Evolutionary numerical approach for solving nonlinear singular periodic boundary value problems

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Abstract. In this approximation study, a nonlinear singular periodic model in nuclear physics is solved by using the Hermite wavelets (HW) technique coupled with a numerical iteration technique such as the Newton Raphson (NR) one for solving the resulting nonlinear system. The stimulation of offering this numerical work comes from the aim of introducing a consistent framework that has as effective structures as Hermite wavelets. Two numerical examples of the singular periodic model in nuclear physics have been investigated to observe the robustness, proficiency, and stability of the designed scheme. The proposed outcomes of the HW technique are compared with available numerical solutions that established fitness of the designed procedure through performance evaluated on a multiple execution.

Keywords: Singular periodic systems in nuclear physics, Hermite wavelets, hybrid approach, Gaussian formula of integration, collocation technique

1. Introduction

Singular boundary value problems arise in many applications including engineering applications [1–10], the modeling of monster beams [11], spline approach [12], plasma in a Magnetic Field [13]. Henceforth, singular boundary value problems have attracted much attention and have been investigated by many researchers. In [14], the authors demonstrated a particular singular boundary value problem by applying FDM, and then got the same results by applying the cubic spline technique in [15]. In [16], the authors investigated the cubic spline method for solving singular boundary value problems. In [17] VIM was investigated for solving nonlinear singular boundary value problems. The authors of [18] solved some singular boundary value problems by reproducing kernel space.

The literature form of the second order nonlinear singular boundary value problem NS-BVPs is given as [19]:

$$W''(t) + \frac{k}{t}W'(t) + f(t, W) = h(t), t, k \geq 1$$ (1)

At the initial conditions (ICs):

$$W(0) = W_0, W'(0) = W_1,$$
where \( h(t), f(x, \mathcal{W}) \) are continuous functions, \( \mathcal{W}_0, \mathcal{W}_1 \) are constants.

If we pick \( f(x, \mathcal{W}) = f(\mathcal{W}) = \mathcal{W}^n \) then Equation (1) is set as:

\[
\mathcal{W}'(x) + \frac{k}{x} \mathcal{W}'(x) + f(\mathcal{W}) = h(x) \tag{2}
\]

with Dirichlet, Neumann, and Neumann-Robin boundary conditions

\[
\mathcal{W}(0) = \mathcal{W}_0, \quad \mathcal{W}(1) = \mathcal{W}_1, \quad \mathcal{W}(0) = c_0, \quad \mathcal{W}'(1) = c_1,
\]

\[
\mathcal{W}'(0) = 0, \quad v_0 \mathcal{W}(1) + v_2 \mathcal{W}'(0) = v_3. \tag{3}
\]

At \( \mathcal{W}_0 = 1 \), this equation is called the NS-BVPs.

Due to singularity and nonlinearity shown in the model (1), numerical and analytical techniques have been proposed for presenting solutions of these models [20–28]. The purpose of the present numerical study is to examine a consistent numerical computing structure for solving the model (1) by using the HW technique. Many researchers are applying various techniques for solving nonlinear systems [29–32]. The remaining splits of the paper are organized as: the second split describes the explanation of design methodology, explanation of the results is provided in the third split, the application of the proposed method is given in split 4, and finally in split 5, the demonstrated results and discussion and the conclusions are reported.

2. The HW technique

The HW \( i,j(t) \) are demonstrated on \([0, 1]\) as [33]:

\[
HW_{i,j}(t) = \begin{cases} 
\frac{\sqrt{2}}{\pi} H_j(2^k t - 2i + 1) & \frac{i-1}{2^{-k+1}} \leq t < \frac{i}{2^{-k+1}} \\
0 & \text{otherwise}
\end{cases} \tag{4}
\]

where \( k \) is an integer, \( i = 1, 2, ..., 2^k - 1 \), \( j = 0, 1, ..., M - 1 \). Thus, we demonstrate our new hybrid as \( \{ HW_{1,0}, HW_{1,1}, ..., HW_{2^k-1,M-1} \} \) and the function can be approximated with them.

The HW defines the orthonormal basis as:

\[
H_0(t) = 1, \quad H_1(t) = 2t, \quad H_{j+2}(t) = 2t, \quad H_{j+1}(t) - 2(j + 1) H_j(t), \quad j = 0, 1, ..., M - 1
\]

2.1. Function approximation using the HW technique

The function \( \mathcal{W}(t) \) [34–40], which is integrable in \([0, 1]\), can be truncated using the HW technique as follows:

\[
\mathcal{W}(t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} HW_{ij}(t), \quad i = 1, 2, ..., \infty, \quad j = 0, 1, 2, ..., \infty, \quad t \in [0, 1], \tag{6}
\]

where the HW coefficients \( c_{ij} \) are calculated as follows:

\[
c_{ij} = \frac{\langle \mathcal{W}(t), HW_{ij}(t) \rangle}{\langle HW_{ij}(t), HW_{ij}(t) \rangle}.
\]

We prune \( \mathcal{W}(t) \) by a series as follows:

\[
\mathcal{W}(t) = \sum_{i=1}^{2^k-1} \sum_{j=0}^{M-1} c_{ij} HW_{ij}(t) = C^T HW(t) \tag{7}
\]

where \( HW(t) \) and \( C \) are \((2^k - 1) (M - 1) \times 1\) vectors given by

\[
HW(t) = [HW_{1,0}, HW_{1,1}, ..., HW_{1,(M-1)}, HW_{2,0}, HW_{2,1}, ..., HW_{2,(M-1)}, ..., HW_{2^k-1,0}, ..., HW_{2^k-1,(M-1)}]^T
\]

and

\[
C = [c_{10}, c_{11}, ..., c_{1(M-1)}, c_{20}, c_{21}, ..., c_{2(M-1)}, ..., c_{2^k-1,0}, ..., c_{2^k-1,(M-1)}]^T \tag{8}
\]

so

\[
D = \langle HW(t), HW(t) \rangle. \tag{9}
\]

where

\[
D = \langle HW(t), HW(t) \rangle, \tag{10}
\]

Then:

\[
(D_n)_{i+1,j+1} = \int_{\frac{i}{2^{-k+1}}}^{\frac{i+1}{2^{-k+1}}} HW_{i,n}(2^{k-1} t - i + 1) dt \tag{11}
\]

We can approximate the function \( k(x, t) \) as follows:

\[
k(x, t) \approx HW^T(x) KHW(t),
\]
where \( K \) is a \((2^{k-1}) (M - 1) \times (2^{k-1}) (M - 1)\) matrix that we attain as:

\[
K = \mathcal{D}^{-1} \left< \mathbf{H} \mathbf{W}(x) \left< k(x, t), \mathbf{H} \mathbf{W}(t) \right> \right> \mathcal{D}^{-1} \tag{12}
\]

2.2. Multiplication of the hybrid functions

We can evaluate \( \mathbf{H} \mathbf{W}_{(2^{k-1})(M - 1)\times 1}(t) \) \( \mathbf{H} \mathbf{W}_{(2^{k-1})(M - 1)\times 1}^T(t) \) for NS-BVPs of the second kind via the \( \mathbf{H} \mathbf{W} \) functions as detailed below.

Let the product of \( \mathbf{H} \mathbf{W}_{(2^{k-1})(M - 1)\times 1}(t) \) and \( \mathbf{H} \mathbf{W}_{(2^{k-1})(M - 1)\times 1}^T(t) \) be given by

\[
\mathbf{H} \mathbf{W}_{(2^{k-1})(M - 1)\times 1}(t) \mathbf{H} \mathbf{W}_{(2^{k-1})(M - 1)\times 1}^T(t)
\]

\[= \mathcal{M}_{(2^{k-1})(M - 1)\times (2^{k-1})(M - 1)}(t) \]

Then, we calculate \( \mathcal{M}_{((2^{k-1})(M - 1)\times (2^{k-1})(M - 1)}(t) \) for any \( k \) and \( M \).

The matrix \( \mathcal{M}_{((2^{k-1})(M - 1)\times (2^{k-1})(M - 1)}(t) \) satisfies the relation:

\[
\mathcal{M}_{((2^{k-1})(M - 1)\times (2^{k-1})(M - 1)}(t) \mathcal{C}_{(2^{k-1})(M - 1)\times 1}
\]

\[= \tilde{\mathcal{C}}_{(2^{k-1})(M - 1)\times 2^{k-1}(M - 1)} \mathbf{H} \mathbf{W}_{(2^{k-1})(M - 1)\times 1}(t) \]

where \( \tilde{\mathcal{C}}_{(2^{k-1})(M - 1)\times 2^{k-1}(M - 1)} \) is the matrix coefficient. We consider the case when \( k = 3 \) and \( M = 4 \). Thus, we have

\[
\mathbf{M}_{16\times 16}(t) = \begin{bmatrix}
\mathbf{H} \mathbf{W}_{10}(t) & \mathbf{H} \mathbf{W}_{10}(t) & \mathbf{H} \mathbf{W}_{20}(t) & \cdots & \mathbf{H} \mathbf{W}_{10}(t) \mathbf{H} \mathbf{W}_{43}(t) \\
\mathbf{H} \mathbf{W}_{20}(t) & \mathbf{H} \mathbf{W}_{10}(t) & \mathbf{H} \mathbf{W}_{20}(t) & \cdots & \mathbf{H} \mathbf{W}_{20}(t) \mathbf{H} \mathbf{W}_{43}(t) \\
\mathbf{H} \mathbf{W}_{30}(t) & \mathbf{H} \mathbf{W}_{10}(t) & \mathbf{H} \mathbf{W}_{20}(t) & \cdots & \mathbf{H} \mathbf{W}_{30}(t) \mathbf{H} \mathbf{W}_{43}(t) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\mathbf{H} \mathbf{W}_{43}(t) & \mathbf{H} \mathbf{W}_{10}(t) & \mathbf{H} \mathbf{W}_{43}(t) & \cdots & \mathbf{H} \mathbf{W}_{43}(t) \mathbf{H} \mathbf{W}_{43}(t)
\end{bmatrix}
\tag{14}
\]

Several of the integral \( \int_a^b F(x, t) \, dt \) using Leibniz rule as:

\[
\int_A^B F(x, t) \, dt = \int_A^B \frac{d}{dx} F(x, t) \, dt
\]

\[
= F(x, b(x)) \frac{db}{dx} - F(x, a(x)) \frac{da}{dx} + \int_A^B \frac{dF}{dx} \, dt,
\]

where \( F(x, t) \) and \( \frac{dF}{dx} \) are continuous in the domain \( D \) in the \( xt \)-plane that contains the region \( R \), \( \alpha \leq x \leq \beta \), \( t_0 \leq t \leq t_n \) and \( a(x), b(x) \) are functions having continuous derivatives for \( \alpha \leq x \leq \beta \). A global

Leibniz rule presented modifies NS-BVPs to several equations.

3. Analysis of the \( \mathbf{H} \mathbf{W} \) technique

In this split, we discuss the NS-BVPs of the shape factor of the form

\[
\mathcal{W}''(x) + \frac{k}{x} \mathcal{W}(x) + f(\mathcal{W}) = h(x),
\]

\[
\mathcal{W}(0) = \alpha, \mathcal{W} (0) = 0, k > 1
\tag{15}
\]

where \( f(\mathcal{W}) \) can take any linear or nonlinear forms.

First, we set

\[
\mathcal{W}''(x) = \alpha - \frac{1}{k-1} \int_0^x (1 - \frac{t^{k-1}}{x^{k-1}}) f(\mathcal{W}(t) - h(t)) \, dt.
\tag{16}
\]

Differentiate Equation (16) twice then using the Leibniz rule, we obtain;

\[
\mathcal{W}'(x) = \int_0^x \mathcal{W}'(t) h(t) \, dt - \int_0^x \frac{x}{t} f(\mathcal{W}(t)) \, dt.
\tag{17}
\]

\[
\mathcal{W}'(x) = h(x) - \int_0^x \frac{x}{k} (\frac{x^k}{x^{k+1}}) h(t) \, dt - f(\mathcal{W}(x)) + \int_0^x \mathcal{W}'(t) h(t) \, dt.
\tag{18}
\]

If we multiply \( \mathcal{W}'(x) \) in (17) by \( \frac{x}{k} \) and add to \( \mathcal{W}''(x) \) in (18) we get the following equation. That is the NS-BVPs (18) is given by:

\[
\mathcal{W}(x) = \int_0^x \mathcal{W}'(t) h(t) \, dt - \int_0^x \frac{x}{k} f(\mathcal{W}(t)) \, dt, \mathcal{W}(0) = \alpha.
\tag{19}
\]
For $k \rightarrow 1$, the integral form of Equation (19) is
\[ W(x) = \alpha + \int_0^x t(\ln(\frac{I}{x})) f(W(t)) - h(t) dt \quad (20) \]

Based on this, the NS-BVPs forms are:
\[
\begin{aligned}
W(x) &= \alpha + \int_0^x t(\ln(\frac{I}{x})) f(W(t)) - h(t) dt, \text{ when } nk = 1, \\
W(x) &= \alpha - \frac{1}{k-1} \int_0^x f(t)(1 - \frac{e^{(k-1)}}{e^{x-1}}) f(W(t)) - h(t) dt, \text{ when } nk > 1.
\end{aligned}
\]

4. Application of \( H W \) technique for NS-BVPs

The unknown function \( W(x) \) in (20) is approximated by the \( H W \) technique as:
\[ W(x) \approx \sum_{i=1}^{\mathcal{M}} \sum_{j=0}^n c_{ij} H W_{ij}(x) = C^T H W(x). \quad (22) \]

First, integrating Equation (20) and from the condition \( W(0) = \alpha \), one gets
\[ W(x) = \alpha + \int_0^x t(\ln(\frac{I}{x})) h(x) dx dz - \sum_{i=1}^{\mathcal{M}} \sum_{j=0}^n c_{ij} H W_{ij}(x) = C^T H W(x). \quad (23) \]

From Equations (22) and (23), we have;
\[ C^T H W(x) = \alpha + \int_0^x t(\ln(\frac{I}{x})) h(x) dx dz - \sum_{i=1}^{\mathcal{M}} \sum_{j=0}^n c_{ij} H W_{ij}(x) = C^T H W(x). \quad (24) \]
\[ = \alpha + \int_0^x H_1(z) dz - \int_0^x H_2(z) dz, \quad (25) \]

where,
\[ H_1(z) = \int_0^x (\frac{z^k}{x^k}) h(x) dx, \quad (26) \]
\[ H_2(z) = \int_0^x (\frac{z^k}{x^k}) f(C^T H W(t)) dt, \quad (27) \]

Now we collocate the Equation (26) at the points
\[ x_i = \frac{(2l-1)}{2((2l-1)M)}, \] yielding
\[ C^T H W(x_i) = \alpha + \int_0^x H_1(z) dz - \int_0^x H_2(z) dz. \quad (29) \]

To apply the Gaussian integration to Equation (26), we use the transformation
\[ t = 2(x_i - z - 1), \quad (30) \]
to transfer the interval of integration into the interval \([-1, 1]\). Equation (26) becomes:
\[ C^T H W(x_i) = \alpha + \frac{x_i}{2} \int_{-1}^{1} H_1(\frac{x_i}{2}(T + 1)) dT - \frac{x_i}{2} \int_{-1}^{1} H_2(\frac{x_i}{2}(T + 1)) dT. \quad (31) \]

Using the Gaussian integration formula, we get
\[ C^T H W(x_i) \approx \alpha + \frac{x_i}{2} \sum_{j=1}^{s_1} W_{j_1} H_1 \left( \frac{x_i}{2} (T_{j_1} + 1) \right) - \frac{x_i}{2} \sum_{j_2=1}^{s_2} W_{j_2} H_2 \left( \frac{x_i}{2} (T_{j_2} + 1) \right), \quad (32) \]

where \( s_1, s_2, T_{j_1} \) and \( T_{j_2} \) are zeros of Legendre polynomials \( p_{s_1+1} (\cdot) \) and \( p_{s_2+1} (\cdot) \) and \( W_{j_1} \) and \( W_{j_2} \) are weights. After that we can use the Newton’s technique to get the values of \( C \) then we get the solution.

**Theorem 4.1.** Let \( X \) be the Banach space with \( ||W|| = \max_{0 \leq t \leq 1} |W(x)|, W \in X \) and \( f(x, W) \) fulfill the Lipschitz constrain, i.e.,
\[ |f(x, r_1) - f(x, r_2)| \leq l |r_1 - r_2|, \forall r_1, r_2 \in X. \quad (33) \]
Let \( \delta \) denote as \( \delta = \frac{b (a+b)}{2(a+1)a} \).
If $\delta < 1$, then $W = N(W)$ has a unique solution in $X$.

**Proof.**

\[
\|N(Y_1) - N(Y_2)\| = \max_{0 \leq t \leq 1} \left\| \int_0^t G(x, \xi) f(x, Y_1) d\xi - \int_0^t G(x, \xi) f(x, Y_2) d\xi \right\| 
\leq \max_{0 \leq t \leq 1} \int_0^t \left| f(x, Y_1) - f(x, Y_2) \right| d\xi 
\leq \delta \max_{0 \leq t \leq 1} \int_0^t \| G(x, \xi) \| d\xi.
\]

(34)

Then

\[
\|N(Y_1) - N(Y_2)\| \leq \max_{0 \leq t \leq 1} |Y_1 - Y_2| = \delta \|Y_1 - Y_2\|.
\]

(35)

\[
\|N(Y_1) - N(Y_2)\| \leq \|Y_1 - Y_2\|.
\]

(36)

**Theorem 4.2.** Let $N(W)$ be the nonlinear operator that fulfills the Lipschitz condition. If $\|W_0\| < \infty$, then $\|W_{k+1}\| \leq \delta \|W_k\|$, $k = 0, 1, 2, \ldots$ and the sequence $Y_n$ characterized by

\[
Y_n(x) = \sum_{j=0}^{n} W_j(x),
\]

converges to the exact solution $W$.

**Proof.** We have

\[
Y_1 = W_0 + W_1, \quad Y_2 = W_0 + W_1 + W_2, \ldots, \quad Y_n = W_0 + W_1 + W_2 + \ldots + W_n, \ldots.
\]

Thus, we find

\[
W_{k+1} = Y_{k+1} - Y_k, \quad k = 1, 2, \ldots.
\]

We now prove that the sequence $\{Y_k\}$ is convergent. We obtain;

\[
\|W_{n+1}\| = \|Y_{n+1} - Y_n\| = \|NY_n - NY_{n-1}\| 
\leq \delta \|Y_{n+1} - Y_{n-1}\|.
\]

and hence

\[
\|Y_{n+1} - Y_n\| = \|W_{n+1}\| \leq \delta \|W_{n+1}\| 
\leq \delta^2 \|W_{n-1}\| \leq \ldots \leq \delta^{n+1} \|W_0\|.
\]

(37)

For $n, m \in \mathbb{N}$, with $n \geq m$, we have,

\[
\|Y_n - Y_m\| = \|(Y_n - Y_{n-1}) + (Y_{n-1} - Y_{n-2}) + \ldots + (Y_{m+1} - Y_m)\|.
\]

(38)

\[
\leq \|Y_n - Y_{n-1}\| + \|Y_{n-1} - Y_{n-2}\| + \ldots + \|Y_{m+1} - Y_m\|.
\]

(39)

\[
\leq \delta^n \|W_0\| + \delta^{n-1} \|W_0\| + \ldots + \delta^{m+1} \|W_0\|.
\]

(40)

\[
= \delta^{m+1} \left( 1 + \delta + \delta^2 + \ldots + \delta^{n-m} \right) \|W_0\|.
\]

(41)

\[
= \frac{\delta^{m+1} (1 + \delta^{n-m})}{1 - \delta} \|W_0\|.
\]

(42)

Since $0 < \delta < 1$, we have $(1 - \delta^{n-m}) < 1$. It readily follows that;

\[
\|Y_n - Y_m\| \leq \frac{\delta^{m+1}}{1 - \delta} \|W_0\|.
\]

(43)

Taking limit as $m \to \infty$, then we have $\|Y_n - Y_m\| \to 0$.

**Theorem 4.3.** Let $W(x)$ be the exact solution of the operator equation $W = NW$. Let $m(x)$ be the sequence of approximate series solutions defined by

\[
W_n(x) = \sum_{j=0}^{m} W_j(x).
\]

Then have

\[
\max_{0 \leq t \leq 1} \left| W(t) - \sum_{j=0}^{m} W_j(x) \right| \leq \frac{\delta^{m+1}}{1 - \delta} \|W_0\|.
\]

(44)

**Proof.** For $n \geq m$, using $\|W - Y_m\| \leq \frac{\delta^{m+1}}{1 - \delta} \|W_0\|$, we have

\[
\|Y_n - Y_m\| \leq \frac{\delta^{m+1}}{1 - \delta} \|W_0\|.
\]

(45)

Since $\lim_{n \to \infty} Y_n = W$, fixing $m$ and letting $n \to \infty$, we obtain

\[
\|W - Y_m\| \leq \frac{\delta^{m+1}}{1 - \delta} \|W_0\|.
\]

(46)

5. Consequence and investigation

In this split, two variants based on the second order NS-BVPs have been taken and the comparison of the present technique with the exact results will also be discussed.
Example 1. Consider the linear second-order NSP-BVP given by:

\[
\begin{align*}
\frac{d^2W(t)}{dt^2} + \frac{2}{p(1-t)^{1/3}} \frac{dW(t)}{dt} + \frac{1}{p(1-t)^{5/3}} W(t) &= f(t), \quad 0 < t < 1, \\
W(0) = W(1), \quad \frac{dW(0)}{dt} = \frac{dW(1)}{dt}.
\end{align*}
\] (47)

The exact solution \( W(x) = x^2 + x^3 \).

Using the HW, the truncate solution \( W(x) \) of (37) at \( 2^{k-1} = 4 \) and \( M = 3 \) technique, we get 16 number algebraic equations with the same number of unknowns at \( 2^{k-1} = 4 \), \( M = 3 \) and these equations are solved by Newton’s technique with maple program, we get the HW coefficients as:

\[ \begin{array}{cccccc}
C &=& 0,0, & 0.02083337654, & 0.078129087, & 0.07812586, \\
& & & 0.1354153427, & 0.22918967655, & 0.3745632, \\
& & & 0.8330000003, & 0.8330000003, & 0.8330000003, \\
& & & 0.1520000007, & 0.1520000007, & 0.25 \times 10^{-15}, 0.
\end{array} \]

So, the truncate of \( W(x) \) is \( W(x) = C^T HW(x) \).

The acquired results have been compared with that of our seventh order (ADM) along with the required solutions and introduced in Table 1. The outcomes reveal that the results by HW, with using only a small number of bases, are very promising and superior to ADM and evaluated absolute errors (AE) by HW for \( W(x) \) will be decreased rapidly in comparison with ADM.

Example 2. Consider the NSP-BVP s given by:

\[
\begin{align*}
\frac{d^2W(t)}{dt^2} + \frac{2}{p(1-t)^{1/3}} \frac{dW(t)}{dt} + \frac{1}{p(1-t)^{5/3}} W(t) + W^2(t) &= f(t), \quad 0 < t < 1, \\
W(0) = W(1), \quad \frac{dW(0)}{dt} = \frac{dW(1)}{dt}.
\end{align*}
\]

This equation can be transformed into the NS-BVPs form as follows

\[ W'(x) = -\int_0^x \frac{t^2}{x^8} W^m(t) \, dt, \quad W(0) = 1, \quad g \geq 1 \]

Table 2 and 3, exhibit the numerical solutions of \( W(t) \) by HW at \( r = m = 0, g = 2 \), and \( r = 0, m = 1, g = 2 \).

**a.** For \( r = m = 0, g = 2 \), the above equation has the accurate solution

\[ W(x) = 1 - \frac{x^2}{6}. \]

Table 3 shows that, for \( g = 2, m = 0 \), the acquired results coincide with the required solution and efficiency of the technique described through the AE.

By applying the HW technique, and taking \( 2^k = 4, M = 3, HW \) then the value of \( C \) is:

\[ [1, 1, 0.879967, 0.9378902, 0.94378094, 0.9849876, 0.9946874, 0.946789, 0.936708, 0.964838, 0.942086, 0.94890328, 0.94289, 0.82478 \ldots, 0.87498727, 0.8347664]. \]
The accurate solutions for this problem are given by \( W(x) = \frac{\sin(x)}{x} \).

Applying the \( \mathbb{H}W \) technique, and taking \( 2^k = 4, M = 3 \), we note that \( \mathbb{H}W \) coefficients \( C \) as
\[
0.965622, 1.766432345, 0.9524433, 0.98642742, 0.9272652, 0.9354247, 0.951467, 0.9642556, 0.9524552, 0.9452464, 0.9642355, 0.9524323, 0.9254334, 0.82345434, 0.8124564, 0.8254336.
\]

Designing this technique and taking \( 2^k = 8, M = 7 \), if \( 2^k, M \) increases, the truncate solution gets the required solution. The truncate solutions acquired by \( \mathbb{H}W \) at \( g = 2 \) and \( 2^k = 4, M = 3 \), at \( 2^k = 8, M = 7 \) with require solutions and \( \text{AE} \) are demonstrated in Tables 2. By increasing, \( 2^k, M \) the computed results have appropriated the exactness better as well, and the truncate solution gets an approximate to the accurate solution as demonstrated in Table 3. The CPU time for running a case may depend on the choice of \( 2^k, M \) for solving the system of linear algebraic equations resulting from the discretized equations. If different values \( 2^k, M \) are used, the CPU time would be different.

6. Conclusions

In this article, the design of the \( \mathbb{H}W \) algorithm was shown to solve the NS-BVPs. The demonstration of the present study was carried out by using wavelets and rapid fine modification of Hermite. The particular merit is that it can be applied very well without change of BVPs into IVPs. The nonlinear singular model based on periodic boundary value problems was assessed effectively by the present approximation technique based on the Hermite technique and accurateness of numerical outcomes was observed. The \( \mathbb{H}W \) scheme’s accuracy was demonstrated by finding the matching outcomes with the exact solutions having 6 decimal places of overlapping for solving nonlinear model based on singular periodic boundary value problem. The important advantage of the suggested technique is to collect scarce terms of the Hermite polynomials so that a higher-order approximation.

References


