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Detection of new multi-wave solutions in an unbounded domain

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We deduce new explicit traveling wave solutions for Zoomeron evolution equation and $(3+1)$-dimensional shallow water wave equation. The reduction process using Lie vectors leads in some cases to ordinary differential equations (ODEs) that having no quadrature. The integrating factor property has been used to derive several new solutions for these nonsolvable ODEs. These solutions have been illustrated with three dimensions plots. Comparison with other works are presented.

Keywords: Integrating factors; Lie transformation; $(2+1)$-dimensional Zoomeron equation; $(3+1)$-dimensional shallow water wave equation.

1. Introduction

Nonlinear differential equations (NLDEs) investigates many phenomena in different sciences, for example, in material science as carbon and fiber materials,[1] blood flow,[2–4] electric field,[5] and some new applications in Ref. 6. The analysis of the

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results of such application in most cases needs the analytical solution for the problem. There are many methods to explore closed-form solutions. Some of these methods are the Bäcklund transformation,[7] Hirota bilinear method,[8] Singular Manifold method,[9,10] tanh–coth strategy,[11] and Lie symmetry reduction method.[12–17] In this paper, we solve two equations, namely the $(2+1)$-dimensional Zoomeron equation and $(3+1)$-dimensional shallow water wave equation using Lie symmetry reduction method with the help of the integrating factor property. Starting with Zoomeron equation, we reduce the order of the equation using its infinitesimals until we reach unsolvable ordinary differential equations (ODEs). Through the property of the integrating factors, we solve these ODEs and generate new solutions. Plots of the resulted solutions and comparison of other works have been presented. In the last section, we apply the same procedure for shallow water wave equation.

2. $(2+1)$-Dimensional Zoomeron Equation

The $(2+1)$-dimensional Zoomeron equation was constructed by Calogero and Degasperis[8];

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{xt} = 0\,, \tag{1}$$

where $u(x,y,t)$ is the amplitude of the relevant wave mode. In Ref. 18, the two sine–cosine methods have been successfully applied to generate exact solutions of Eq. (1). Applying the (G'/G)-expansion method in Ref. 19, explicit traveling wave solutions are generated. Using Lie symmetries in Refs. 20 and 21, some form of traveling-wave solutions is created by examination of various combinations of reductions which would lead to an ODE in terms of a similarity variable of traveling-wave form as an independent variable. Here, we apply the symmetry reduction method to reduce the Zoomeron equation to ODEs, but during the reduction process, some of the obtained ODE's had no quadrature. We solve these equations using the corresponding Lie integrating factors. Equation (1) has five Lie vectors, we will choose some of these vectors to reduce the equation

$$X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = y\frac{\partial}{\partial y} - 0.5u\frac{\partial}{\partial u}. \tag{2}$$

2.1. *Reduction of the independent variables in Zoomeron equation using X_2 Lie vector*

Equation (1) is transformed through the optimal vector $X_2 = \frac{\partial}{\partial t}$ to

$$F_{rrrs}F^2 - 2F_rF_{rrs}F + 2F_{rs}F_r^2 - FF_{rs}F_{rr} = 0\,. \tag{3}$$

This equation has no closed-form solution, but possesses six Lie vectors; we will choose here to work only with V_3

$$V_3 = \frac{\partial}{\partial r} + \frac{\partial}{\partial s}\,. \tag{4}$$

This Lie vector leads to an ODE with no analytic solution.

- **Reduction using V_3**

Using V_3 transform Zoomeron to a nonlinear fourth-degree ODE of the form

$$\theta^2\theta_{\eta\eta\eta\eta} - 2\theta_{\eta\eta\eta}\theta_\eta\theta + 2{\theta_\eta}^2\theta_{\eta\eta} - {\theta_{\eta\eta}}^2\theta = 0\,. \tag{5}$$

- **Using integrating factor to get an explicit solution**

We first deduce Eq. (5) integrating factors as

$$\mu_1 = \eta, \quad \mu_2 = 1\,. \tag{6}$$

The integrating factors reduce Eq. (5) to two solvable ODEs

$$\theta_{\eta\eta} + c_1\theta = 0, \tag{7}$$

$$\ln(\theta_{\eta\eta}) - \ln(\theta) + c_1 = 0\,. \tag{8}$$

Equation (7) has a solution in the form

$$\theta(\eta) = c_2\sin(\sqrt{c}_1\eta) + c_3\cos(\sqrt{c}_1\eta) \tag{9}$$

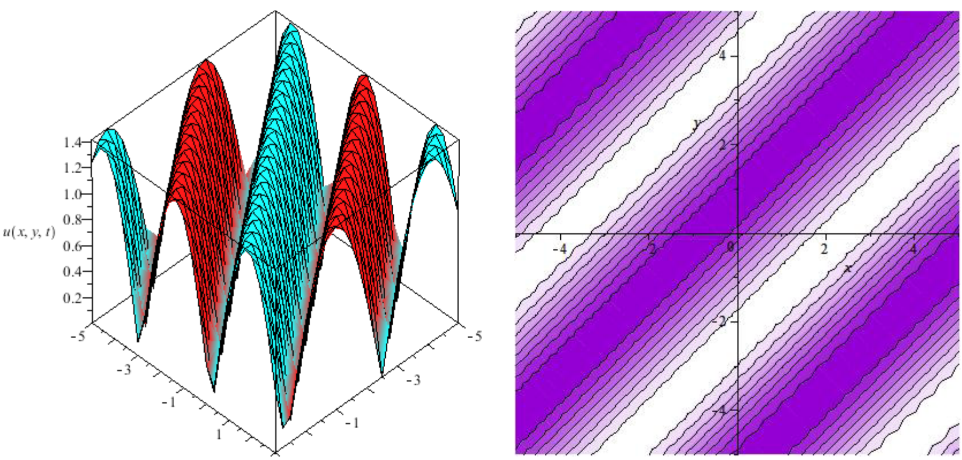
and Eq. (8) has a closed-form solution of the form

$$\theta(\eta) = c_2 e^{e^{-0.5c_1}\eta} + c_3 e^{e^{-0.5c_1}\eta}, \tag{10}$$

where $\eta = -r + s$, $\theta(\eta) = F(r,s)$ and c_1, c_2, c_3 are integration constants. Back substituting to (x,y,t) for $r = x, s = y, F(r,s) = u(x,y,t)$, we obtain

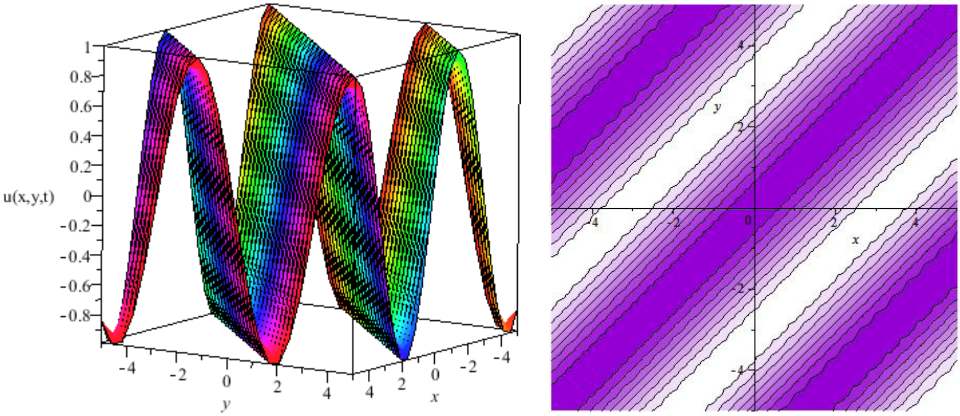
$$u(x,y,t) = c_2\sin(\sqrt{c}_1(-x+y)) + c_3\cos(\sqrt{c}_1(-x+y)\,. \tag{11}$$

This result is shown in Fig. 1 in a complex domain.



(a) Three dimensions plot for $u(x,y,t)$ (b) Contour plot for $u(x,y,t)$

Fig. 1. (Color online) $u(x,y,t)$ in Eq. (11) with $c_1 = 1$, $c_2 = 1$, $c_3 = 1$.



(a) (b)

Fig. 2. (Color online) $u(x,y,t)$ in Eq. (10) at $c_1 = \ln(-1)$, $c_2 = 0.5$, $c_3 = 0.5$. (a) Three dimensions plot. (b) Contour plot for $u(x,y,t)$.

Corresponding to Eq. (10), we obtain

$$u(x,y,t) = c_2 e^{e^{-0.5c_1(-x+y)}} + c_3 e^{e^{-0.5c_1(-x+y)}}. \tag{12}$$

This solution is plotted in complex domain as depicted in Fig. 2.

2.2. *Reduction of the independent variables in Zoomeron equation using X_4 Lie vector*

Equation (1) is transformed through the optimal vector $X_4 = y\frac{\partial}{\partial y} - 0.5u\frac{\partial}{\partial u}$ to

$$\begin{aligned} &F_{ssr}F^2 - 2F_sF_{rS}F + 2F_rF_s^2 - FF_rF_{ss} - F_{rrr}F^2 + 3FF_{rr}F_r \\ &\quad - 2F_r^3 - 8F_rF_sF^3 - 8F^4F_{rs} = 0. \end{aligned} \tag{13}$$

This equation has no closed-form solution, but possesses three Lie vectors

$$V_1 = \frac{\partial}{\partial s}, \quad V_2 = \frac{\partial}{\partial r}, \quad V_3 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s} - 0.5F\frac{\partial}{\partial F}. \tag{14}$$

We will choose here to work only with V_3 Lie vector as it leads to an ODE with no analytic solution.

- **Reduction using V_3**

Using V_3 transform Zoomeron to a nonlinear fourth-degree ODE of the form

$$\begin{aligned} &3\eta\theta_{\eta\eta}\theta_\eta\theta - 3\eta^3\theta\theta_{\eta\eta}\theta_\eta + \theta^3 + 6\eta\theta_\eta\theta^2 + 6\eta^2\theta^2\theta_{\eta\eta} + \eta^3\theta^2\theta_{\eta\eta\eta} + 2{\theta_\eta}^3\eta^3 + 16\theta^4\theta_\eta \\ &\quad + 8\eta\theta_{\eta\eta}\theta^4 - \eta\theta_{\eta\eta\eta}\theta^2 - 6\eta^2\theta\theta_\eta^2 + 8\eta{\theta_\eta}^2\theta^3 + 2{\theta_\eta}^2\theta = 0. \end{aligned} \tag{15}$$

- **Using integrating factor to get an explicit solution**

We first deduce Eq. (15) integrating factors as

$$\mu_1 = \frac{1}{\eta^2}, \quad \mu_2 = \frac{1}{\eta^3}. \tag{16}$$

The integrating factors reduce Eq. (15) to two solvable ODEs

$$2\eta^2\theta_\eta + \eta\theta - 2\theta_\eta + 8\theta^3 = 0. \tag{17}$$

Equation (17) has a solution in the form

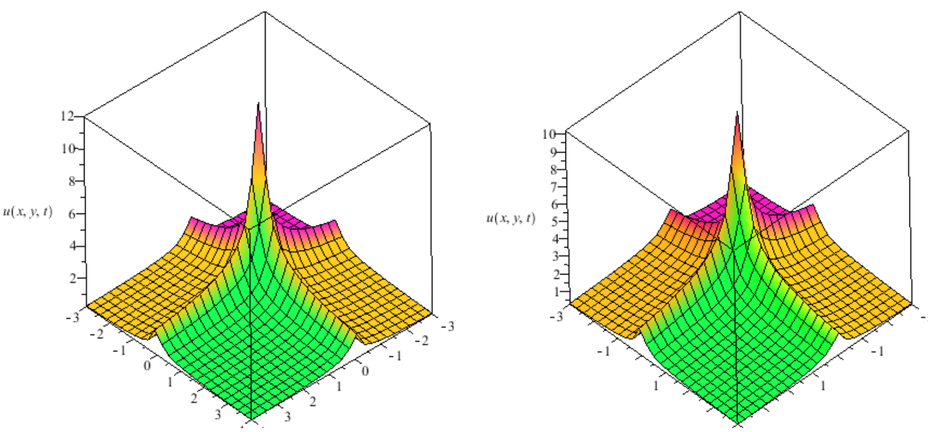
$$\theta(\eta) = \mp\frac{1}{\sqrt{-8\eta + c_1\sqrt{\eta+1} * \sqrt{\eta-1}}}, \tag{18}$$

where c_1 is integration constant, $\eta = \frac{s}{r}$, $\theta(\eta) = F(r,s)\sqrt{r}$ and c_1, c_2, c_3 are integration constants. Back substituting to (x,y,t) for $r = x$, $s = t$, $F(r,s) = u(x,y,t)\sqrt{y}$, we obtain

$$u(x,y,t) = \mp\frac{1}{\sqrt{x}\sqrt{y}\sqrt{-8\frac{t}{x} + c_1\sqrt{\frac{t}{x}+1} * \sqrt{\frac{t}{x}-1}}}. \tag{19}$$

This result is depicted in Fig. 3 in a complex domain.

The wave that plotted in Fig. 3 decayed rapidly with time.

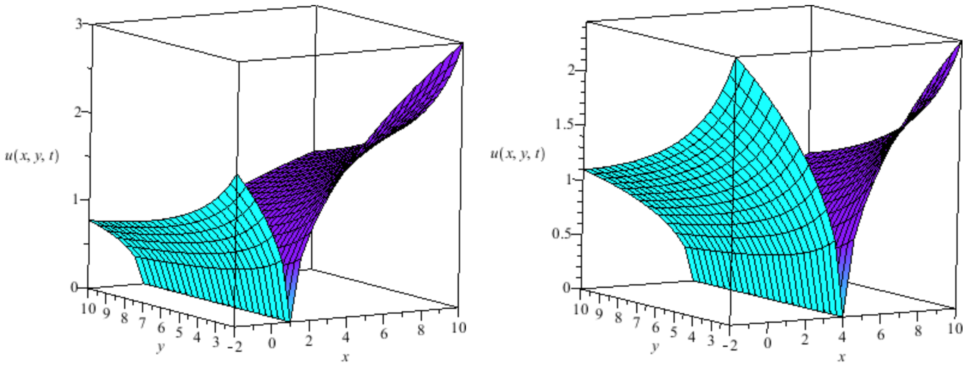


(a) The positive value of $u(x,y,t)$ at $c_1 = 1, \mathrm{t} = 0$. (b) The positive value of $u(x,y,t)$ at $c_1 = 1, \mathrm{t} = 0.01$.

Fig. 3. (Color online) Three dimensions plots for positive $u(x,y,t)$ in Eq. (18) for different values of time.

2.3. *Reduction of the independent variables in Zoomeron equation using $X_2 + X_3 + X_4$ Lie vector*

Using the optimal vector $X_{234} = \frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{\partial}{\partial t} - 0.5u\frac{\partial}{\partial u}$ and through the same procedure in Secs. 2.1 and 2.2, Eq. (1) will reduce to a nonlinear fourth-degree ODE



(a) The positive value of $u(x,y,t)$ at $c_1 = c_2 = 1, \mathrm{t} = 0$. (b) The positive value of $u(x,y,t)$ at $c_1 = c_2 = 1, t = 3$.

Fig. 4. (Color online) Three dimensions plots for positive $u(x,y,t)$ in Eq. (22) for different values of time.

of the form

$$\theta\theta_{\eta\eta} + {\theta_\eta}^2 = 0\,. \tag{20}$$

Equation (20) has a solution in the form

$$\theta(\eta) = \mp\sqrt{2c_1\eta + 2c_2}\,, \tag{21}$$

where c_1 is integration constant, $\eta = s$, $\theta(\eta) = F(r,s)\sqrt{r}$ and c_1, c_2 are integration constants. Back substituting to (x,y,t) for $r = ye^{-x}$, $s = -x+t$, $F(r,s) = u(x,y,t)e^{0.5x}$, we obtain

$$u(x,y,t) = \mp\frac{\sqrt{2c_1t - 2c_1x + 2c_2}}{\sqrt{y}}\,. \tag{22}$$

This result is depicted in Fig. 4 in a complex domain.

Table 1. Analysis of our results with Ref. 21.

Selected symmetry	Our result	Result of Ref. 21
X_1 or Δ_1	$u(x,y,t) = c_2\sin(\sqrt{c}_1(-x+y))$ $+ c_3\cos(\sqrt{c}_1(-x+y)$ and $u(x,y,t) = c_2e^{e^{-0.5c_1(-x+y)}}$ $+ c_3e^{e^{-0.5c_1(-x+y)}}$	$u(x,y,t) = m_3Ai\big[m_1^{\frac{-2}{3}}(m_2 + m_1(x+y))\big]$ $+ m_4Bi\big[m_1^{\frac{-2}{3}}(m_2 + m_1(x+y))\big]$
X_4 or Δ_5	$u(x,y,t)$ $= \mp\dfrac{1}{\sqrt{x}\sqrt{y}\sqrt{-8\frac{t}{x} + c_1\sqrt{\frac{t}{x}+1} * \sqrt{\frac{t}{x}-1}}}$	$u(x,y,t) = \Bigg[\sqrt{\frac{2\pi}{A}}\exp\left(\frac{-2(m_2+m_1(x+t))^2}{A}\right)$ $\times\,\mathrm{erfi}\left(\sqrt{\frac{2}{A}}(m_2+m_1(x+t))\right)$ $+ m_3\exp\left(\frac{-2m_1(x+t)(2m_2+m_1(x+t))}{A}\right)\Bigg]^{\frac{-1}{2}}$

2.4. *Comparison and analysis*

Morris *et al.* in Ref. 21 generated explicit solutions for Eq. (1) using the Lie point symmetries reduction method. We use the same infinitesimals with the help of the integrating factors, then we explore new and different solutions that resulted in Ref. 21 as shown in Table 1.

3. (3 + 1)-Dimensional Shallow Water Wave Equation

Shallow water equation has the form[22,23];

$$u_{yzt}+u_{xxxyz}-6u_x u_{xyz}-6u_{xz}u_{xy}=0\,. \tag{23}$$

Using the simplified Hirota's method, multiple soliton solutions and generalized multiple singular soliton solutions were generated in Ref. 23. Hirota's bilinear method was used to determine the multiple-soliton solutions for this equation in Ref. 22. In Ref. 24, some dromion solitons, periodic dromion solitons and interaction between two solitary waves were generated though d'Lambert transformation and G'/G-expansion method. Applying Hirota method, using an auxiliary function Lump soliton and interaction of Lump with one-stripe soliton solutions had been explored in Ref. 25.

Here, we investigate 20 Lie vectors for Eq. (23) and we will choose only X_5 to reduce it

$$X_5=\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\,. \tag{24}$$

The original partial differential equation (PDE) (23) consisted of four independent variables; $(x, y, t; z)$ are first reduced to a PDE in three independent variables, (l, h, o) using the Lie vector X_5, then reduced to two independent variables (r, s) and then one independent variable η.

3.1. *Reduction of the independent variables in the shallow water equation using X_5 Lie vector*

Equation (23) is transformed through the vector $X_5=\frac{\partial}{\partial x}+\frac{\partial}{\partial t}$ to

$$K_{olh}-K_{lohhh}-6K_{loh}K_h-6K_{lh}K_{ho}=0\,. \tag{25}$$

This equation has no closed-form solution but possesses 14 Lie vectors; we choose V_1, V_2.

3.1.1. *Reduction of* (25) *using V_1 Lie vector*

$$V_1=\frac{\partial}{\partial h}+\frac{\partial}{\partial o}\,. \tag{26}$$

Equation (25) is transformed to

$$F_{rssss}-F_{ssr}-6F_sF_{ssr}-6F_{sr}F_{ss}=0\,. \tag{27}$$

This equation has no closed-form solution but possesses eight Lie vectors; we will choose here to work with the vectors e_5 to lead to non-solvable ODE's while the rest of the vectors lead to solvable ODE's.

- **Reduction of (27) using e_5 Lie vector**

$$e_5 = \frac{\partial}{\partial r} + s\frac{\partial}{\partial s} - \left(\frac{s}{3} + F\right)\frac{\partial}{\partial F}\,. \tag{28}$$

We use e_5 to transform **shallow water** to a nonlinear fifth-degree ordinary differential equation of the form

$$6\eta\theta_{\eta\eta\eta}\theta_\eta + 30\theta_\eta\theta_{\eta\eta} - 5\theta_{\eta\eta\eta\eta} - \eta\theta_{\eta\eta\eta\eta\eta} + 6\eta\theta_{\eta\eta}^2 = 0\,. \tag{29}$$

- **Using integrating factor to get an explicit solution**

We first deduce Eq. (29) integrating factors as

$$\mu_1 = 1, \quad \mu_2 = \eta^4\,. \tag{30}$$

The integrating factors reduce (29) to

$$2\theta_\eta^3 - \theta_{\eta\eta}^2 = 0\,. \tag{31}$$

This equation has a closed-form solution

$$\theta(\eta) = -\frac{2}{\eta + c_1} + c_2\,, \tag{32}$$

where $\eta = se^{-r}$, $\theta(\eta) = \frac{F(r,s)+\frac{1}{6}s}{e^{-r}}$. Then back to (l, h, o) coordinates with

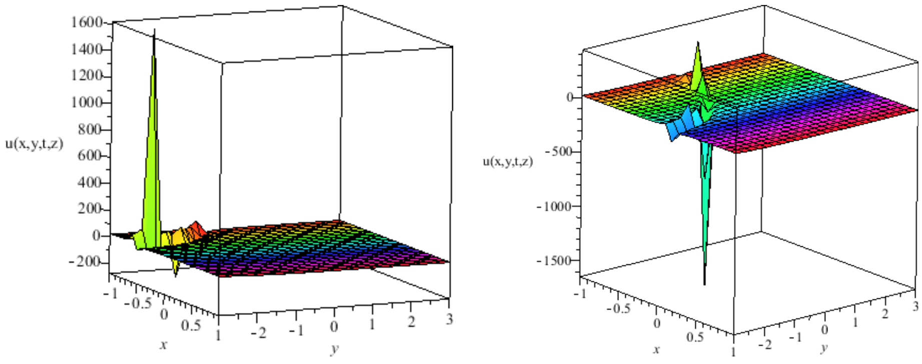
$$r = l, \quad s = -h + o, \quad F(r,s) = K(l,h,o), \quad h = -x + t, \quad l = y\,,$$
$$o = z, \quad K(l,h,o) = u(x,y,t,z)\,.$$

Leads to

$$u(x,y,t,z) = \frac{-2e^{-y}}{(x - t + z)e^{-y} + c_1} + c_2e^{-y} + \frac{1}{6}(-x + t) - \frac{1}{6}z\,. \tag{33}$$

This result is plotted in Fig. 5.

Peakon's are distributed on a parabola drifting to the left with time and the peaks of the waves decay with time.



(a) $u(x,y,t,z)$ at $c_1 = 1, c_2 = 1, z = 0.5, t = 0.7$ s. (b) $u(x,y,t,z)$ at $c_1 = 1, c_2 = 1, z = 0.5, t = 0.7$ s.

Fig. 5. (Color online) $u(x,y,t,z)$ for various values of time.

3.1.2. *Reduction of* (25) *using* V_2 *Lie vector*

Through the vector $V_2 = \frac{\partial}{\partial h}$, Eq. (25) is transformed to

$$6sF_{sr}F_s - 6F_{ssr} - 6sF_{sssr} - s^2F_{ssssr} + 12F_{sr}F_s + 6sF_{sr}F_{ss} = 0\,. \tag{34}$$

This equation has eight Lie vectors, we choose the fifth vector.

$$e_5 = \frac{\partial}{\partial r} + s\ln(s)\frac{\partial}{\partial s} - \left(\frac{\ln(s)}{3} + F\right)\frac{\partial}{\partial F}\,. \tag{35}$$

We use e_5 to transform (34) to a nonlinear fifth-degree ordinary differential equation of the form

$$-6\eta\theta_{\eta\eta\eta}\theta_\eta - 30\theta_\eta\theta_{\eta\eta} + 5\theta_{\eta\eta\eta\eta} + \eta\theta_{\eta\eta\eta\eta\eta} - 6\eta\theta_{\eta\eta}^2 = 0\,. \tag{36}$$

- **Using integrating factor to get an explicit solution**

We first deduce Eq. (36) integrating factors as

$$\mu_1 = 1, \quad \mu_2 = \eta^4\,. \tag{37}$$

The integrating factors reduce Eq. (36) to

$$-2{\theta_\eta}^3 + {\theta_{\eta\eta}}^2 = 0\,. \tag{38}$$

This equation has a closed-form solution of the form

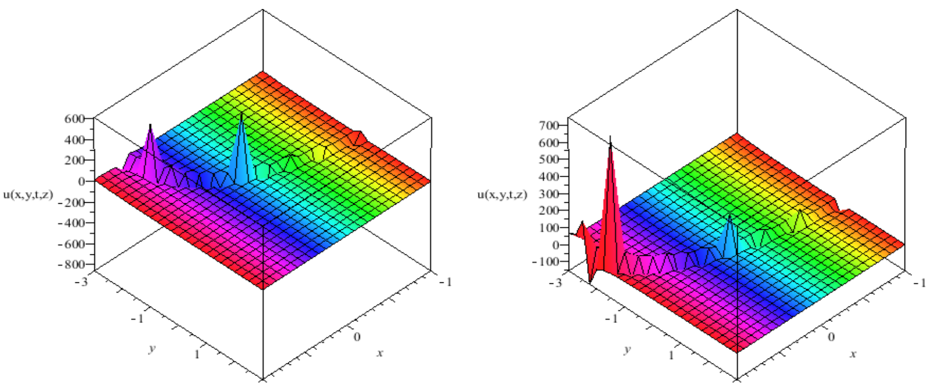
$$\theta(\eta) = -\frac{2}{\eta + c_1} + c_2, \quad \text{where } \eta = \ln(s)e^{-r}, \quad \theta(\eta) = \frac{1}{6}\frac{6F(r,s) + \ln(s)}{e^{-r}}\,. \tag{39}$$

Then back to (l,h,o) with $r = l$, $s = oe^{-h}$, $F(r,s) = K(l,h,o)$, $h = -x + t$, $l = y$, $o = z$, $K(l,h,o) = u(x,y,t,z)$ we obtain

$$u(x,y,t,z) = \frac{-2e^{-y}}{(\ln(z) + (x-t))e^{-y} + c_1} + c_2e^{-y} - \frac{1}{6}\ln(ze^{x-t})\,. \tag{40}$$

This solution is plotted in Fig. 6 with $c_1 = c_2 = 1$.

The wave axis drifts left as time increases.



(a) $u(x,y,t,z)$ at $c_1 = 1, c_2 = 1, z = 0.5, t = 0$ s. (b) $u(x,y,t,z)$ at $c_1 = 1, c_2 = 1, z = 0.5, t = 0.4$ s.

Fig. 6. (Color online) $u(x,y,t,z)$ for different values of time and constant value of z.

4. Conclusions

Starting with the infinitesimals's reduction with integrating factor property, we generate new closed-form solutions for two nonlinear PDEs, namely the $(2+1)$-dimensional Zoomeron equation and $(3+1)$-dimensional shallow water wave equation by reducing these PDEs to non-solvable ODEs. Here, we deduce the following:

- We were able to solve these ODEs using the integrating factors property.
- New analytic solutions have been generated compared with other researchers' work as Ref. 11.
- Reduce the number of the reduction stages using the Lie symmetry method.
- In some cases, we cannot solve the reduced ODEs using Lie vectors only.

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