Chinese Journal of Physics 65 (2020) 198–206

Contents lists available at ScienceDirect

Chinese Journal of Physics

journal homepage: www.elsevier.com/locate/cjph

New exact solutions of Bratu Gelfand model in two dimensions using Lie symmetry analysis

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ARTICLE INFO

Keywords:
Two dimensions Bratu equation
Boundary value problems
Lie symmetry method
Exact solutions

ABSTRACT

We explore new analytical solutions for the two-dimensional nonlinear elliptic Bratu equation. Through the point transformation, the integrable form of Bratu equation was investigated then we obtain the Lie infinitesimals for the new equation. These vectors reduce the integrable equation to solvable ODEs then we use the boundary conditions (BCs) to spin two new exact solutions for Bratu equation in a unit square domain. A three-dimensional plot illustrates some resulting solutions. Comparison with other work has been presented.

1. Introduction

In the present study, we find out a new analytical solution for the two-dimensional nonlinear an elliptic boundary-value problem: the Bratu model [1–4]. Bratu equation represents in a large variety of application areas, for example, nanotechnology, radiative heat transfer and chemical reactor theory [5]. It is likewise utilized as a model for the thermal reaction process [6]. The higher-dimensional model can be used in the sun core temperatures [7–8]. The Bratu model appears in various cases such as elasticity theory [9] and thermo-electro-hydrodynamics models [10].The importance of Bratu model become from the variety of its applications as; heat transfer, electrical conducting model for solids material, nanotechnology and chemical responses [11–15]. This equation at the stationary state has the form;

$$\frac{\partial^2}{\partial x^2}v(x,y)+\frac{\partial^2}{\partial y^2}v(x,y)+\lambda^2 e^{v(x,y)}=0,\ \ x,y\in D=[0,1]\times[0,1] \tag{1}$$

with (BCs);

$$v(0,\ y)=\ v(1,\ y)=\ v(x,\ 0)=\ v(x,\ 1)=\ 0,\ \ x,\ \ y\ \in\ D. \tag{2}$$

where λ announce to the response expression or control parameter and $\lambda > 0$.

Several numerical and analytical techniques are applied for LBG equation in one and two dimensions[16–20] .

Odejide and Aregbesola [16] introduced a brief discussion for the properties of Bratu equation and they obtained near exact

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https://doi.org/10.1016/j.cjph.2020.01.008
Received 22 June 2019; Received in revised form 5 December 2019; Accepted 15 January 2020
Available online 06 February 2020
0577-9073/

solution for a collection point at $x = 0.5,\ y = 0.5$. Liao [17] solved Bratu equation in the two and three dimensions using the Homotopy method. Al and Hadhoudb [18] apply both (HOBW) with (MLSDQM) to acquire an approximate solution for Eq. (1). Recently, Iqbal and Andries Zegeling [19] applied multigrid approach to examine the numerical bifurcation behavior of the Bratu equation. There are many ways to find the exact solution [20–25]. Lie symmetry analysis is one of the efficient approaches for obtaining exact solutions of nonlinear partial differential equations (NLPDEs). In decades time, Lie's method has been described and applied in different textbooks and several physical and engineering models are analyzed [26–31]. As Bratu equation haven't Lie vectors due to the nonlinearity in the exponential term, we use a potential transformation to transfer the equation to an integrable one. We investigate Lie generators for the new equation then we explore new solutions by applying Lie symmetry method. The rest of the paper develops as follows;

In Section 2. Lie symmetry vectors, are derived using Maple and reduce the Bratu equation to an (ODE) that have quadrature. For each Lie vector the following steps apply;

- Bratu (PDE) variables (x, y) are reduced to an ODE in to one variable (η).
- These ODE equations are solved, and we back substitution to the original variables.
- For each Lie vector, the resultant solutions are registered in Table 1.

In Section 3. We apply the Dirichlet (BCs) on a unit square domain and the various numbers of the critical turning point, versus the constants of integration are listed in Table 2.

In Section 4. Another analytical method's solutions were contrasted with our results.

In Section 5. Conclusions.

2. Lie symmetry reduction and unbounded solution of Bratu equation

Bratu model (1) is a non-integrable equation and haven't Lie infinitesimals; so, we simplify it through a point transformation;

$$u(x, y) = ln\left(\frac{\partial}{\partial x}v(x, y)\right). \tag{3}$$

Substituting from Eq. (3) into Eq. (1) leads to a new higher nonlinear PDE;

$$\left(\frac{\partial^2}{\partial x^2}u(x, y)\right)^2 - \frac{\partial^3}{\partial x^3}u(x, y)\frac{\partial}{\partial x}u(x, y) - \frac{\partial^3}{\partial y^2\partial x}u(x, y)\frac{\partial}{\partial x}u(x, y) + \left(\frac{\partial^2}{\partial y\,\partial x}u(x, y)\right)^2 - \lambda^2\left(\frac{\partial}{\partial x}u(x, y)\right)^3 = 0. \tag{4}$$

Through the Maple program, we investigate six Lie infinitesimals;

$$\begin{aligned} X_1 &= \tfrac{\partial}{\partial y} + \tfrac{\partial}{\partial u}, \quad X_2 = \tfrac{\partial}{\partial y} + y\tfrac{\partial}{\partial u},\\ X_3 &= \tfrac{\partial}{\partial x} + \tfrac{\partial}{\partial u},\\ X_4 &= \tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial u},\\ X_5 &= x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + (-u + 1)\tfrac{\partial}{\partial u},\\ X_6 &= x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + (-u + y)\tfrac{\partial}{\partial u}. \end{aligned} \tag{5}$$

These vectors have been utilized to reduce (4) to an ODE. We tested all of them, as a combination of vectors; X_2 and X_3 and X_1, X_3. The results listed in Table 1, concern only the solvable ODEs.

2.1. *Using of new infinitesimal* $X_2 + X_3$ *in reducing and solving Eq. (4)*

Consider the Lie vector as;

$$X_2 + X_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + (y + 1)\frac{\partial}{\partial u}. \tag{6}$$

The characteristic equation corresponding to this Lie vector is;

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{(y + 1)}. \tag{7}$$

Solving this equation, results the similarity variables;

$$\eta = -x + y, \theta(\eta) = u(x, y) + 0.5x^2 - (y + 1)x. \tag{8}$$

Using these variables to transform (4) to a nonlinear third order ordinary differential equation;

Table 1
The reduced equations and the solutions for each Lie vector.

vectors	ansatz	Reduction of Eq. (4)	Solution of eq. (1)
X_1	$\eta = x,\ \theta(\eta) = u(x,y) - y$	$-(\frac{d^3}{d\eta^3}\theta(\eta))(\frac{d}{d\eta}\theta(\eta)) + (\frac{d^2}{d\eta^2}\theta(\eta))^2 - \lambda^2(\frac{d}{d\eta}\theta(\eta))^3 = 0$	$v(x,y) = \ln(\frac{1}{2\lambda^2 c_1^2} - \frac{1}{2}(-\frac{sinh(\frac{1}{2}\frac{x}{c_1} + \frac{1}{2}\frac{c_2}{c_1})^2}{\lambda^2 c_1^2 \cosh(\frac{1}{2}\frac{x}{c_1} + \frac{1}{2}\frac{c_2}{c_1})^2}))$
X_2	$\eta = x,\ \theta(\eta) = u(x,y) - 0.5y^2$		
X_5	$\eta = \frac{y}{x},\ \theta(\eta) = (u(x,y) - 1)x.$	$(\frac{d}{d\eta}\theta(\eta))^2(2\eta^2 - 4 - 3\lambda^2\eta^2\theta(\eta))$ $+(\frac{d}{d\eta}\theta(\eta))(8\eta\theta(\eta) - 3\lambda^2\eta\theta(\eta)^2 + \eta) - 2\theta(\eta)^2$ $+(\frac{d^2}{d\eta^2}\theta(\eta))((\eta)^3(\frac{d}{d}\theta(\eta)) + 5\eta^2\theta(\eta) - 1 + 3\theta(\eta))$ $+(\frac{d^3}{d\eta^3}\theta(\eta))(\eta\theta(\eta)(\eta^2+1) + (-\eta^4 + \eta^2))$ $-(\frac{d^2}{d\eta^2}\theta(\eta))^2(\eta^4 + \eta^2)$ $-\lambda^2(\frac{d}{d\eta}\theta(\eta))^3\eta^3$ $-\lambda^2\lambda\theta(\eta)^3 = 0.$	$v(x,y) = \ln(\frac{-1}{2}\frac{1 + tan(\frac{1}{2}\frac{\arctan(\frac{y}{x}) - c_2}{c_1})^2}{x^2(1 + \frac{y^2}{x^2})c_1^2\lambda}).$
X_6	$\eta = \frac{y}{x},\ \theta(\eta) = (u(x,y) - 0.5y)x.$		
$X_2 + X_3$	$\eta = -x + y,\ \theta(\eta) = u(x,y) + 0.5x^2 - (y+1)x$	$(\frac{d}{d\eta}\theta(\eta))^3\lambda^2 - 3(\frac{d}{d\eta}\theta(\eta))^2\lambda^2(\eta+1) + 3(\frac{d}{d\eta}\theta(\eta))\lambda^2(\eta^2 + 2\eta + 1) - \lambda^2 - 3\eta^2\lambda^2$ $-2(\frac{d^3}{d\eta^3}\theta(\eta))((\frac{d}{d\eta}\theta(\eta)) - \eta - 1) + 2(\frac{d^2}{d\eta^2}\theta(\eta))^2 - 3\eta\lambda^2 - 4(\frac{d^2}{d\eta^2}\theta(\eta)) - \lambda^2\eta^3 + 2 = 0.$	$v(x,y) = ln(-\frac{1}{c_1^2\lambda^2\cos(\frac{1}{2}\frac{x - y + c_2}{c_1})^2})$
$X_1 + X_3$	$\eta = -x + y,\ \theta(\eta) = u(x,y) - x.$	$-2(\frac{d^3}{d\eta^3}\theta(\eta))(\frac{d}{d\eta}\theta(\eta) - 1) + 2(\frac{d^2}{d\eta^2}\theta(\eta))^2 + (\frac{d}{d\eta}\theta(\eta))^3\lambda^2 - 3\lambda^2(\frac{d}{d\eta}\theta(\eta))^2 + 3\lambda^2(\frac{d}{d\eta}\theta(\eta)) - \lambda^2 = 0.$	$v(x,y) = ln(\frac{1}{c_1^2\lambda^2\cosh(\frac{1}{2}\frac{-x + y + c_2}{c_1})^2})$

Table 2
Values of c_1 and c_3 versus λ.

λ	$(X_2 + X_3)$ *Solution* Lower branch c_1	Upper branch c_1	$(X_1 + X_3)$ *Solution* Lower branch c_3	Upper branch c_3
1	0.9675208458I	0.076697I	0.9675208458	−0.076697
2	0.424168969I	0.11754708I	0.424168969	−0.11754708
$\sqrt{7.02766}$	0.20838914I	0.20838914I	0.20838914	0.20838914

$$\left(\frac{d}{d\eta}\theta(\eta)\right)^3\lambda^2 - 3\left(\frac{d}{d\eta}\theta(\eta)\right)^2\lambda^2(\eta+1) + 3\lambda^2\left(\frac{d}{d\eta}\theta(\eta)\right)(\eta^2+2\eta+1) - \lambda^2(1+3\eta^2)$$
$$-2\left(\frac{d^3}{d\eta^3}\theta(\eta)\right)\left(\left(\frac{d}{d\eta}\theta(\eta)\right)-\eta-1\right) + 2\left(\frac{d^2}{d\eta^2}\theta(\eta)\right)^2$$
$$-3\eta\lambda^2 - 4\left(\frac{d^2}{d\eta^2}\theta(\eta)\right) - \lambda^2\eta^3 + 2 = 0. \tag{9}$$

Eq. (9) has a solution in the form;

$$\theta(\eta) = \frac{-\left(4\sin\left(\frac{-\eta+c_2}{2c_1}\right) - \eta^2\lambda^2 c_1\cos\left(\frac{-\eta+c_2}{2c_1}\right) + \lambda^2 c_1\cos\left(\frac{-\eta+c_2}{2c_1}\right)(c_2^2-2\eta)\right) + 2\lambda^2 c_1\cos\left(\frac{-\eta+c_2}{2c_1}\right)(c_2-c_3)}{2\lambda^2 c_1\cos\left(\frac{-\eta+c_2}{2c_1}\right)-2}. \tag{10}$$

Substituting from Eqs. (3) and (8) into (10) leads to;

$$v(x,y) = \ln\left(-\frac{1}{c_1^2\lambda^{\ 2}\cos^2\left(\frac{1}{2}\,\frac{x-y+c_2}{c_1}\right)}\right) \tag{11}$$

where, $v(x, y)$ is a solution of Eq. (1), c_1, c_2 are integration constants that will be acquired using (BCs) and λ is the control parameter. For each Lie vector, the reduced equations are written besides the solutions for Bratu equation in Table 1.

3. Applying the boundary conditions

In this section, we apply the (BCs) in (2) to some results that explored from two different Lie vectors as follow;

3.1. *The solution of vector* $X_2 + X_3$ *(third row)*

Eq. (11) is general solution of Bratu equation. The constants of integration are evaluated by substituting from the (BCs) as follows;

- From; $v(0, y) = v(1, y) = 0$, yields a system of two nonlinear algebraic equations:

$$\lambda^2 c_1{}^2\cos^2\left(\frac{-y+c_2}{2c_1}\right) = -1, \tag{12}$$

$$\lambda^2 c_1^2\cos^2\left(\frac{1-y+c_2}{2c_1}\right) = -1. \tag{13}$$

By solving the system; $y = c_2 + 0.5$ and by substituting the value of y in (12) or (13), we find that,

$$c_1^2 = -\frac{1}{\lambda^2\cos^2\left(\frac{1}{4c_1}\right)}, \tag{14}$$

from this new relative equation, c_1 will be determined later and we expect that the value of c_1 is complex. At this edge, by substituting from the new relation of c_1 in Eq. (11) we obtain a solution satisfy Eq. (1) and the (BCs);

$$v(x,y) = ln\left(-\frac{1}{c_1^2\lambda^2\cos^2\left(\frac{x-0.5}{2c_1}\right)}\right). \tag{15}$$

- From; $v(x, 0) = v(x, 1) = 0$, a system of two equations in a trigonometric form are generated as;

$$\lambda^2 c_1^2 \cos^2\left(\frac{x + c_2}{2c_1}\right) = -1. \tag{16}$$

$$\lambda^2 c_1^2 \cos^2\left(\frac{-1 + x + c_2}{2c_1}\right) = -1. \tag{17}$$

This system yields a solution at $x = -c_2 + 0.5$ and $c_1^2 = -\frac{1}{\lambda^2 \cos^2(\frac{1}{4c_1})}$, the solution of (1) at this edge, gives $c_2 = 0.5$.

$$v(x, y) = ln\left(-\frac{1}{c_1^2 \lambda^2 \cos^2(\frac{-y+0.5}{2c_1})}\right). \tag{18}$$

From the above results, the (BCs) leads to two different curves (15) and (18). The solution of 2D Bratu is the surface resulting from the mesh of the two curves; Eqs. (15) and (18) in the unit square boundary $\{x, y \in D = [0; 1] \times [0; 1]$, which documented as;

$$v(x, y) = \frac{2\,.\,\ln\left(-\frac{1}{c_1^2\lambda^2\cos^2\left(\frac{x-0.5}{2c_1}\right)}\right).\,2\ln\left(-\frac{1}{c_1^2\lambda^2\cos^2\left(\frac{-y+0.5}{2c_1}\right)}\right)}{\ln\left(-\frac{1}{c_1^2\lambda^2\cos^2\left(\frac{x-0.5}{2c_1}\right)}\right) + \ln\left(-\frac{1}{c_1^2\lambda^2\cos^2\left(\frac{-y+0.5}{2c_1}\right)}\right)}. \tag{19}$$

This exact solution satisfies both Bratu equation and the boundary conditions. The value of the critical point of constant of integration c_1 and the control parameter λ are determined (see Appendix A);

3.2. The solution of vector $X_1 + X_3$ (fourth row)

The solution given in the (4th row) in Table 1 is a solution of (1). The constants of integration are evaluated by substituting from the (BCs) as follows;

- From; $v(0, y) = v(1, y) = 0$, by the same procedure, the system will be;

$$\lambda^2\ c_3^2\, cosh^2\left(\frac{y + c_2}{2c_3}\right) = 1, \tag{20}$$

$$\lambda^2\ c_3^2\, cosh^2\left(\frac{-1 + y + c_2}{2c_3}\right) = 1, \tag{21}$$

By settling the system; $y = -c_2 + 0.5$ and $c_3^2 = \frac{1}{\lambda^2 \cosh^2(\frac{1}{4c_3})}$, c_3 will be determined later and the solution of (1) at this edge is;

$$v(x, y) = ln\left(\frac{1}{c_3^2\ \lambda \cosh^2(\frac{-x+0.5}{2c_3})}\right). \tag{22}$$

- From; $v(x, 0) = v(x, 1) = 0$, the nonlinear equations are;

$$\lambda^2 c_3^2 \cosh^2\left(\frac{-x + c_2}{2c_3}\right) = 1, \tag{23}$$

$$\lambda^2 c_3^2 \cosh^2\left(\frac{-x + 1 + c_2}{2c_3}\right) = 1, \tag{24}$$

This system yields a solution at $x = c_2 + 0.5$ and $c_3^2 = \frac{1}{\lambda^2 \cosh^2(\frac{1}{4c_3})}$, from this new relative equation c_3 will be obtained later and the

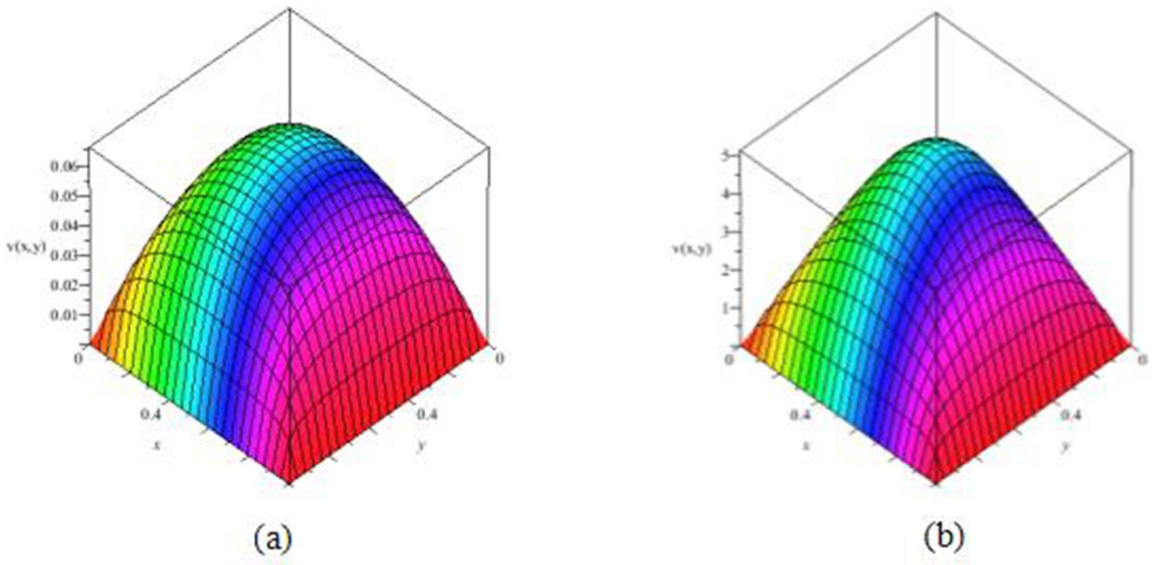

Fig. 1. Solution of Bratu equation for $\lambda = 1$.
(**a**) Lower branch at $c_1 = 0.9675208458I$ and $c_3 = 0.9675208458$. **(b)** Upper branch at $c_1 = 0.076697I$and $c_3 = -0.076697$.

solution of the equation at this edge is;

$$v(x, y) = ln\left(\frac{1}{c_3^2\lambda^2\cosh^2\left(\frac{y-0.5}{2c_3}\right)}\right). \tag{25}$$

From the above results the (BCs) lead to two different curves Eqs. (22) and (25) and we deduce the solution of Eq. (1) as;

$$v(x, y) = \frac{2.\ ln\left(\frac{1}{c_3^2\lambda^2\cosh^2\left(\frac{y-0.5}{2c_3}\right)}\right).\ 2\,ln\ \frac{1}{c_3^2\lambda^2\cosh^2\left(\frac{-x+0.5}{2c_3}\right)}}{ln\left(\frac{1}{c_3^2\lambda^2\cosh^2\left(\frac{-x+0.5}{2c_3}\right)}\right) + ln\left(\frac{1}{c_3^2\lambda^2\cosh^2\left(\frac{y-0.5}{2c_3}\right)}\right)}. \tag{26}$$

Eq. (26) is the exact solution of (1), that satisfies the (BCs), where;

$$c_3^2 = \frac{1}{\lambda^2\cosh^2\left(\frac{1}{4c_3}\right)}. \tag{27}$$

For some values of λ, the consistent values of c_1 and c_3 are generated and recorded in Table 2. We notice that if $\lambda < \lambda_c$ (2DPBG) has two solutions and if $\lambda > \lambda_c$, it has no solution and at $\lambda = \lambda_c$, the equation has one solution.

The two new solutions, Eqs. (19) and (26)- of the 2D Bratu equation is plotted for various values of λ as depicted in Figure 1, 2 and 3–3.

The acquired solutions are plotted for various estimations of λ, demonstrating that one digression point at $\lambda_c =$ 7.02766, two convergence points for $\lambda < \lambda_c$ and no purposes of intersection point for $\lambda > \lambda_c$. The solution of the two dimensional models is plotted for various estimations of λ.

4. Results analysis

The results obtained here are compared with those obtained in [32],

$$v(x, y) = \frac{-2\ \ ln\left(\frac{\sqrt{\lambda}}{c_3}\cosh\left(\frac{c_3(y-0.5)}{2}\right)\right)(-2)\ \ ln\left(\frac{\sqrt{\lambda}}{c_3}\cosh\left(\frac{c_3(x-0.5)}{2}\right)\right)}{-ln\left(\frac{\sqrt{\lambda}}{c_3}\cosh\left(\frac{c_3(y-0.5)}{2}\right)\right) - ln\left(\frac{\sqrt{\lambda}}{c_3}\cosh\left(\frac{c_3(x-0.5)}{2}\right)\right)}.$$

we investigate that;

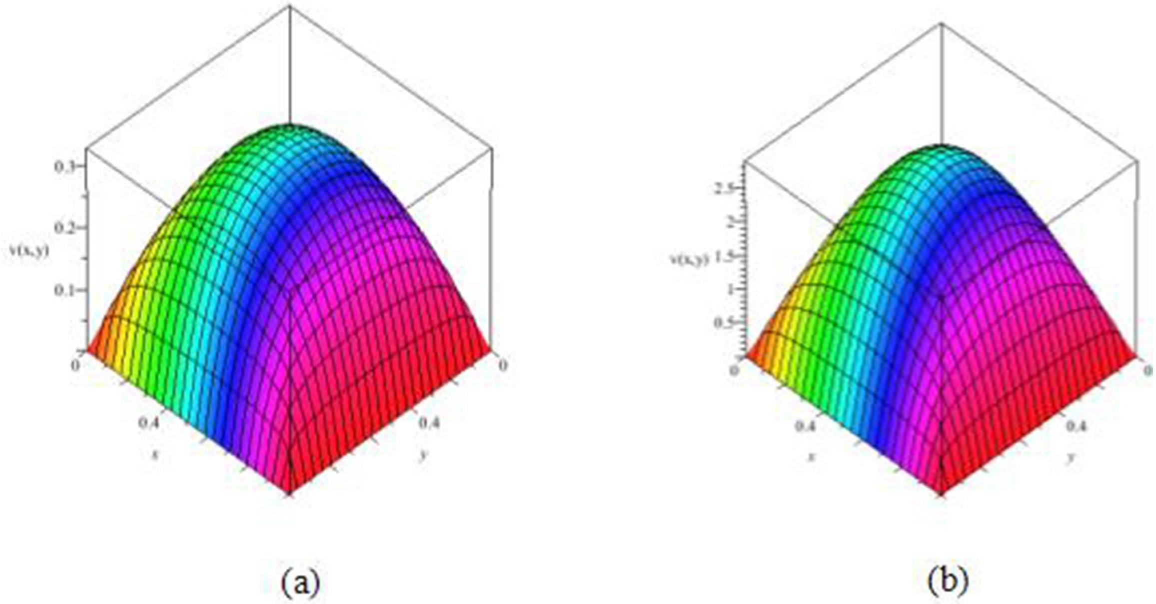

Fig. 2. Solution of Bratu equation for $\lambda = 2$.
(b) Lower branch at $c_1 = 0.424168969I$ and $c_3 = 0.424168969$ **(b)** Upper branch at c_1=0.11754708I and $c_3 = -0.11754708$.

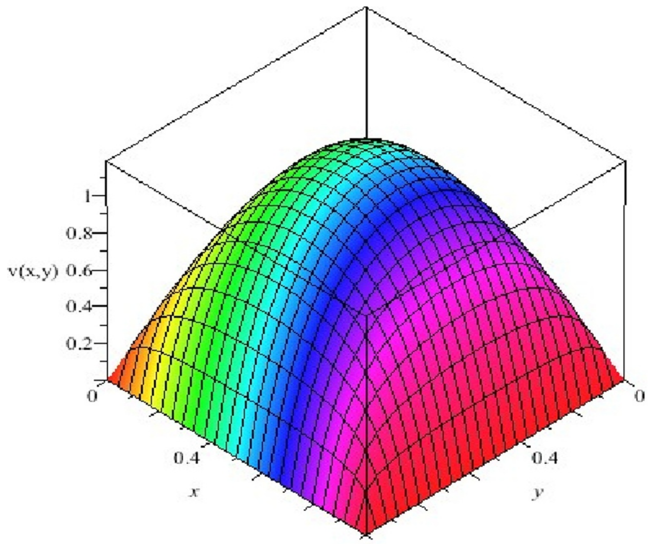

Fig. 3. Solution of Bratu equation for $\lambda = \sqrt{7.02766}$, c_1=0.20838914I and $c_3 = 0.20838914$.

- Our solution in (26) is different than the result in [32].
- By using this method there are two solutions one in real domain and other in complex domain.

5. Conclusions

The non-integrable 2D Bratu equation transferred to integrable equation through the point transformation then we investigate the Lie infinitesimals for the integrable one. We explore new and exclusive solutions for nonlinear elliptic Bratu equation using Lie symmetry reduction method. We apply the (BCs) and deduce the net surface that represent a solution of (1) in unit square domain. Comparing our results with [32], conclude that our solutions are new. We use Newton method as a numerical treatment to get the intersecting points of two different curves.

Declaration of Competing Interest

None.

Appendix A

From the procedure in [16], the value of λ and the corresponding values of c_1 and c_3 are acquired by using Newton Raphson method. For the case of $X_1 + X_3$ vector;

Rewrite the relative equation between λ and c_3 in Eq. (27) as;

$$\lambda = \frac{1}{\phi^2 \cosh\left(\frac{1}{4\phi}\right)^2} \tag{A1}$$

To obtain the critical points for λ, differentiate λ w. r. t. ϕ to get $\frac{d\lambda}{d\phi}$ then equaling the result to zero;

$$\tanh\left(\frac{1}{4\phi_c}\right) - 4\phi_c = 0 \tag{A2}$$

Using Newton Raphson method to obtain the intersecting points for (A.2). Starting from initial value of $\phi_0 = 0.1$ and apply the relation;

$$\phi_{n+1} = \phi_n - \frac{f(\phi)}{f'(\phi)} \tag{A3}$$

After five iterations we obtain;

$\phi_c = 0.20838914$ substitute in (A.1) to get the critical value of λ; we find that $\lambda_c = 7.02766$. Following the same procedure we get the values of $c_{1c}$and λ_c that listed in Table 2.

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