimal values for these functions first, by evaluating the commutative product of these infinitesimals as listed in Table 1. where;

$$\begin{aligned} &a_1=e^{-t}\left(\frac{f_3'}{4}+\frac{ye^{-t}f_1'}{4}-\frac{ye^{-t}f_1''}{4}-f_2'\right),\ a_2=\frac{1}{4}f_1+\frac{e^{-t}}{4}f_5'-\frac{3}{8}-\frac{1}{4}f_1', \\ &a_3=\frac{1}{3}f_1-f_1',\ a_4=\frac{e^{-t}f_7'}{4}+\frac{ye^{-t}f_1'}{12}-\frac{1}{3}f_2-\frac{1}{4}ye^{-t}f_1''-f_2', \\ &a_5=e^{-t}\left(\frac{-1}{4}ye^{-t}f_5'+\frac{1}{4}ye^{-t}f_5''+f_6'\right)-\frac{1}{4}f_3', \\ &a_6=e^{-t}\left(\frac{-1}{4}ye^{-t}f_7'+\frac{1}{4}ye^{-t}f_7''+f_8'+\frac{yf_3'}{6}-\frac{yf_3''}{4}\right)-f_4'-\frac{1}{3}f_4 \\ &a_7=\frac{1}{4}f_7'+\frac{1}{12}ye^{-t}f_5'-\frac{1}{6}x-\frac{1}{4}y-\frac{2y}{t^2}f_5'-\frac{1}{3}f_6-\frac{1}{4}f_7-\frac{1}{4}ye^{-t}f_5''-f_6' \end{aligned} \tag{5}$$

Simplifying Table 1 by setting the values for a's, generates a nonlinear system of ODEs;

$$\begin{aligned} &f_1'=0,\ -\frac{1}{4}\ e^{-t}f_7'+\frac{2}{3}f_2+f_2'=0, \\ &e^{-t}f_5'+\frac{1}{4}f_3=0,\ e^{-t}f_6'=f_2 \\ &-\frac{1}{3}f_6-\frac{1}{4}f_7'+\frac{1}{4}f_7+f_6'=0, \\ &-\frac{2}{3}f_3+e^{-t}f_7'-f_3'=0,\ -\frac{4}{3}f_4+f_4'+e^{-t}f_8'=0, \end{aligned} \tag{6}$$

Solving this system of differential equations manually and the assumption of some values, results in;

$$\begin{aligned} &f_1=c_1, \\ &f_5=e^t,\ f_4=\frac{3}{4}+e^{\frac{4}{3}t},\ f_7=\frac{8}{3}(c_1-1)e^t \\ &f_3=4(c_1-1),\ f_6=(c_1-1)e^t+3e^{\frac{1}{3}t},\ f_2=(c_1-1)+e^{-\frac{2}{3}t},\ f_8=e^t \end{aligned} \tag{7}$$

Substituting from (7) in (4), we explore the four unknown Lie infinitesimals then simplify Table 1 to an optimized form described in Table 2.

Table 2
Commutator table after optimization.

	$\boldsymbol{X}_1$	$\boldsymbol{X}_2$	$\boldsymbol{X}_3$	$\boldsymbol{X}_4$
$\boldsymbol{X}_1$	0	$-f_1'\ e^{-t}\frac{\partial}{\partial x}+a_1\frac{\partial}{\partial \nu}$	$a_2\frac{\partial}{\partial \nu}$	$\frac{1}{3}\boldsymbol{X}_1$
$\boldsymbol{X}_2$	$f_1'\ e^{-t}\frac{\partial}{\partial x}-a_1\frac{\partial}{\partial \nu}$	0	$\boldsymbol{X}_1$	$\boldsymbol{X}_2$
$\boldsymbol{X}_3$	$-a_2\frac{\partial}{\partial \nu}$	$-\boldsymbol{X}_1$	0	$-\frac{2}{3}\boldsymbol{X}_3$
$\boldsymbol{X}_4$	$-\frac{1}{3}\boldsymbol{X}_1$	$-\boldsymbol{X}_2$	$\frac{2}{3}\boldsymbol{X}_3$	0

$$\begin{cases} X_1=c_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\left((c_1-1)+e^{-\frac{2}{3}t}\right)\frac{\partial}{\partial \nu}, \\ X_2=4(c_1-1)\frac{\partial}{\partial x}+e^{-t}\frac{\partial}{\partial t}+\left(\frac{3}{4}+e^{\frac{4}{3}t}\right)\frac{\partial}{\partial \nu}, \\ X_3=e^t\frac{\partial}{\partial x}+e^t\frac{\partial}{\partial y}+\left((c_1-1)e^t+3e^{\frac{1}{3}t}+\frac{1}{4}x-\frac{1}{8}y\right)\frac{\partial}{\partial \nu}, \\ X_4=\left(\frac{1}{3}x+\frac{8}{3}(c_1-1)e^t\right)\frac{\partial}{\partial x}+\frac{1}{3}y\frac{\partial}{\partial y}+\frac{\partial}{\partial t}+\left(\frac{2}{3}(c_1-1)y+e^t-\frac{1}{3}\nu\right)\frac{\partial}{\partial \nu}. \end{cases} \tag{8}$$

2.2. Case II

Assume $\alpha(t)=2,\ \beta(t)=-1,$ so, $\gamma(t)=12$ and $\delta(t)=\theta(t)=-4$. Eq. (3) admits the following Lie infinitesimals;

$$\begin{cases} X_1=f_1(t)\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\left(-\frac{1}{4}yf_1'(t)+f_2(t)\right)\frac{\partial}{\partial \nu}, \\ X_2=f_3(t)\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\left(-\frac{1}{4}yf_3'(t)+f_4(t)\right)\frac{\partial}{\partial \nu}, \\ X_3=f_5(t)\frac{\partial}{\partial x}+t\frac{\partial}{\partial y}+\left(-\frac{(f_5'(t)y}{4}+f_6(t)-\frac{1}{4}x-\frac{3}{4}y\right)\frac{\partial}{\partial \nu}, \\ X_4=\left(\frac{1}{3}x+f_7(t)\right)\frac{\partial}{\partial x}+\frac{1}{3}y\frac{\partial}{\partial y}+t\frac{\partial}{\partial t}+\left(-\frac{1}{4}yf_7'(t)+f_8(t)-\frac{1}{3}\nu\right)\frac{\partial}{\partial \nu}. \end{cases} \tag{9}$$

Follow the same procedure the previous cases, so;

$$\begin{aligned} &f_3=1, f_1=t^{\frac{-2}{3}},\ f_5=t,\ f_4=t^{\frac{-2}{3}},\ f_7=1\ f_8=1,\ f_6=-\frac{4}{3} \\ &\quad +t^{\frac{1}{3}},\ f_2=t^{\frac{-4}{3}}. \end{aligned} \tag{10}$$

Substituting from (10) in (9), we generate new Lie infinitesimals as in Eq. (11).

$$\begin{cases} X_1=t^{\frac{-2}{3}}\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\left(\frac{1}{6}yt^{\frac{-5}{3}}+t^{\frac{-4}{3}}\right)\frac{\partial}{\partial \nu}, \\ X_2=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\left(t^{\frac{-2}{3}}\right)\frac{\partial}{\partial \nu}, \\ X_3=t\frac{\partial}{\partial x}+t\frac{\partial}{\partial y}+\left(-\frac{y}{4}-\frac{4}{3}+t^{\frac{1}{3}}-\frac{1}{4}x-\frac{3}{4}y\right)\frac{\partial}{\partial \nu}, \\ X_4=\left(\frac{1}{3}x+1\right)\frac{\partial}{\partial x}+\frac{1}{3}y\frac{\partial}{\partial y}+t\frac{\partial}{\partial t}+\left(1-\frac{1}{3}\nu\right)\frac{\partial}{\partial \nu}. \end{cases} \tag{11}$$

3. An optimal equations of dimensional subalgebras

3.1. Case I

3.1.1. Using Lie vector X_1 in Eq. (8);

$$X_1=c_1\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\left((c_1-1)+e^{-\frac{2}{3}t}\right)\frac{\partial}{\partial \nu} \tag{12}$$

Therefore, the characteristic equation will be;

$$\frac{dx}{c_1}=\frac{dy}{1}=\frac{dt}{0}=\frac{d\nu}{\left((c_1-1)+e^{-\frac{2}{3}t}\right)} \tag{13}$$

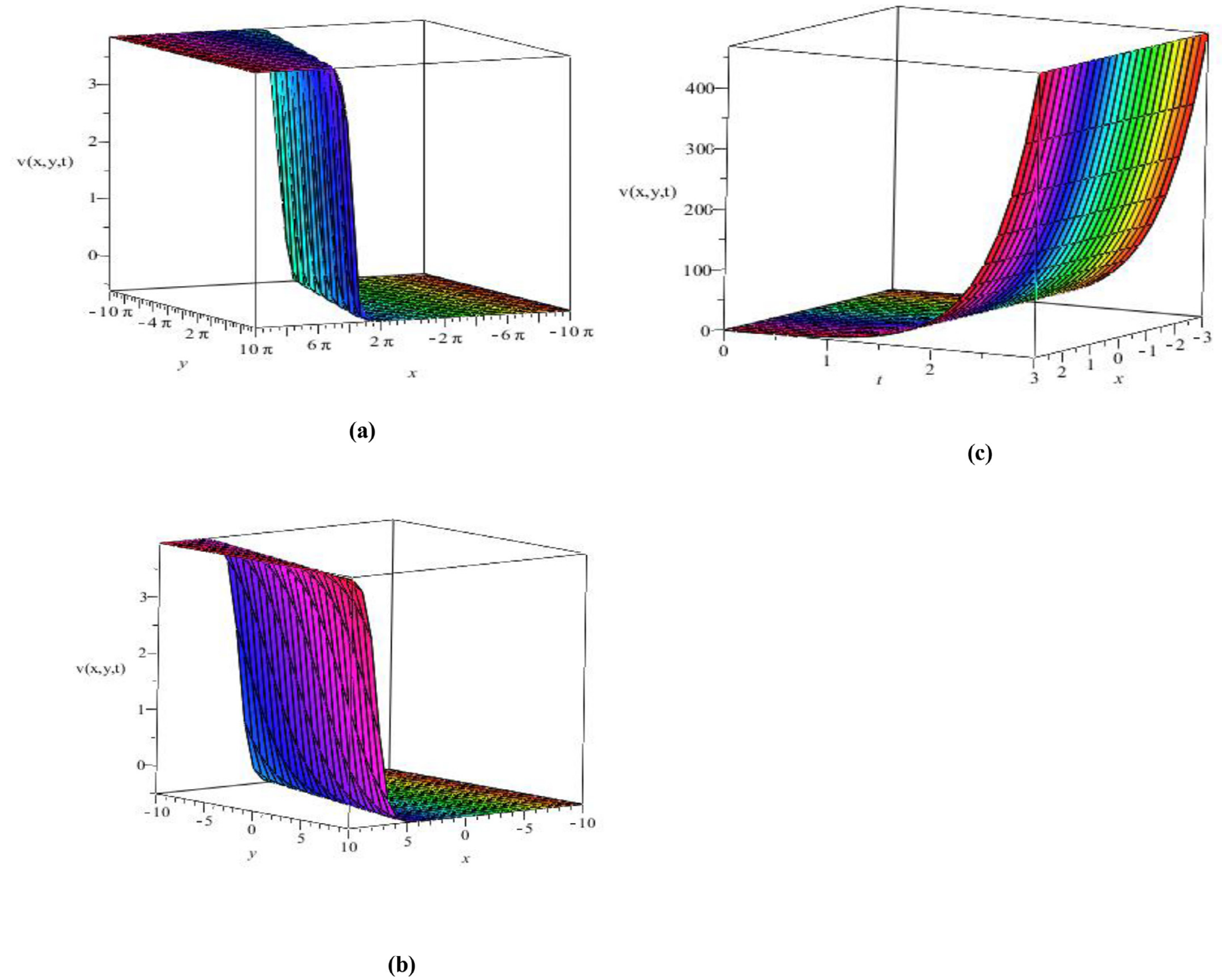

Fig. 2. Three dimensional plots for $\nu(x, y, t)$ at $c = 1,\ c_3 = 1,\ c_4 = 1$ (a) $t = 0$, (b) $t = 0.1$ and (c) $y = 0$.

Solving this equation leads to;

$$\nu(x, y, t) = F(r, s) + \left((c_1 - 1) + e^{-\frac{2}{3}t}\right)y. \tag{14}$$

where $F(r, s)$ is a new independent variable and;

$$r = x - c_1 y,\ \ s = t \tag{15}$$

Using the new similarity and the dependent variables, Eq. (3) will be reduced to;

$$F_{rs} - (c_1 - 1)e^s F_{rrrr} + 6e^s F_r F_{rr} - 8\ c_1 e^s F_r F_{rr} + 4(c_1 - 1)e^s F_{rr} + 4e^{\frac{1}{3}s} F_{rr} = 0, \tag{16}$$

Put $c_1 = 0$ in Eq. (16) then explore its generators; one of these symmetries has the form $V_8 = e^{-\frac{2}{3}s}\frac{\partial}{\partial r} + e^{-s}\frac{\partial}{\partial s} + s\frac{\partial}{\partial F}$;

Where,

$$\eta = -r + 12e^{\frac{1}{3}s}, \theta(\eta) = F(r, s) - 3e^s\left(8\ln(2) + 4\ln(3) + \frac{1}{3}s + 1\right), \tag{17}$$

Eq. (16) will reduced to fourth order nonlinear ODE that has no closed form solution;

$$\theta_{\eta\eta\eta\eta} + 4\theta_{\eta\eta} - 6\theta_{\eta\eta}\theta_\eta = 0. \tag{18}$$

Applying the integration factors method explore new solutions for (18).

Ø Integration method

Integrate Eq. (18) once with respect to η, set the integration constant equal to zero;

$$\theta_{\eta\eta\eta} = 3{\theta_\eta}^2 + 4\theta_\eta, \tag{19}$$

Secondly, multiply Eq. (18) by (θ_η) then integrate once with respect to η,

$$\theta_{\eta\eta\eta} = \frac{1}{2(\theta_\eta)}\left(4\theta_\eta^3 + \ \theta_{\eta\eta}^2 + 4\theta_\eta^2\right), \tag{20}$$

Equating Eqs. (19) to (20), results in;

$$10\theta_\eta^3 + 12\theta_\eta^2 + \theta_{\eta\eta}^2 = 0, \tag{21}$$

By solving this equation;
$\theta(\eta) = -\frac{2}{5}\sqrt{3}\tan(\sqrt{3}\eta + \sqrt{3}c) + \frac{2}{5}\sqrt{3}\tan^{-1}(\tan(\sqrt{3}\eta + \sqrt{3}c)) - \frac{6}{5}\eta + c_2$,(22) Back substitution using Eqs. (17), (15) and (14);

$$X_2 = 4\frac{\partial}{\partial x} + e^{-t}\frac{\partial}{\partial t} + \left(\frac{3}{4} + e^{\frac{4}{3}t}\right)\frac{\partial}{\partial \nu}, \tag{23}$$

This solution plotted as depicted in Fig. 1.

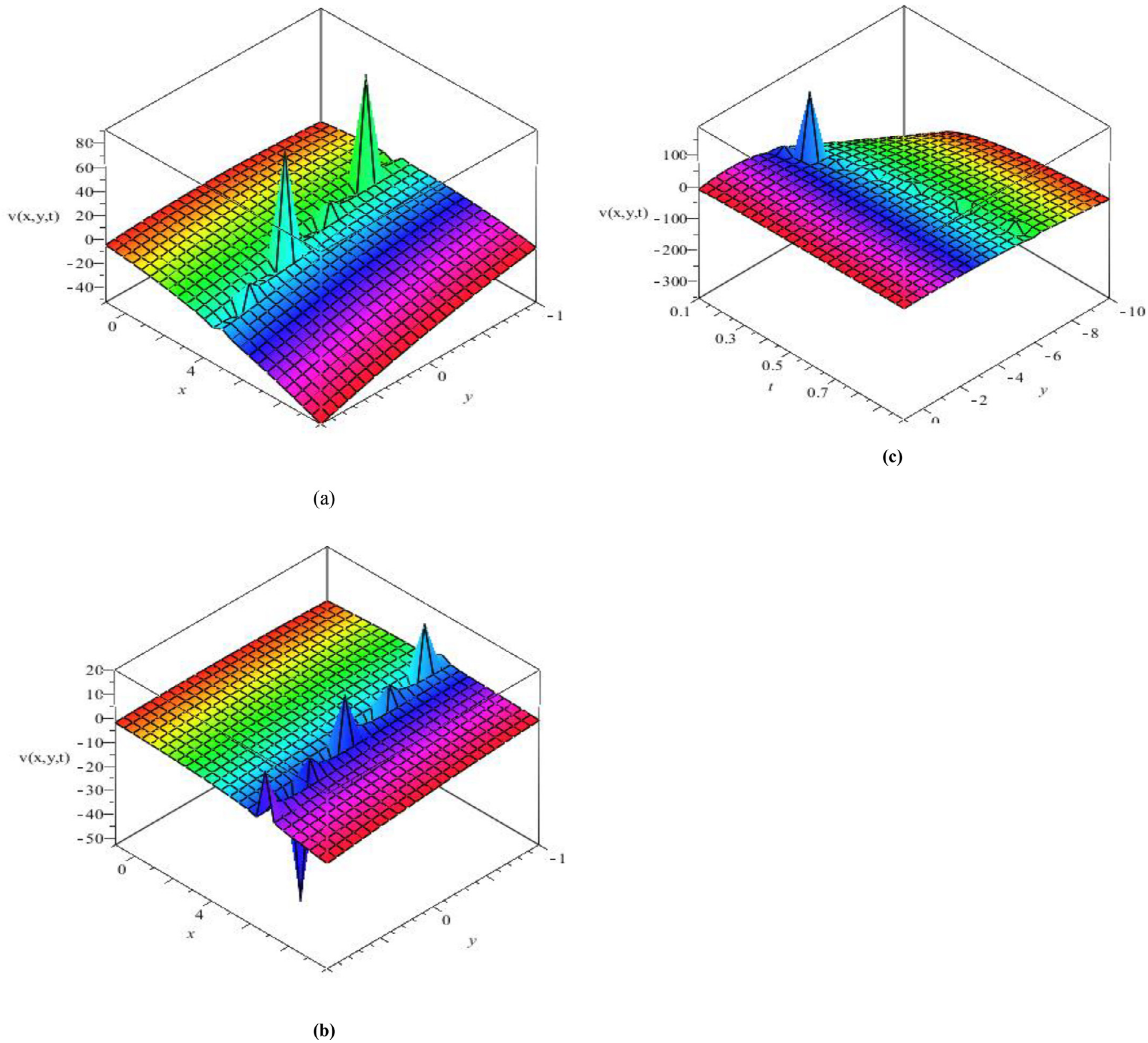

Fig. 3. Three dimensional plots for $v(x,y,t)$ for $c_1=1,\ c_2=1$ (a) $t=0.1$, (b) $t=0.5$ and (c) $x=1$.

3.1.2. Using Lie vector X_2 in Eq. (8) and set $c_1=2$;

$$\boldsymbol{X}_2 = 4\frac{\partial}{\partial x} + e^{-t}\frac{\partial}{\partial t} + \left(\frac{3}{4} + e^{\frac{4}{3}t}\right)\frac{\partial}{\partial v}, \tag{24}$$

Therefore, the characteristic equation will be;

$$\frac{dx}{4} = \frac{dy}{0} = \frac{dt}{e^{-t}} = \frac{dv}{\left(\frac{3}{4} + e^{\frac{4}{3}t}\right)}, \tag{25}$$

Solving this equation leads to;

$$v(x,y,t) = F(r,s) + \frac{3\ 8^{\frac{1}{3}}}{14}e^{\frac{7}{3}t} + \frac{3}{16}x. \tag{26}$$

where $F(r,s)$ is a new independent variable and;

$$r = y,\ \ s = 4e^{t} - x, \tag{27}$$

Using the new similarity and the dependent variables, Eq. (3) will be reduced to;

$$-23F_{ss} + F_{ssss} - 8F_{sssr} - 48F_{ss}F_{s} + 32F_{s}F_{rs} - 6F_{rs} + 32F_{r}F_{ss} = 0, \tag{28}$$

Eq. (28) has an analytic solution;

$$F(r,s) = c_4 + 87\frac{c_3\tanh\left(C + c_3 s + \frac{c_3\left(32c_2{}^2-23\right)r}{2\left(16c_3{}^2+3\right)}\right)}{16c_3{}^2 - 55}. \tag{29}$$

Using the similarity variables in (27) and (26) then we can back substitution to the original variables;

$$v(x,y,t) = c_4 + 87\frac{c_3\tanh\left(c + c_3\left(4e^{t} - x\right) + \frac{c_3\left(32c_2{}^2-23\right)y}{2\left(16c_3{}^2+3\right)}\right)}{16c_3{}^2 - 55} + \frac{3\ 8^{\frac{1}{3}}}{14}e^{\frac{7}{3}t} + \frac{3}{16}x \tag{30}$$

This result is plotted in Fig. 2

3.2. Case II

Using Lie vector X_3 in Eq. (9)

$$X_3 = t\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \left(-\frac{y}{4} - \frac{4}{3} + t^{\frac{1}{3}} - \frac{1}{4}x - \frac{3}{4}y\right)\frac{\partial}{\partial \nu}, \tag{31}$$

This vector generates the characteristic equation.

$$\frac{dx}{t} = \frac{dy}{t} = \frac{d\nu}{-\frac{y}{4} - \frac{4}{3} + t^{\frac{1}{3}} - \frac{1}{4}x - \frac{3}{4}y}, \tag{32}$$

Solving Eq. (32) produces the similarity variables;

$$\nu(x,y,t) = F(r,s) + \frac{3}{8}\frac{x^2}{t} + \left(-\frac{y}{t} + \frac{1}{t^{\frac{2}{3}}} - \frac{3}{4t}\right)x. \tag{33}$$

where $F(r,s)$ is a new independent variable and;

$$r = -x + y, \;\; s = t. \tag{34}$$

Using the new similarity and the dependent variables, Eq. (3) will be reduced to;

$$27s^{\frac{2}{3}} - 37s + 36s^{\frac{2}{3}}r + 48s^{\frac{5}{3}}F_r + 36s^{\frac{5}{3}}F_{rr} - \;48s^2F_{rr} + 48s^{\frac{5}{3}}rF_{rr} + 60s^{\frac{8}{3}}F_rF_{rr} + 3s^{\frac{8}{3}}F_{rs} - 3s^{\frac{8}{3}}F_{rrrr} = 0 \tag{35}$$

Therefore, the Eq. (35) eight Lie vectors, we choose to work with $V_7 = -2s^{\frac{-2}{3}}\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + (\frac{17}{15}\frac{r}{s^{\frac{5}{3}}} + \frac{3}{5}\frac{r}{s^2} + \frac{2}{5}\frac{r^2}{s^2} + 1)\frac{\partial}{\partial F}$,Where;

$$\eta = r + 6s^{\frac{1}{3}}, \theta(\eta) = F(r,s) + \frac{2}{5}\frac{r^2}{s} - \left(\frac{7}{10s^{\frac{2}{3}}} - \frac{3}{5s}\right)r - s + \frac{21}{5s^{\frac{1}{3}}} - \frac{9}{5s^{\frac{2}{3}}}, \tag{36}$$

Eq. (36) will reduced to;

$$\theta_{\eta\eta\eta\eta} - 20\theta_{\eta\eta}\theta_{\eta} = 0. \tag{37}$$

This condition has no scientific solution, so we attempt to address it utilizing the integrating factors as follow;

3.3. Integration method

Integrate Eq. (37) once with respect to η, set the integration constant equal to zero;

$$\theta_{\eta\eta\eta} \;= 10{\theta_\eta}^2, \tag{38}$$

Secondly, multiply Eq. (37) by (θ_η) then integrate once with respect to η,

$$\theta_{\eta\eta\eta} = \frac{1}{6(\theta_\eta)}\left(40\theta_\eta^3 + 3\;\theta_{\eta\eta}^2\right), \tag{39}$$

Equating Eqs. (38) to (39), results in;

$$-20\theta_\eta^3 + 3\theta_\eta^2 = 0, \tag{40}$$

By solving this equation;

$$\theta(\eta) = \frac{-3}{5(\eta + c_1)} + c_2, \tag{41}$$

Back substitution using Eqs. (36), (34) and (33);

$$\nu(x,y,t) = \frac{-3}{5\left(-x + y + 6t^{\frac{1}{3}} + c_1\right)} + c_2 - \frac{2}{5}\frac{(-x+y)^2}{t} + \left(\frac{7}{10t^{\frac{2}{3}}} - \frac{3}{5t}\right)(-x+y) + t - \frac{21}{5t^{\frac{1}{3}}} + \frac{9}{5t^{\frac{2}{3}}} + \frac{3}{8}\frac{x^2}{t} + \left(-\frac{y}{t} + \frac{1}{t^{\frac{2}{3}}} - \frac{3}{4t}\right)x. \tag{42}$$

The result is depicted in Fig. 3.

The resulted waves consist of some solitons by increasing the time, the amplitude of the wave is reduced and moved to the right.

4. Conclusions

The present work is to study a (2 + 1dimensional Bogoyavlensky Konopelchenko equation with variable coefficients through the Lie Symmetry approach, which results in the implementation of new exact solutions for the vcBK equation.The obtained solutions are organized, accurate, interesting and huge scheme for future authors in ruling an important solution to other nonlinear mathematical models in the fields of ocean engineering and sciences. First, the enhanced infinitesimal generators and then group classification is achieved with the assistance of the Lie symmetry group. Although, the presence of functional arbitrary elements in infinitesimal generators makes the Lie group classification a difficult and tedious task. Furthermore, the associated subalgebras are being utilized to reduce the considered equation into lower-dimensional equations, which are then deciphered to obtain the corresponding ordinary differential equation. It is remarkable to notify that the derived solutions in this study have not been reported in the literature. Additionally, the wide diversity of features and physical parameters of these acquired solutions are elucidated with the support of three-dimensional plots, considering the appropriate choice of involved functional parameters and other constant parameters shown by Figs. 1–3. Such a type of investigation is highly recommended in the areas of progressive research and development. We showed that the (2 + 1) dimensional VCBK equation admits an infinite number of infinitesimals of Lie point symmetries. We determined an optimal system of one-dimensional sub-algebras of Lie symmetries. Abundant exact travelling wave solutions, two-soliton type solutions and three-soliton type solutions were presented. The summary of our results;

- The investigated Lie vectors are different and new compared with [10,16,17].
- Based on the new Lie vectors, we reduce Eq. (3) to some ODEs and solved these equations using the direct method and the integration method.
- We can analyze our solutions as:

(I) Figs. (2-a,2-b,3-a,3-b,3-c) imply one-soliton and three- soliton solutions depend on the increase in time value. The amplitude decreases with increasing time and the wave peaks move to a lift direction.

(II) Figs. (1-a) represent multi- peaks of waves moves together and this means the physical properties do not change after the interaction.

(III) In Figs. (3-c), exponential waves are presented.

- In the future, the design of the wavelet method can be used to solve new applications in Ocean Engineering and Science.

Declaration of Competing Interest

The authors declare no competing interests.

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