Journal of Intelligent & Fuzzy Systems 39 (2020) 7723–7731
DOI:10.3233/JIFS-201045
IOS Press

Evolutionary numerical approach for solving nonlinear singular periodic boundary value problems

Mohamed R. Ali[a], Adel R. Hadhoud[b] and Wen-Xiu Ma[c,d,e,f,*]
[a]*Department of Basic Science, Faculty of Engineering at Benha, Benha University, Egypt*
[b]*Department of Mathematics, Faculty of Science, Menoufia University, Egypt*
[c]*Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, China*
[d]*Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia*
[e]*Department of Mathematics and Statistics, University of South Florida, Tampa, USA*
[f]*School of Mathematics, South China University of Technology, Guangzhou, China*

Abstract. In this approximation study, a nonlinear singular periodic model in nuclear physics is solved by using the Hermite wavelets (HW) technique coupled with a numerical iteration technique such as the Newton Raphson (NR) one for solving the resulting nonlinear system. The stimulation of offering this numerical work comes from the aim of introducing a consistent framework that has as effective structures as Hermite wavelets. Two numerical examples of the singular periodic model in nuclear physics have been investigated to observe the robustness, proficiency, and stability of the designed scheme. The proposed outcomes of the HW technique are compared with available numerical solutions that established fitness of the designed procedure through performance evaluated on a multiple execution.

Keywords: Singular periodic systems in nuclear physics, Hermite wavelets, hybrid approach, Gaussian formula of integration, collocation technique

1. Introduction

Singular boundary value problems arise in many applications including engineering applications [1–10], the modeling of monster beams [11], spline approach [12], plasma in a Magnetic Field [13]. Henceforth, singular boundary value problems have attracted much attention and have been investigated by many researchers. In [14], the authors demonstrated a particular singular boundary value problem by applying FDM, and then got the same results by applying the cubic spline technique in [15]. In [16], the authors investigated the cubic spline method for solving singular boundary value problems. In [17] VIM was investigated for solving nonlinear singular boundary value problems. The authors of [18] solved some singular boundary value problems by reproducing kernel space.

The literature form of the second order nonlinear singular boundary value problem NS-BVPs is given as [19]:

$$\mathcal{W}''(t) + \frac{k}{t}\mathcal{W}'(t) + f(t, \mathcal{W}) = \hbar(t), \; t, k \geq 1 \quad (1)$$

At the initial conditions (ICs) :

$$\mathcal{W}(0) = \mathcal{W}_0, \mathcal{W}'(0) = \mathcal{W}_1,$$

*Corresponding author. Wen-Xiu Ma. E-mail: mawx@cas.usf.edu.

where $\hbar(t)$, $f(x,\mathcal{W})$ are continuous functions, $\mathcal{W}_0$, $\mathcal{W}_1$ are constants.

If we pick $f(x,\mathcal{W}) = f(\mathcal{W}) = \mathcal{W}^n$ then Equation (1) is set as:

$$\mathcal{W}''(x) + \frac{k}{x}\mathcal{W}'(x) + f(\mathcal{W}) = \hbar(x) \tag{2}$$

with Dirichlet, Neumann, and Neumann-Robin boundary conditions

$$\mathcal{W}(0)=\mathcal{W}_0,\ \mathcal{W}(1)=\mathcal{W}_1,\ \mathcal{W}(0) = c_0,\ \mathcal{W}'(1) = c_1,$$

$$\mathcal{W}'(0) = 0,\ v_0\mathcal{W}(1) + v_2\mathcal{W}'(0) = v_3. \tag{3}$$

At $\mathcal{W}_0 = 1$, this equation is called the NS-BVPs.

Due to singularity and nonlinearity shown in the model (1), numerical and analytical techniques have been proposed for presenting solutions of these models [20–28]. The purpose of the present numerical study is to examine a consistent numerical computing structure for solving the model (1) by using the HW technique. Many researchers are applying various techniques for solving nonlinear systems [29–32]. The remaining splits of the paper are organized as: the second split describes the explanation of design methodology, explanation of the results is provided in the third split, the application of the proposed method is given in split 4, and finally in split 5, the demonstrated results and discussion and the conclusions are reported.

2. The ӇW technique

The $HW_{i,j}(t)$ are demonstrated on [0, 1) as [33]:

$$HW_{i,j}(t) = \begin{cases} 2^{\frac{k}{2}}\frac{\sqrt{2}}{\pi}H_j(2^k t - 2i + 1)^{\blacksquare} & \frac{i-1}{2^{k-1}} \le t < \frac{i}{2^{k-1}} \\ 0 & ot\hbar erewise \end{cases} \tag{4}$$

where k is an integer, $i = 1, 2, ..., 2^{k-1}$, $j = 0, 1, ..., \mathcal{M} - 1$. Thus, we demonstrate our new hybrid as $\{\mathrm{HW}_{1,0}, \mathrm{HW}_{1,1}, ..., \mathrm{HW}_{2^{k-1},\mathcal{M}-1}\}$ and the function can be approximated with them.

The ӇW defines the orthonormal basis as:

$$H_0(t) = 1,\ H_1(t) = 2t,\ H_{j+2}(t) = 2t, H_{j+1}(t) - 2(j+1)H_j(t),\ j=0, 1, ..., \mathcal{M}-1 \tag{5}$$

2.1. Function approximation using the ӇW technique

The function $\mathcal{W}(t)$ [34–40], which is integrable in [0, 1), can be truncated using the ӇW technique as follows:

$$\mathcal{W}(t) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{ij}HW_{ij}(t),\ i = 1, 2, ..., \infty,$$
$$j = 0, 1, 2, ..., \infty, t \in [0, 1), \tag{6}$$

where the ӇW coefficients c_{ij} are calculated as follows:

$$c_{ij} = \frac{\left(\mathcal{W}(t), \text{ӇW}_{ij}(t)\right)}{\left(\text{ӇW}_{ij}(t), \text{ӇW}_{ij}(t)\right)}$$

We prune $\mathcal{W}(t)$ by a series as follows:

$$\mathcal{W}(t) = \sum_{i=1}^{2^{k-1}}\sum_{j=0}^{\mathcal{M}-1} c_{ij}HW_{ij}(t) = \mathcal{C}^T\text{ӇW}(t) \tag{7}$$

where ӇW (t) and $\mathcal{C}$ are $\left(2^{k-1}\right)(\mathcal{M}-1)\times 1$ vectors given by

$$\text{ӇW}(t) = [\text{ӇW}_{10}, \text{ӇW}_{11}, ..., \text{ӇW}_{1(\mathcal{M}-1)}, \text{ӇW}_{20}, \text{ӇW}_{21}, ..., \text{ӇW}_{2(\mathcal{M}-1)}, ..., \text{ӇW}_{2^{k-1}0}, ..., \text{ӇW}_{2^{k-1}(\mathcal{M}-1)}].^T$$

and

$$\mathcal{C} = \left[c_{10}, c_{11}, ..., c_{1(\mathcal{M}-1)}\right), c_{20}, c_{21}, ..., c_{2(\mathcal{M}-1)}, ..., c_{2^{k-1}0}, ..., c_{2^{k-1}(\mathcal{M}-1)}]^T, \tag{8}$$

so

$$\mathcal{C} = \mathcal{D}^{-1}\left\langle \mathcal{W}(t), \text{ӇW}(t)\right\rangle. \tag{9}$$

where

$$\mathcal{D} = \left\langle \text{ӇW}(t), \text{ӇW}(t)\right\rangle, \tag{10}$$

Then;

$$(\mathcal{D}_n)_{i+1,j+1} = \int_{\frac{i-1}{2^{k-1}}}^{\frac{i}{2^{k-1}}} \text{ӇW}_{i,n}\left(2^{k-1}t - i + 1\right) \text{ӇW}_{j,n}\left(2^{k-1}t - i + 1\right) dt \tag{11}$$

We can approximate the function $k(x,t)$ as follows:

$$k(x,t) \approx \text{ӇW}^T(x)\, K\text{ӇW}(t),$$

where K is a $\left(2^{k-1}\right)(\mathcal{M}-1)\times\left(2^{k-1}\right)(\mathcal{M}-1)$ matrix that we attain as:

$$K=\mathcal{D}^{-1}\left\langle \text{ḤW}(x)\left\langle k(x,t),\text{ḤW}(t)\right\rangle\right\rangle \mathcal{D}^{-1} \quad (12)$$

2.2. Multiplication of the hybrid functions

We can evaluate $\text{ḤW}_{((2^{k-1})(\mathcal{M}-1)\times 1)}(t)$ $\text{ḤW}^{T}_{((2^{k-1})(\mathcal{M}-1)\times 1)}(t)$ for NS-BVPs of the second kind via the ḤW functions as detailed below.

Let the product of $\text{ḤW}_{((2^{k-1})(\mathcal{M}-1)\times 1)}(t)$ and $\text{ḤW}^{T}_{((2^{k-1})(\mathcal{M}-1)\times 1)}(t)$ be given by

$$\text{ḤW}_{((2^{k-1})(\mathcal{M}-1)\times 1)}(t)\,\text{ḤW}^{T}_{((2^{k-1})(\mathcal{M}-1)\times 1)}(t) \cong \mathcal{M}_{((2^{k-1})(\mathcal{M}-1)\times(2^{k-1})(\mathcal{M}-1))}(t) \quad (13)$$

Then, we calculate $\mathcal{M}_{((2^{k-1})(\mathcal{M}-1)\times(2^{k-1})(\mathcal{M}-1))}(t)$ for any k and $\mathcal{M}$.

The matrix $\mathcal{M}_{((2^{k-1})(\mathcal{M}-1)\times(2^{k-1})(\mathcal{M}-1))}(t)$ satisfies the relation:

$$\mathcal{M}_{(2^{k-1}(\mathcal{M}-1)\times 2^{k-1}(\mathcal{M}-1))}(t)\,\mathcal{C}_{(2^{k-1}(\mathcal{M}-1)\times 1)} = \tilde{\mathcal{C}}_{(2^{k-1}(\mathcal{M}-1)\times 2^{k-1}(\mathcal{M}-1))}\text{ḤW}_{(2^{k-1}(\mathcal{M}-1)\times 1)}(t)$$

where $\tilde{\mathcal{C}}_{(2^{k-1}(\mathcal{M}-1)\times 2^{k-1}(\mathcal{M}-1))}$ is the matrix coefficient. We consider the case when $k=3$ and $\mathcal{M}=4$. Thus, we have

$$M_{(16)\times(16)}(t)=\begin{bmatrix} \text{ḤW}_{10}(t)\,\text{ḤW}_{10}(t) & \text{ḤW}_{10}(t)\,\text{ḤW}_{20}(t) & \cdots & \text{ḤW}_{10}(t)\,\text{ḤW}_{43}(t) \\ \text{ḤW}_{20}(t)\,\text{ḤW}_{10}(t) & \text{ḤW}_{20}(t)\,\text{ḤW}_{20}(t) & \cdots & \text{ḤW}_{20}(t)\,\text{ḤW}_{43}(t) \\ \text{ḤW}_{30}(t)\,\text{ḤW}_{10}(t) & \text{ḤW}_{30}(t)\,\text{ḤW}_{20}(t) & \cdots & \text{ḤW}_{30}(t)\,\text{ḤW}_{43}(t) \\ \vdots & \vdots & \cdots & \vdots \\ \text{ḤW}_{43}(t)\,\text{ḤW}_{10}(t) & \text{ḤW}_{43}(t)\,\text{ḤW}_{20}(t) & \cdots & \text{ḤW}_{43}(t)\,\text{ḤW}_{43}(t) \end{bmatrix} \quad (14)$$

Several of the integral $\int\limits_{a(x)}^{b(x)} F(x,t)dt$ using Leibniz rule as:

$$\frac{d}{dx}\int\limits_{a(x)}^{b(x)} F(x,t)\,dt = F(x,b(x))\frac{db}{dx}-F(x,a(x))\frac{da}{dx}+\int\limits_{a(x)}^{b(x)}\frac{dF}{dx}dt,$$

where $F(x,t)$ and $\frac{dF}{dx}$ are continuous in the domain D in the xt- plane that contains the region $R, \alpha \le x \le \beta, t_0 \le t \le t_n$ and $a(x)$,$b(x)$ are functions having continuous derivatives for $\alpha \le x \le \beta$. A global Leibniz rule presented modifies NS-BVPs to several equations.

3. Analysis of the ḤW technique

In this split, we discuss the NS-BVPs of the shape factor of the form

$$\mathcal{W}''(x)+\frac{k}{x}\mathcal{W}'(x)+f(\mathcal{W})=\hbar(x),$$
$$\mathcal{W}(0)=\alpha, \mathcal{W}'(0)=0, k>1 \quad (15)$$

where $f(\mathcal{W})$ can take any linear or nonlinear forms.

First, we set

$$\mathcal{W}'(x)=\alpha-\frac{1}{k-1}\int\limits_{0}^{x}(t(1-\frac{t^{k-1}}{x^{k-1}})f(\mathcal{W}(t))-\hbar(t))dt. \quad (16)$$

Differentiate Equation (16) twice then using the Leibniz rule, we obtain;

$$\mathcal{W}'(x)=\int\limits_{0}^{x}(\frac{t^k}{x^k})\hbar(t)\,dt-\int\limits_{0}^{x}(\frac{t^k}{x^k})f(\mathcal{W}(t))\,dt, \quad (17)$$

$$\mathcal{W}''(x)=\hbar(x)-\int\limits_{0}^{x}k(\frac{t^k}{x^{k+1}})\hbar((t))\,dt-f(\mathcal{W}(x)) + \int\limits_{0}^{x}k(\frac{t^k}{x^{k+1}})f(\mathcal{W}(t))\,dt. \quad (18)$$

If we multiply $\mathcal{W}'(x)$ in (17) by $\frac{k}{x}$ and add to $\mathcal{W}''(x)$ in (18) we get the following equation. That is the NS-BVPs (18) is given by:

$$\mathcal{W}'(x)=\int\limits_{0}^{x}(\frac{t^k}{x^k})\hbar(t)\,dt-\int\limits_{0}^{x}(\frac{t^k}{x^k})f(\mathcal{W}(t))\,dt, \mathcal{W}(0)=\alpha. \quad (19)$$

For $k \to 1$, the integral form of Equation (19) is

$$\mathcal{W}(x) = \alpha + \int_0^x t(\ln(\frac{t}{x}) f(\mathcal{W}(t)) - \hbar(t))dt \quad (20)$$

Based on this, the NS-BVPs forms are:

$$\begin{cases} W(x) = \alpha + \int_0^x t(\ln(\frac{t}{x}) f(W(t)) - \hbar(t))dt, \; when k = 1, \\ W(x) = \alpha - \frac{1}{k-1} \int_0^x (t(1 - \frac{t^{k-1}}{x^{k-1}}) f(W(t)) - \hbar(t))dt, \; when k > 1. \end{cases} \quad (21)$$

4. Application of ḤW technique for NS-BVPs

The unknown function $\mathcal{W}(x)$ in (20) is approximated by the ḤW technique as:

$$\mathcal{W}(x) \cong \sum_{i=1}^{\mathcal{M}} \sum_{j=0}^{n} c_{ij} \mathbf{H}\mathrm{W}_{ij}(x) = \mathcal{C}^T \mathbf{H}\mathrm{W}(x). \quad (22)$$

First, integrating Equation (20) and from the condition $\mathcal{W}(0) = \alpha$, one gets

$$\mathcal{W}(x) = \alpha + \int_0^x [\int_0^z (\frac{x^k}{z^k}) \hbar(x)\, dx] dz - \int_0^x [\int_0^z (\frac{t^k}{z^k}) f(\mathcal{W}(t))\, dt] dz, \; k \geq 1. \quad (23)$$

From Equations (22) and (23), we have;

$$\mathcal{C}^T \mathbf{H}\mathrm{W}(x) = \alpha + \int_0^x [\int_0^z (\frac{x^k}{z^k}) \hbar(x)\, dx] dz - \int_0^x [\int_0^z (\frac{t^k}{z^k}) f(\mathcal{W}(t))\, dt] dz, \; k \geq 1. \quad (24)$$

$$= \alpha + \int_0^x \mathrm{H}_1(z) dz - \int_0^x \mathrm{H}_2(z) dz, \quad (25)$$

where,

$$\mathrm{H}_1(z) = \int_0^z \left(\frac{x^k}{z^k} \right) \hbar(x) dx, \quad (26)$$

where $\mathcal{D}$ is a $\left(2^{\mathrm{k}-1}\right)(\mathcal{M}-1) \times \left(2^{\mathrm{k}-1}\right)(\mathcal{M}-1)$ matrix, and is said the dual matrix of ḤW (x)

$$\mathcal{D} = \langle HW(x), HW(x) \rangle = \int_0^1 HW(x)\, HW^T(x)\, dx \quad (27)$$

$$\mathrm{H}_2(z) = z \int_0^z \left(\frac{t^k}{z^k} \right) f(C^T \mathbf{H}\mathrm{W}(t)) dt, \quad (28)$$

Now we collocate the Equation (26) at the points $x_i = \frac{(2i-1)}{2((2^{k-1})(\mathcal{M}))}$ yielding

$$C^T \mathbf{H}\mathrm{W}(x_i) = \alpha + \int_0^{x_i} \mathrm{H}_1(z) dz - \int_0^{x_i} \mathrm{H}_2(z) dz, \quad (29)$$

To apply the Gaussian integration to Equation (26), we use the transformation

$$\tau = \frac{2}{x_i} z - 1, \quad (30)$$

to transfer the interval of integration into the interval [–1, 1]. Equation (26) becomes:

$$\mathcal{C}^T \mathbf{H}\mathrm{W}(x_i) = \alpha + \frac{x_i}{2} \int_{-1}^{1} \mathrm{H}_1(\frac{x_i}{2}(\mathcal{T}+1) d\tau - \frac{x_i}{2} \int_{-1}^{1} \mathrm{H}_2(\frac{x_i}{2}(\mathcal{T}+1) d\mathcal{T}, \quad (31)$$

Using the Gaussian integration formula, we get

$$\mathcal{C}^T \mathbf{H}\mathrm{W}(x_i) \cong \alpha + \frac{x_i}{2} \sum_{j_1=1}^{s_1} \mathcal{W}_{j_1} \mathrm{H}_1 \left(\frac{x_i}{2} \left(\mathcal{T}_{j1} + 1 \right) \right) - \frac{x_i}{2} \sum_{j_2=1}^{s_2} \mathcal{W}_{j_2} \mathrm{H}_2 \left(\frac{x_i}{2} \left(\mathcal{T}_{j2} + 1 \right) \right), \quad (32)$$

where s_1 and s_2, $\mathcal{T}_{j1}$ and $\mathcal{T}_{j2}$ are zeros of Legendre polynomials $p_{s_1+1}(.)$ and $p_{s_2+1}(.)$ and $\mathcal{W}_{j_1}$ and $\mathcal{W}_{j_2}$ are weights. After that we can use the Newton's technique to get the values of C then we get the solution.

Theorem 4.1. *Let X be the Banach space with $\|\mathcal{W}\| = \max_{0 \leq x \leq 1} |\mathcal{W}(x)|$, $\mathcal{W} \in X$ and $f(x, \mathcal{W})$ fulfill the Lipschitz constrain, i.e.,*

$$|f(x, \Upsilon_1) - f(x, \Upsilon_2) \leq l | \Upsilon_1 - \Upsilon_2|, \forall \Upsilon_1, \Upsilon_2 \in X. \quad (33)$$

Let δ be denote as $\delta = \frac{l(a+b)}{2a(1+\alpha)}$.

If $\delta < 1$, then $\mathcal{W} = \mathcal{N}(\mathcal{W})$ has a unique solution in X.

Proof.

$$\|\mathcal{N}(\Upsilon_1) - \mathcal{N}(\Upsilon_2)\|$$
$$= \max_{0\le x\le 1} \left| \int_0^1 G(x,\zeta)\zeta^\alpha \left[f(\zeta,\Upsilon_1) - f(\zeta,\Upsilon_1) \right] \right|$$
$$\le \max_{0\le\zeta\le 1} \int_0^1 |f(\zeta,\Upsilon_1) - f(\zeta,\Upsilon_2)| \max_{0\le x\le 1} \left| \int_0^1 G(x,\Upsilon)\zeta^\alpha d\zeta \right|. \tag{34}$$

Then

$$\|\mathcal{N}(\Upsilon_1) - \mathcal{N}(\Upsilon_2)\| \le \max_{0\le\zeta\le 1} |\Upsilon_1 - \Upsilon_2| = \zeta \|\Upsilon_1 - \Upsilon_2\|, \tag{35}$$

$$\|\mathcal{N}(\Upsilon_1) - \mathcal{N}(\Upsilon_2)\| \le \|\Upsilon_1 - \Upsilon_2\|, \tag{36}$$

Theorem 4.2. *Let $N(\mathcal{W})$ be the nonlinear operator that fulfills the Lipschitz condition. If $\|\mathcal{W}_0\| < \infty$, then $\|\mathcal{W}_{k+1}\| \le \delta \|\mathcal{W}_k\|$,*

$k = 0, 1, 2...$ and the sequence Y_n characterized by $Y_n(x) = \sum_{j=0}^{n} \mathcal{W}_j(x)$, converges to the exact solution $\mathcal{W}$.

Proof. We have

$$Y_1 = \mathcal{W}_0 + \mathcal{W}_1,\ Y_2 = \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2, \ldots\ Y_n = \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{W}_2 + \ldots + \mathcal{W}_n, \ldots.$$

Thus, we find

$$\mathcal{W}_{k+1} = Y_{k+1} - Y_k,\ k = 1, 2, \ldots.$$

We now prove that the sequence $\{Y_k\}$ is convergent. We obtain;

$$\|\mathcal{W}_{n+1}\| = \|Y_{n+1} - Y_n\| = \|\mathcal{N}Y_n - \mathcal{N}Y_{n-1}\| \le \delta \|Y_n - Y_{n-1}\|,$$

and hence

$$\|Y_{n+1} - Y_n\| = \|\mathcal{W}_{n+1}\| \le \delta \|\mathcal{W}_{n+1}\| \le \delta^2 \|\mathcal{W}_{n-1}\| \le \ldots \le \delta^{n+1} \|\mathcal{W}_0\|. \tag{37}$$

For $n, m \in N$, with $n \ge m$, we have,

$$\|Y_n - Y_m\| = \|(Y_n - Y_{n-1}) + (Y_{n-1} - Y_{n-2}) + \ldots + (Y_{m+1} - Y_m)\|, \tag{38}$$

$$\le \|(Y_n - Y_{n-1})\| + \|(Y_{n-1} - Y_{n-2})\| + \ldots + \|(Y_{m+1} - Y_m)\|, \tag{39}$$

$$\le \delta^n \|\mathcal{W}_0\| + \delta^{n-1} \|\mathcal{W}_0\| + \ldots + \delta^{m+1} \|\mathcal{W}_0\|, \tag{40}$$

$$= \delta^{m+1} \left(1 + \delta + \delta^2 + \ldots + \delta^{n-m-1}\right) \|\mathcal{W}_0\|, \tag{41}$$

$$= \frac{\delta^{m+1}\left(1 + \delta^{n-m}\right)}{1-\delta} \|\mathcal{W}_0\|. \tag{42}$$

Since $0 < \delta < 1$, we have $\left(1 - \delta^{n-m}\right) < 1$. It readily follows that;

$$\|(Y_n - Y_m)\| \le \frac{\delta^{m+1}}{1-\delta} \|\mathcal{W}_0\|. \tag{43}$$

Taking limit as $m \to \infty$, then we have $\|(Y_n - Y_m)\| \to 0$.

Theorem 4.3. *Let $\mathcal{W}(x)$ be the exact solution of the operator equation $\mathcal{W} = \mathcal{N}\mathcal{W}$. Let $m(x)$ be the sequence of approximate series solutions defined by $Y_n(x) = \sum_{j=0}^{n} \mathcal{W}_j(x)$. Then have*

$$\max_{0\le t\le 1} \left| \mathcal{W}(t) - \sum_{j=0}^{m} \mathcal{W}_j(x) \right| \le \frac{\delta^{m+1}}{1-\delta} \|\mathcal{W}_0\|. \tag{44}$$

Proof. For $n \ge m$, using $\|\mathcal{W} - Y_m\| \le \frac{\delta^{m+1}}{1-\delta} \|\mathcal{W}_0\|$, we have

$$\|Y_n - Y_m\| \le \frac{\delta^{m+1}}{1-\delta} \|\mathcal{W}_0\|. \tag{45}$$

Since $\lim_{n\to\infty} Y_n = \mathcal{W}$, fixing m and letting $n \to \infty$, we obtain

$$\|\mathcal{W} - Y_m\| \le \frac{\delta^{m+1}}{1-\delta} \|\mathcal{W}_0\|. \tag{46}$$

5. Consequence and investigation

In this split, two variants based on the second order NS-BVPs have been taken and the comparison of the present technique with the exact results will also be discussed.

Table 1
The ḤW solutions at $2^{k-1} = 4$, $\mathcal{M} = 3$ and ADM for Example 1

x	ḤOBW	ADM	Exact	AE of ḤW	AE of ADM
0.1	0.011	0.011	0.011	0	0
0.2	0.048	0.04800000004	0.048	6.5×10^{-14}	4×10^{-11}
0.3	0.117	0.1170000002	0.117	1.2×10^{-15}	2×10^{-10}
0.4	0.224	0.2240000003	0.224	8.7×10^{-14}	3×10^{-10}
0.5	0.375	0.3750000001	0.375	1.4×10^{-16}	1×10^{-10}
0.6	0.576	0.5760000004	0.576	3.1×10^{-14}	4×10^{-10}
0.7	0.833	0.83300000003	0.833	2.5×10^{-15}	0
0.8	1.152	1.1520000007	1.152	1.6×10^{-14}	1×10^{-9}
0.9	1.539	1.5390000009	1.539	2.310^{-14}	9×10^{-9}

Example 1. Consider the linear second-order NSP-BVP given by:

$$\begin{cases} \frac{d^2\mathcal{W}(t)}{dt^2} + \frac{2}{t^4(1-t)^{1.5}}\frac{d\mathcal{W}(t)}{dt} + \frac{1}{t^3(1-t)^{1.5}}W(t) = f(t), 0 < t < 1. \\ W(0) = W(1), \frac{d\mathcal{W}(0)}{dt} = \frac{d\mathcal{W}(1)}{dt}. \end{cases} \tag{47}$$

The exact solution $\mathcal{W}(x) = x^2 + x^3$.

Using the ḤW, the truncate solution $\mathcal{W}(x)$ of (37) at $2^{k-1} = 4$ and $\mathcal{M} = 3$ technique, we get 16 number algebraic equations with the same number of unknowns at $2^{k-1} = 4$, $\mathcal{M} = 3$ and these equations are solved by Newton's technique with maple program, we get the *HW* coefficients as:

$\mathcal{C}$=[0,0, 0.02083337654, 0.078129087, 0.078125 8976, 0.1354153427, 0.22918967655, 0.3745632, 0.378987, 0.52068574, 0.71870796785, 0.984375 8647563, 0.98437685764, 1.258764, 1.5863452313 3, 2]

So, the truncate of $\mathcal{W}(x)$ is $\mathcal{W}(x) = \mathcal{C}^T$ḤW$(x)$

The acquired results have been compared with that of our seventh order (ADM) along with the required solutions and introduced in Table 1. The outcomes reveal that the results by HW, with using only a small number of bases, are very promising and superior to ADM and evaluated absolute errors (AE) by HW for $\mathcal{W}(x)$ will be decreased rapidly in comparison with ADM.

Example 2. Consider the NSP-BVP s given by:

$$\begin{cases} \frac{d^2\mathcal{W}(t)}{dt^2} + \frac{2}{t^2(1-t)}\frac{d\mathcal{W}(t)}{dt} + \frac{1}{t(1-t)}W(t) + \mathcal{W}^2(t) = f(t), 0 < t < 1, \\ W(0) = W(1), \frac{d\mathcal{W}(0)}{dt} = \frac{d\mathcal{W}(1)}{dt}. \end{cases}$$

This equation can be transformed into the NS-BVPs form as follows

$$\mathcal{W}'(x) = -\int_0^x (\frac{t^g}{x^g}) t^r \mathcal{W}^m(t)\, dt,\ \ \mathcal{W}(0) = 1,\ \ g \geq 1$$

Tables 2 and 3, exhibit the numerical solutions of $\mathcal{W}(t)$ by ḤW at $r = m = 0$, $g = 2$, and $r = 0$, $m = 1$, $g = 2$.

a. For $r = m = 0$, and $g = 2$, the above equation has the accurate solution

$$\mathcal{W}(x) = 1 - \frac{x^2}{6},$$

Table 3 shows that, for $g = 2$, $m = 0$, the acquired results coincide with the required solution and efficiency of the technique described through the AE.

By applying the ḤW technique, and taking $2^k = 4$, $\mathcal{M} = 3$, *HW* then the value of $\mathcal{C}$ is:

[1, 1, 0.879967, 0.9378902, 0.94378094,

0.9894876, 0.9946874, 0.946789, 0.936708,

0.964838, 0.942086, 0.94890328, 0.94289,

0.82478 - , 0.87448277, 0.8347664].

b. For $r = 0$, $m = 1$, and $k = 2$, we have:

$$\mathcal{W}'(x) = -\int_0^x (\frac{t^2}{x^2})\mathcal{W}(t)\, dt,\ \ \mathcal{W}(0) = 1.$$

Table 2
The HW solutions for Example.2 when $g = 2, m = 0$

x	Accurate solution	HW solution	AE
0.1	0.9983333333	0.9983333334	1×10^{-10}
0.2	0.9933333333	0.9933333321	1.2×10^{-9}
0.3	0.9850000000	0.9850000002	1×10^{-10}
0.4	0.9733333333	0.9733333364	3.1×10^{-9}
0.5	0.9583333333	0.9583333336	1×10^{-10}
0.6	0.9400000000	0.9400000015	1.5×10^{-9}
0.7	0.9183333333	0.9183333331	1×10^{-10}
0.8	0.8933333333	0.8933333344	1.1×10^{-9}
0.9	0.8650000000	0.8650000056	5.6×10^{-9}

Table 3
The HW solutions for Example.2 when $g = 2, m = 1$

x	Accurate Solution	HW Solution at $2^k = 4, \mathcal{M} = 3$	AE at $2^k = 4$, $\mathcal{M} = 3$	AE at $2^k = 8$, $\mathcal{M} = 7$
0.1	0.9983341665	0.99834534345	2.37×10^{-8}	4.61×10^{-16}
0.2	0.9933466540	0.9933468760	1.260×10^{-7}	7.02×10^{-16}
0.3	0.9850673556	0.9850677866	1.173×10^{-8}	2.35×10^{-16}
0.4	0.9735458558	0.9735456548	2.161×10^{-8}	5.11×10^{-16}
0.5	0.9588510772	0.9588515722	7.278×10^{-8}	2.73×10^{-16}
0.6	0.9410707892	0.9410777892	1.80×10^{-8}	6.35×10^{-16}
0.7	0.9203109820	0.92031769820	1.086×10^{-7}	1.05×10^{-16}
0.8	0.8966951136	0.8966951661	9.66×10^{-8}	3.24×10^{-16}
0.9	0.8703632328	0.8703632628	1.12×10^{-8}	4.01×10^{-16}

The accurate solutions for this problem are given by $\mathcal{W}(x) = \frac{\sin(x)}{x}$.

Applying the **HW** technique, and taking $2^k = 4, \mathcal{M} = 3$, we note that **HW** coefficients $\mathcal{C}$ as

[0.965622 1.766432345, 0.9524433, 0.98642742, 0.9272652, 0.9354247, 0.951467, 0.9642556, 0.9524552, 0.9452464, 0.9642355, 0.9524323, 0.9254334, 0.82345434, 0.8124564, 0.825433]

Designing this technique and taking $2^k = 8, \mathcal{M} = 7$, if $2^k, \mathcal{M}$ increases, the truncate solution gets the required solution. The truncate solutions acquired by HW at $g = 2$ and $2^k = 4, \mathcal{M} = 3$, at $2^k = 8, \mathcal{M} = 7$ with require solutions and AE are demonstrated in Tables 2. By increasing, $2^k, \mathcal{M}$ the computed results have appropriated the exactness better as well, and the truncate solution gets an approximate to the accurate solution as demonstrated in Table 3. The CPU time for running a case may depend on the choice of $2^k, \mathcal{M}$ for solving the system of linear algebraic equations resulting from the discretized equations. If different values $2^k, \mathcal{M}$ are used, the CPU time would be different.

6. Conclusions

In this article, the design of the **HW** algorithm was shown to solve the NS-BVPs. The demonstration of the present study was carried out by using wavelets and rapid fine modification of Hermite. The particular merit is that it can be applied very well without change of BVPs into IVPs. The nonlinear singular model based on periodic boundary value problems was assessed effectively by the present approximation technique based on the Hermite technique and accurateness of numerical outcomes was observed. The **HW** scheme's accuracy was demonstrated by finding the matching outcomes with the exact solutions having 6 decimal places of overlapping for solving nonlinear model based on singular periodic boundary value problem. The important advantage of the suggested technique is to collect scarce terms of the Hermite polynomials so that a higher-order approximation.

References

[1] I.T. Kiguradze, Two-Point Boundary Value Problems for Essentially Singular Nonlinear Second-Order Differential Equations, *Differ Equ* **55** (2019), 776–786.

[2] I.T. Kiguradze, Two-Point Boundary Value Problems for Essentially Singular Second-Order Linear Differential Equations, *Differ Equ* **55** (2019), 591–608.

[3] A.K. Verma and B. Pandit, Existence and Nonexistence Results of Radial Solutions to Singular BVPs Arising in Epitaxial Growth Theory. arXiv 2019, arXiv:1911.04979.

[4] S. Biagi and T. Isernia, On the solvability of singular boundary value problems on the real line in the critical growth case. *Disc Cont Dyn Syst (A)* **40** (2020), 1131–1157.

[5] P. Roul, Doubly Singular Boundary Value Problems with Derivative Dependent Source Function: A Fast-Converging Iterative Approach, *Math Meth Appl Sci* **42** (2019), 354–374.

[6] R. Singh, Analytic Solution of Singular Emden-Fowler-type Equations by Green's Function and Homotopy Analysis Method, *Eur Phys J Plus* **134** (2019), 583.

[7] W.X. Ma, M.M. Mousa and M.R. Ali, Application of a new hybrid method for solving singular fractional Lane–Emden-type equations in astrophysics, *Modern Physics Letters B* **34**(03) (2020), 2050049.

[8] M.R. Ali, M.M. Mousa and W.-X. Ma, Solution of Nonlinear Volterra Integral Equations with Weakly Singular Kernel by Using the HOBW Method, *Advances in Mathematical Physics* (2019), 1–10.

[9] R. Singh, N. Das and J. Kumar, The Optimal Modified Variational Iteration Method for the Lane–Emden Equations with Neumann and Robin Boundary Conditions, *Eur Phys J Plus* **132** (2017), 251.

[10] R. Singh, H. Garg and V. Guleria, Haar Wavelet Collocation Method for Lane–Emden Equations with Dirichlet. Neumann and Neumann-Robin boundary conditions, *J Comput Appl Math* **346** (2019), 150–161.

[11] W. Glabisz, The use of Walsh-wavelets packets in linear boundary value problems, *Comput Struct* **82** (2004), 131–141.

[12] Siraj-ul-Islam M.A. Noor, I.A. Tirmizi and M.A. Khan, Quadratic non-polynomial spline approach to the solution of a system of second-order boundary-value problems, *Appl Math Comput* **179** (2006), 153–160.

[13] E. Wiebel, Confinement of a Plasma Column by Radiation Pressure in the Plasma in a Magnetic Field, Stanford University Press, California, 1958.

[14] A.S.V.R. Kanth and Y.N. Redd, Higher order finite difference method for a class of singular boundary value problems, *Appl Math Comput* **155** (2004), 249–258.

[15] A.S.V.R. Kanth and Y.N. Reddy, Cubic spline for a class of singular boundary value problems, *Appl Math Comput* **170** (2005), 733–740.

[16] E. Al-Said, The use of cubic splines in the numerical solution of system of second-order boundary-value problems, *Int J Comput Math Appl* **42** (2001), 861–869.

[17] A.M. Wazwaz, The variational iteration method for solving nonlinear singular boundary value problems arising in various physical models, *Commun Nonlin Sci Num Simult* **16**(388) (2011), 1–6.

[18] F.Z. Geng and M.G. Cui, Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space, *Appl Math Comput* **192** (2007), 389–398.

[19] M.A. Ramadan, A.R. Hadhoud and M.R. Ali, Numerical solutions of singular initial value problems in the second-order ordinary differential equations using Hybrid Orthonormal Bernstein and Block-Pulse Functions, *Journal of the Egyptian Mathematical Society* **26** (2018), 156–166.

[20] K.-Z. Lu, L.-B. Liu, H. Fang and L. Liu, A dual mutation differential evolution algorithm for singularly perturbed problems with two small parameters, *Journal of Intelligent & Fuzzy Systems* **36**(6) (2019), 6579–6587.

[21] E.-ul-H. Qazi, M. Hussain and H. Aboalsamh, An efficient intelligent system for the classification of electroencephalography (EEG) brain signals using nuclear features for human cognitive tasks, *Journal of Intelligent & Fuzzy Systems* **37**(1) (2019), 913–928.

[22] Z. Jin, Q. Zhang and X. Meng, The stability analysis and control of uncertain singular biological economic system with invasion of alien species based on sliding mode control, *Journal of Intelligent & Fuzzy Systems* **34**(6) (2018), 4079–4091.

[23] O.F. Hassan, A. Jamal and S. Abdel-Khalek, Genetic algorithm and numerical methods for solving linear and nonlinear system of equations: a comparative study, *Journal of Intelligent & Fuzzy Systems* **38**(3) (2020), 2867–2872.

[24] O.F. Hassan, A. Jamal and S. Abdel-Khalek, Genetic algorithm and numerical methods for solving linear and nonlinear system of equations: a comparative study, *Journal of Intelligent & Fuzzy Systems* **38**(3) (2020), 2867–2872.

[25] H. Luo, J. Liu and X. Li, Periodic behavior of a class of nonlinear dynamic systems based on the Runge-Kutta algorithm, *Journal of Intelligent & Fuzzy Systems* **38**(2) (2020), 1471–1476.

[26] H. Wang, Two-point boundary value problems for first-order nonlinear fuzzy differential equation, *Journal of Intelligent & Fuzzy Systems* **30**(6) (2016), 3335–3347.

[27] S. Owyed, M.A. Abdou, A.-H. Abdel-Aty, A.A. Ibraheem, R. Nekhili and D. Baleanu, New optical soliton solutions of space-time fractional nonlinear dynamics of microtubules via three integration schemes, *Journal of Intelligent & Fuzzy Systems* **38**(3) (2020), 2859–2866.

[28] J. Gong, B. Jiang and Q. Shen, Distributed adaptive output-feedback fault tolerant control for nonlinear systems with sensor faults, *Journal of Intelligent & Fuzzy Systems* **38**(4) (2020), 4173–4190.

[29] H. Yang, Y. Jiang and S. Yin, Adaptive Fuzzy Fault Tolerant Control for Markov Jump Systems with Additive and Multiplicative Actuator Faults, in *IEEE Transactions on Fuzzy Systems*, doi: 10.1109/TFUZZ.2020.2965884

[30] S. Yin, H. Yang, H. Gao, J. Qiu and O. Kaynak, An Adaptive NN-Based Approach for Fault-Tolerant Control of Nonlinear Time-Varying Delay Systems With Unmodeled Dynamics, in *IEEE Transactions on Neural Networks and Learning Systems* **28**(8) (2017), pp. 1902–1913, doi: 10.1109/TNNLS.2016.2558195

[31] K. Sun, L. Liu, J. Qiu and G. Feng, Fuzzy Adaptive Finite-Time Fault-Tolerant Control for Strict-Feedback Nonlinear Systems, in *IEEE Transactions on Fuzzy Systems*, doi: 10.1109/TFUZZ.2020.2965890

[32] S. K., Q. Jianbin, H.R. Karimi and Y. Fu, Event-Triggered Robust Fuzzy Adaptive Finite-Time Control of Nonlinear Systems with Prescribed Performance, in *IEEE Transactions on Fuzzy Systems*, doi: 10.1109/TFUZZ.2020.2979129

[33] A. Kheirabadi, A.M. Vaziri and S. Effati, Solving optimal control problem using Hermite wavelet, *Numerical Algebra,Control and Optimization* **9**(1) (2019), 101–112.

[34] M.R. Ali and W.X. Ma, New exact solutions of Bratu Gelfand model in two dimensions using Lie symmetry analysis, *Chinese Journal of Physics* **65** (2020), 198–206.

[35] M.R. Ali and W.X. Ma, Detection of a new multi-wave solutions in an unbounded domain, *Modern Physics Letters B* **33**(34) (2019), 1–10.

[36] M.R. Ali and D. Baleanu, Haar wavelets scheme for solving the unsteady gas flow in four-dimensional, *Thermal Science* **2019**(23), 292–301.

[37] R. Saleh, R. Sadat and M. Kassem, Optimal solutions of a (3 + 1)-dimensional B-Kadomtsev-Petviashvii equation, *Mathematical Methods in the Applied Sciences* **43**(4) (2020), 1775–1787.

[38] M.R. Ali, A Truncation Method for Solving the Time-Fractional Benjamin-Ono Equation, *Journal of Applied Mathematics* **2019**(18), 1–7.

[39] M.R. Ali and W.X. Ma, New Exact Solutions of Nonlinear (3 + 1)-Dimensional Boiti-Leon-Manna-Pempinelli Equation, *Advances in Mathematical Physics* (2019), 1–8.

[40] M.R. Ali and A.R. Hadhood, Hybrid Orthonormal Bernstein and Block-Pulse functions wavelet scheme for solving the 2D Bratu problem, *Results in Physics* **13** (2019), 12–21.