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Exploration of soliton structures in the Hirota–Maccari system with stability analysis

Noor Alam [*,††], Wen-Xiu Ma [†,‡,§,¶,‡‡,***], Mohammad Safi Ullah [‖,§§,***],
Aly R. Seadawy [**,¶¶] and Mahinur Akter [‖,‖‖]

[*] *Department of Mathematics,*
Bangabandhu Sheikh Mujibur Rahman University,
Kishoreganj 2300, Bangladesh

[†] *Department of Mathematics,*
Zhejiang Normal University,
Jinhua 321004, Zhejiang, China

[‡] *Department of Mathematics,*
King Abdulaziz University,
Jeddah 21589, Saudi Arabia

[§] *Department of Mathematics and Statistics,*
University of South Florida, Tampa, FL, USA

[¶] *Department of Mathematical Sciences,*
Material Science Innovation and Modelling,
North-West University,
Mafikeng Campus, Mmabatho 2735, South Africa

[‖] *Department of Mathematics,*
Comilla University, Cumilla 3506, Bangladesh

[**] *Faculty of Science, Mathematics Department,*
Taibah University,
Al-Madinah Al-Munawarah 41411, Kingdom of Saudi Arabia
[††] *nooralam.cou@gmail.com*
[‡‡] *mawx@cas.usf.edu*
[§§] *safi.ru1985@gmail.com*
[¶¶] *aly742001@yahoo.com*
[‖‖] *mahi1991cu@gmail.com*

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In this research, the modified extended tanh-function (METF) and the extended Jacobi elliptic function expansion (EJEFE) techniques are used to investigate the generation and detection of soliton structures in the Hirota–Maccari (HM) model. Consequently, we obtain soliton solutions with advanced structures, including singular bright soliton, dark soliton, periodic

[***] Corresponding authors.

waves, breather waves, periodic breather waves, and multiple bright and dark breather waves. In addition, a lump-type breather wave is also included in the presented solutions. Stability analysis of the obtained solutions is addressed by employing the Hamiltonian technique. 3D surfaces and 2D visuals of the outcomes are represented with the help of a computer application. These findings contribute to understanding nonlinear wave phenomena with potential applications in optics, fluid dynamics, and plasma physics.

Keywords: Modified extended tanh-function technique; extended Jacobi elliptic function expansion technique; soliton; breather wave; periodic wave.

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1. Introduction

Over the recent decades, nonlinear time-dependent models have become an integral part of different branches of science namely, physical science,[1] hydrodynamics,[2] electromagnetism,[3] condensed matter physics,[4] and others.[5–7] The experts in these sectors have established a huge number of nonlinear models including the Bogoyavlenskii equation,[8] the $(3+1)$-D Burger system,[9,10] the generalized KP equation,[11] the Zoomeron equation,[12] the Hirota–Maccari (HM) equation,[13] the KdV equation,[14,15] the WBBM equation,[16] fractional generalized CBS-BK equation,[17] etc. The Hirota nonlinear model with soliton solution was established by Hirota.[18] Subsequently, Maccari showed a 2D expansion of the Hirota model.[19] This physical structure is used to develop different nonlinear models in different fields such as optics, mathematical physics, electric communications, and hydrodynamics.[20]

The nonlinear equations are solved by several powerful techniques: the $(\frac{G'}{G'+G+A})$ method,[21] the unified technique,[22] the Hirota bilinear technique,[23,24] the generalized KP equation,[25] the improved Kudryashov scheme,[26] the generalized CBS-BK model,[27] (G'/G)-expansion technique,[28] the new Kudryashov technique,[29,30] the lie group technique,[31] the modified rational sine-cosine method,[32] the modified extended tanh-function (METF) method,[33] the extended Jacobi elliptic function expansion (EJEFE) method,[34] etc. The solutions of the nonlinear equations represent different types of wave profiles. Among these, dark soliton, bright soliton, breather waves, periodic waves, and kink waves have important appearances in nonlinear systems dynamics.[35]

This work proposes to explore the soliton outcomes of the HM model utilizing the METF technique and the EJEFE technique. Moreover, to improve the accuracy of the governing model the stability condition is calculated.

The framework of this study is as follows: Sec. 2 covers the conversion of ordinary differential equations from the HM model. In Sec. 3, we discuss the solution process and the utilization of the METF technique. In addition, the result procedure for the EJEFE technique is shown in Sec. 4. In the next part, the condition of existence is developed for the proposed model. Section 6 represents the discussion based on figures. The novelty of our work is figured out in Sec. 7. Finally, some completion statements are written in Sec. 8.

The novelty of this work is that the finding outcome from our proposed model according to the mentioned method does not describe any other researcher before.

2. Ordinary Differential Equation Form of the HM Model

If $\chi(x, y, t)$ performs as real, $\xi(x, y, t)$ indicates complex field, the temporal variable is denoted by t and space variables are x, y, then the proposed model[36] can be written as

$$i\xi_t + \xi_{xy} + i\xi_{xxx} + \xi\chi - i|\xi|^2\xi_x = 0, \quad 3\chi_t + (|\xi|^2)_y = 0. \tag{1}$$

To determine the ODE form of (1), assume the transformation as follows:

$$\xi(x, y, t) = f(\zeta)e^{-i(px+qy+rt)}, \quad \chi(x, y, t) = g(\zeta), \quad \zeta = x + y + st, \tag{2}$$

where p and s represent frequency and velocity sequentially f and g are real functions, and lastly q and r indicate free constants and $i = \sqrt{-1}$.

Now from Eqs. (1) and (2), we get the desired ODE form:

$$3(1 - 3p)f'' + 3(p^3 - pq - r)f + (3p - 1)f^3 = 0, \quad g = -\frac{f^2}{3}. \tag{3}$$

3. METF Method for the Suggested Model

Suppose the trial solution with the ancillary equation of Ref. 37 is written as

$$f(\zeta) = \sum_{j=0}^{m} a_j\phi^j(\zeta) + \sum_{j=1}^{m} b_j\phi^{-j}(\zeta) \tag{4}$$

and

$$\phi'(\zeta) = b + \phi^2(\zeta), \tag{5}$$

with here two arbitrary parameters b and m.

The solution of (5) implies that

$$\phi(\zeta) = \begin{cases} -\sqrt{-b}\tanh(\sqrt{-b}\zeta), & \text{for } b < 0, \\ -\sqrt{-b}\coth(\sqrt{-b}\zeta), & \text{for } b < 0, \\ -\dfrac{1}{\zeta}, & \text{for } b = 0, \\ \sqrt{b}\tan(\sqrt{b}\zeta), & \text{for } b > 0, \\ -\sqrt{b}\cot(\sqrt{b}\zeta), & \text{for } b > 0. \end{cases} \tag{6}$$

By balancing f'' and f^3, one reaches $m = 1$. Now from (4) we get

$$f(\zeta) = a_0 + a_1\phi(\zeta) + b_1\phi^{-1}(\zeta). \tag{7}$$

Now using (5) and (7) in (3), we get

$$b = b, \quad p = p, \quad r = p^3 - 6bp - pq + 2b, \quad a_0 = 0, \quad a_1 = \pm\sqrt{6}, \quad b_1 = 0, \tag{8}$$

$$b = b, \quad p = p, \quad r = p^3 - 6bp - pq + 2b, \quad a_0 = 0, \quad a_1 = 0, \quad b_1 = \pm\sqrt{6}b. \tag{9}$$

From Eqs. (2), (3), (6), (7), and (8), we get

$$\begin{aligned}
\xi_1(x,y,t) &= \pm\sqrt{-6b}\tanh(\sqrt{-b}\zeta)e^{-i(px+qy+rt)},\\
\xi_2(x,y,t) &= \pm\sqrt{-6b}\coth(\sqrt{-b}\zeta)e^{-i(px+qy+rt)},\\
\xi_3(x,y,t) &= \pm\frac{\sqrt{6}}{\zeta}e^{-i(px+qy+rt)},\\
\xi_4(x,y,t) &= \pm\sqrt{6b}\tan(\sqrt{b}\zeta)e^{-i(px+qy+rt)},\\
\xi_5(x,y,t) &= \pm\sqrt{6b}\cot(\sqrt{b}\zeta)e^{-i(px+qy+rt)},\\
\chi_1(x,y,t) &= 2b\tanh^2(\sqrt{-b}\zeta),\\
\chi_2(x,y,t) &= 2b\coth^2(\sqrt{-b}\zeta),\\
\chi_3(x,y,t) &= -\frac{2}{\zeta^2},\\
\chi_4(x,y,t) &= -2b\tan^2(\sqrt{b}\zeta),\\
\chi_5(x,y,t) &= -2b\cot^2(\sqrt{b}\zeta),
\end{aligned}$$

where $\zeta = x + y + st$.

From Eqs. (2), (3), (6), (7), and (9), we get

$$\begin{aligned}
\xi_6(x,y,t) &= \pm\sqrt{6}b\{-\sqrt{-b}\tanh(\sqrt{-b}\zeta)\}^{-1}e^{-i(px+qy+rt)},\\
\xi_7(x,y,t) &= \pm\sqrt{6}b\{-\sqrt{-b}\coth(\sqrt{-b}\zeta)\}^{-1}e^{-i(px+qy+rt)},\\
\xi_8(x,y,t) &= \pm\sqrt{6}b\{\sqrt{b}\tan(\sqrt{b}\zeta)\}^{-1}e^{-i(px+qy+rt)},\\
\xi_9(x,y,t) &= \pm\sqrt{6}b\{-\sqrt{b}\cot(\sqrt{b}\zeta)\}^{-1}e^{-i(px+qy+rt)},\\
\chi_6(x,y,t) &= -2b^2\{(-\sqrt{-b}\tanh(\sqrt{-b}\zeta))^{-1}\}^2,\\
\chi_7(x,y,t) &= -2b^2\{(-\sqrt{-b}\coth(\sqrt{-b}\zeta))^{-1}\}^2,\\
\chi_8(x,y,t) &= -2b^2\{(\sqrt{b}\tan(\sqrt{b}\zeta))^{-1}\}^2,\\
\chi_9(x,y,t) &= -2b^2\{(-\sqrt{b}\cot(\sqrt{b}\zeta))^{-1}\}^2,
\end{aligned}$$

where $\zeta = x + y + st$.

4. EJEFE Method for the Proposed System

Suppose the trial solution with the ancillary equation of Ref. 38 is written as

$$f(\zeta) = \sum_{j=-n}^{n} a_j \eta(\zeta)^j. \tag{10}$$

Here, the Jacobi elliptic function (JEF) is denoted by η and define by $\eta(\zeta) = sn(\theta) = sn(\zeta,\rho)$ or $cn(\zeta,\rho)$ or $dn(\zeta,\rho)$ with $0 < \rho < 1$.

Note: For ρ tends to 1, JEF is converted to the hyperbolic pattern, that is $sn(\zeta,\rho)$ will be $\tanh(\zeta)$, $cn(\zeta,\rho)$ will be $\mathrm{sech}(\zeta)$, $dn(\zeta,\rho)$ will be $\mathrm{sech}(\zeta)$.

Also, for ρ tends to 0, JEF is changed to the hyperbolic pattern, that is $sn(\zeta,\rho)$ will be $\sin(\zeta)$, $cn(\zeta,\rho)$ will be $\cos(\zeta)$, $dn(\zeta,\rho)$ will be 1.

By balancing f'' and f^3, one reaches $n = 1$. Now from (10) we get

$$f(\zeta) = a_0 + a_1\eta(\zeta) + a_{-1}\eta(\zeta)^{-1}. \tag{11}$$

First case: For $\eta(\zeta) = sn(\zeta,\rho)$, we get the following outcomes from Eqs. (3), (10), and (11):

$$\begin{cases} r = \dfrac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)}, \quad \mathrm{p}=\mathrm{p}, \quad \rho=\rho, \quad q = -a_{-1}^2(\rho^2+1)p, \quad a_{-1}=a_{-1}, \\ a_0 = 0, \quad a_1 = 0, \end{cases} \tag{12}$$

$$\begin{cases} r = \dfrac{\mathrm{pa}_1^2}{\rho^2(\mathrm{p}^2-1)}, \quad \mathrm{p}=\mathrm{p}, \quad \rho=\rho, \quad q = -\dfrac{a_1^2(\rho^2+1)p}{\rho^2}, \quad a_0=0, \\ a_1 = a_1, \quad a_{-1} = 0, \end{cases} \tag{13}$$

$$\begin{cases} r = \dfrac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)}, \quad \mathrm{p}=\mathrm{p}, \quad \rho=\rho, \quad q = -a_{-1}^2(\rho^2+6\rho+1)p, \quad a_0=0, \\ a_1 = a_{-1}\rho, \quad a_{-1} = a_{-1}, \end{cases} \tag{14}$$

$$\begin{cases} r = \dfrac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)}, \quad \mathrm{p}=\mathrm{p}, \quad \rho=\rho, \quad q = -a_{-1}^2(\rho^2-6\rho+1)p, \quad a_0=0, \\ a_1 = -a_{-1}\rho, \quad a_{-1} = a_{-1}. \end{cases} \tag{15}$$

From Eqs. (2), (3), (11), and (12) with $\eta(\zeta) = sn(\zeta,\rho)$, we get the following two results:

$$\xi_{10}(x,y,t) = a_{-1}(sn(\zeta,\rho))^{-1}e^{-i(px - a_{-1}^2(\rho^2+1)py + \frac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)}t)},$$
$$\chi_{10}(x,y,t) = -\frac{\{a_{-1}(sn(\zeta,\rho))^{-1}\}^2}{3}.$$

For $\rho \to 1$, we found the following outputs:

$$\xi_{10,1}(x,y,t) = a_{-1}(\tanh(\zeta))^{-1}e^{-i(px - a_{-1}^2 2py + \frac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)}t)},$$
$$\chi_{10,1}(x,y,t) = -\frac{\{a_{-1}(\tanh(\zeta))^{-1}\}^2}{3}.$$

For $\rho \to 0$, we found the following outputs:

$$\xi_{10,2}(x,y,t) = a_{-1}(\sin(\zeta))^{-1}e^{-i(px - a_{-1}^2 py + \frac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)}t)},$$
$$\chi_{10,2}(x,y,t) = -\frac{\{a_{-1}(\sin(\zeta))^{-1}\}^2}{3},$$

where $\zeta = x + y + st$.

From Eqs. (2), (3), (11), and (13) with $\eta(\zeta) = sn(\zeta, \rho)$, we get the following two results:

$$\xi_{11}(x,y,t) = a_1 sn(\zeta,\rho) e^{-i\left(px - \frac{a_1^2(\rho^2+1)p}{\rho^2} y + \frac{\mathrm{pa}_1^2}{\rho^2(\mathrm{p}^2-1)} t\right)},$$
$$\chi_{11}(x,y,t) = -\frac{\{a_1 sn(\zeta,\rho)\}^2}{3}.$$

For $\rho \to 1$, we found the following outputs:

$$\xi_{11,1}(x,y,t) = a_1 \tanh(\zeta) e^{-i\left(px - a_1^2 2py + \frac{\mathrm{pa}_1^2}{(\mathrm{p}^2-1)} t\right)},$$
$$\chi_{11,1}(x,y,t) = -\frac{\{a_1 \tanh(\zeta)\}^2}{3},$$

where $\zeta = x + y + st$.

From Eqs. (2), (3), (11), and (14) with $\eta(\zeta) = sn(\zeta, \rho)$, we get the following two results:

$$\xi_{12}(x,y,t) = \{a_{-1}\rho sn(\zeta,\rho) + a_{-1}(sn(\zeta,\rho))^{-1}\} e^{-i\left(px - a_{-1}^2(\rho^2+6\rho+1)py + \frac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)} t\right)},$$
$$\chi_{12}(x,y,t) = -\frac{\{a_{-1}\rho sn(\zeta,\rho) + a_{-1}(sn(\zeta,\rho))^{-1}\}^2}{3}.$$

For $\rho \to 1$, we found the following outputs:

$$\xi_{12,1}(x,y,t) = \{a_{-1}\tanh(\zeta) + a_{-1}(\tanh(\zeta))^{-1}\} e^{-i\left(px - a_{-1}^2 8py + \frac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)} t\right)},$$
$$\chi_{12,1}(x,y,t) = -\frac{\{a_{-1}\tanh(\zeta) + a_{-1}(\tanh(\zeta))^{-1}\}^2}{3}.$$

For $\rho \to 0$, we found the following outputs:

$$\xi_{12,2}(x,y,t) = \{a_{-1}(\sin(\zeta))^{-1}\} e^{-i\left(px - a_{-1}^2 py + \frac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)} t\right)},$$
$$\chi_{12,2}(x,y,t) = -\frac{\{a_{-1}(\sin(\zeta))^{-1}\}^2}{3},$$

where $\zeta = x + y + st$.

From Eqs. (2), (3), (11), and (15) with $\eta(\zeta) = sn(\zeta, \rho)$, we get the following two results:

$$\xi_{13}(x,y,t) = \{-a_{-1}\rho sn(\zeta,\rho) + a_{-1}(sn(\zeta,\rho))^{-1}\} e^{-i\left(px - a_{-1}^2(\rho^2-6\rho+1)py + \frac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)} t\right)},$$
$$\chi_{13}(x,y,t) = -\frac{\{-a_{-1}\rho sn(\zeta,\rho) + a_{-1}(sn(\zeta,\rho))^{-1}\}^2}{3}.$$

For $\rho \to 1$, we found the following outputs:

$$\xi_{13,1}(x,y,t) = \{-a_{-1}\tanh(\zeta) + a_{-1}(\tanh(\zeta))^{-1}\} e^{-i\left(px + a_{-1}^2 4py + \frac{\mathrm{pa}_{-1}^2}{(\mathrm{p}^2-1)} t\right)},$$
$$\chi_{13,1}(x,y,t) = -\frac{\{-a_{-1}\tanh(\zeta) + a_{-1}(\tanh(\zeta))^{-1}\}^2}{3}.$$

For $\rho \to 0$, we found the following outputs:

$$\xi_{13,2}(x,y,t) = \{a_{-1}(\sin(\zeta))^{-1}\}e^{-i(px - a_{-1}^{2}py + \frac{\mathrm{pa}_{-1}^{2}}{(\mathrm{p}^2-1)}t)},$$
$$\chi_{13,2}(x,y,t) = -\frac{\{a_{-1}(\sin(\zeta))^{-1}\}^2}{3},$$

where $\zeta = x + y + st$.

Second case: For $\eta(\zeta) = cn(\zeta, \rho)$, we get the following outcomes from Eqs. (3), (10) and (11):

$$\begin{cases} r = -\dfrac{pa_{-1}^{2}}{(\rho^2-1)(p^2-1)}, \quad p = p, \quad \rho = \rho, \quad q = -\dfrac{a_{-1}^{2}(2\rho^2-1)p}{(\rho^2-1)}, \\ a_0 = 0, \quad a_1 = 0, \quad a_{-1} = a_{-1}, \end{cases} \tag{16}$$

$$\begin{cases} r = -\dfrac{pa_1^2}{\rho^2(p^2-1)}, \quad p = p, \quad \rho = \rho, \quad q = -\dfrac{a_1^2(2\rho^2-1)p}{\rho^2}, \quad a_0 = 0, \\ a_1 = a_1, \quad a_{-1} = 0. \end{cases} \tag{17}$$

From Eqs. (2), (3), (11), and (16) with $\eta(\zeta) = cn(\zeta, \rho)$, we get the following two results:

$$\xi_{14}(x,y,t) = a_{-1}(cn(\zeta,\rho))^{-1}e^{-i(px - \frac{a_{-1}^{2}(2\rho^2-1)p}{(\rho^2-1)}y - \frac{\mathrm{pa}_{-1}^{2}}{(\rho^2-1)(\mathrm{p}^2-1)}t)},$$
$$\chi_{14}(x,y,t) = -\frac{\{a_{-1}(cn(\zeta,\rho))^{-1}\}^2}{3}.$$

For $\rho \to 0$, we found the following outputs:

$$\xi_{14,1}(x,y,t) = a_{-1}(\cos(\zeta))^{-1}e^{-i(px - a_{-1}^{2}py + \frac{\mathrm{pa}_{-1}^{2}}{(\mathrm{p}^2-1)}t)},$$
$$\chi_{14,1}(x,y,t) = -\frac{\{a_{-1}(\cos(\zeta))^{-1}\}^2}{3},$$

where $\zeta = x + y + st$.

From Eqs. (2), (3), (11), and (17) with $\eta(\zeta) = cn(\zeta, \rho)$, we get the following two results:

$$\xi_{15}(x,y,t) = a_1 cn(\zeta,\rho)e^{-i(px - \frac{a_1^2(2\rho^2-1)p}{\rho^2}y - \frac{\mathrm{pa}_1^2}{\rho^2(\mathrm{p}^2-1)}t)},$$
$$\chi_{15}(x,y,t) = -\frac{\{a_1 cn(\zeta,\rho)\}^2}{3}.$$

For $\rho \to 1$, we found the following outputs:

$$\xi_{15,1}(x,y,t) = a_1 \mathrm{sech}(\zeta)e^{-i(px + a_1^2 py - \frac{\mathrm{pa}_1^2}{(\mathrm{p}^2-1)}t)},$$
$$\chi_{15,1}(x,y,t) = -\frac{\{a_1 \mathrm{sech}(\zeta)\}^2}{3},$$

where $\zeta = x + y + st$.

Third case: For $\eta(\zeta) = dn(\zeta, \rho)$, we get the following outcomes from Eqs. (3), (10) and (11):

$$\begin{cases} r = -\dfrac{pa_1^2}{(p^2-1)}, \quad p = p, \quad \rho = \rho, \quad q = a_1^2(\rho^2 - 2)p, \quad a_0 = 0, \\ a_1 = a_1, \quad a_{-1} = 0. \end{cases} \tag{18}$$

From Eqs. (2), (3), (11), and (18) with $\eta(\zeta) = dn(\zeta, \rho)$, we get the following two results:

$$\xi_{16}(x, y, t) = a_1 dn(\zeta, \rho) e^{-i(px + a_1^2(\rho^2-2)py - \frac{\mathrm{pa}_1^2}{(\mathrm{p}^2-1)}t)},$$
$$\chi_{16}(x, y, t) = -\frac{\{a_1 dn(\zeta, \rho)\}^2}{3}.$$

For $\rho \to 1$, we found the following outputs:

$$\xi_{16,1}(x, y, t) = a_1 \mathrm{sech}(\zeta) e^{-i(px - a_1^2 py - \frac{\mathrm{pa}_1^2}{(\mathrm{p}^2-1)}t)},$$
$$\chi_{16,1}(x, y, t) = -\frac{\{a_1 \mathrm{sech}(\zeta)\}^2}{3},$$

where $\zeta = x + y + st$.

5. Stability Exploration

The stability of the proposed model is discussed here. With the help of the Hamiltonian method,[39] from Eq. (1) we obtain

$$R = \frac{1}{2}\int_{-\infty}^{\infty} \chi^2(x, y, t)dx, \tag{19}$$

where R and χ represent momentum and capacity respectively. The stability holds only for

$$\frac{\partial R}{\partial s} > 0, \tag{20}$$

with the velocity component s. Considering χ_4 and taking -2 to 2 as the limit of x, then we get from Eq. (19)

$$R = \frac{1}{2}\int_{-2}^{2} (2b\tan^2(\sqrt{b}(x + y + st)))^2 dx, \tag{21}$$

which implies that

$$\begin{aligned} R = & -\frac{2}{3}b^{\frac{3}{2}}\tan^3(\sqrt{b}(-2 + y + st)) - \tan^3(\sqrt{b}(2 + y + st)) \\ & - 3\tan(\sqrt{b}(-2 + y + st)) + 3\sqrt{b}(-2 + y + st) + 3\tan(\sqrt{b}(2 + y + st)) \\ & - 3\sqrt{b}(2 + y + st). \end{aligned} \tag{22}$$

Taking $y = t = 2$, $b = 1$ and then from Eqs. (20) and (22), we conclude the required stability condition of Eq. (1) as follows:

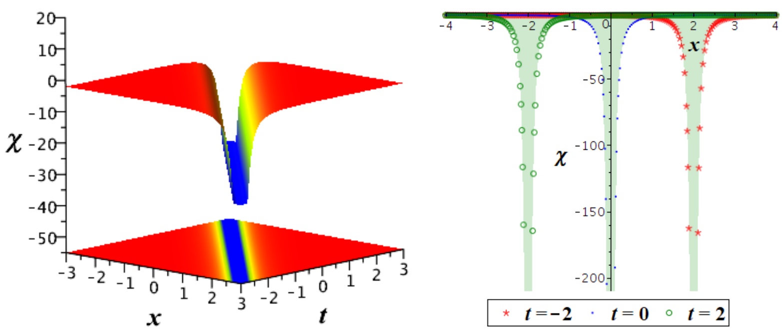
$$\frac{\partial R}{\partial s} = \frac{\begin{array}{c}-4(2\cos^4)(2s)\cos^2(2s+4) - 2\cos^2(2s)\cos^4(2s+4)\\ -\cos^4(2s) + \cos^4(2s+4)\end{array}}{\cos^4(2s)\cos^4(2s+4)} > 0. \tag{23}$$

Consequently, we get $\frac{\partial R}{\partial s}|_{s=0.1} = 39.94831046$ which is positive.

Hence, the solution of χ_4 is stable and Eq. (23) is the required condition for it. Similarly, we can determine the conditions of stability for other variables.

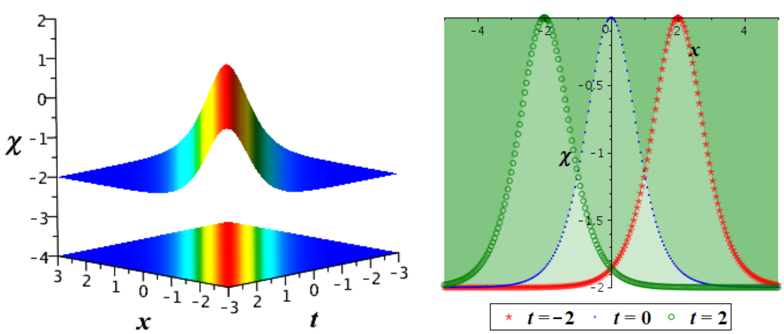
6. Graphical Representation

According to the graphical investigation, we see that the results χ_2, χ_3, χ_6, $\chi_{10,1}$, $\chi_{12,1}$, $\chi_{13,1}$, $\chi_{15,1}$, $\chi_{16,1}$ represent a dark soliton with singularity, χ_1, χ_7, $\chi_{11,1}$ denote a bright soliton and ξ_1, ξ_2, ξ_6, ξ_7, $\xi_{11,1}$ show double periodic wave. For $b = -1$, $p = q = 1$ and at $y = 0$, we draw the graph of χ_2 (Fig. 1), χ_7 (Fig. 2), and ξ_7 (Fig. 3).



(a) (b)

Fig. 1. (Color online) Soliton wave of χ_2: (a) 3D curve and (b) equivalent 2D curve.



(a) (b)

Fig. 2. (Color online) Soliton wave of χ_7: (a) 3D curve and (b) equivalent 2D curve.

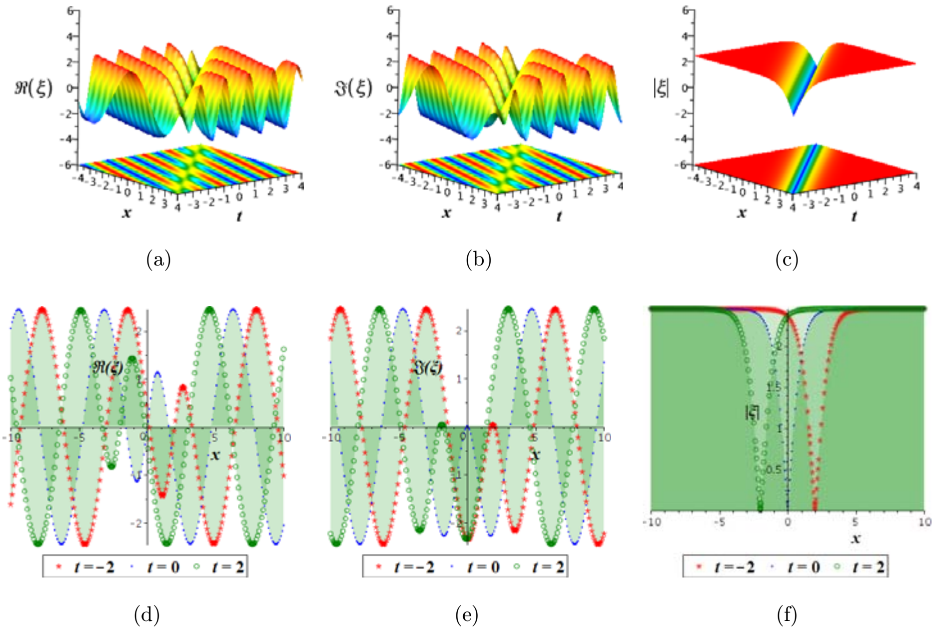


Fig. 3. (Color online) Wave profile of ξ_7: (a–c) 3D curve and (d–f) equivalent 2D plot.

Dark periodic waves with singularity, periodic waves, and multiple bright dark breather waves can be found for (χ_4, χ_5, χ_8, χ_9, χ_{10}, $\chi_{10,2}$, χ_{12}, $\chi_{12,2}$, χ_{13}, $\chi_{13,2}$, χ_{14}, $\chi_{14,1}$), (ξ_{11}, ξ_{16}, ξ_{15}, χ_{11}, χ_{15}, χ_{16}), and (ξ_4, ξ_5, ξ_8, ξ_9, ξ_{10}, $\xi_{10,2}$, ξ_{12}, $\xi_{12,2}$, ξ_{13}, $\xi_{13,2}$, ξ_{14}, $\xi_{14,1}$) respectively. Among them, χ_{13}, ξ_{11}, and ξ_{14} are attached in Figs. 4–6, respectively for $\rho = p = 0.5$, $a_{-1} = 1$ at $y = 0$.

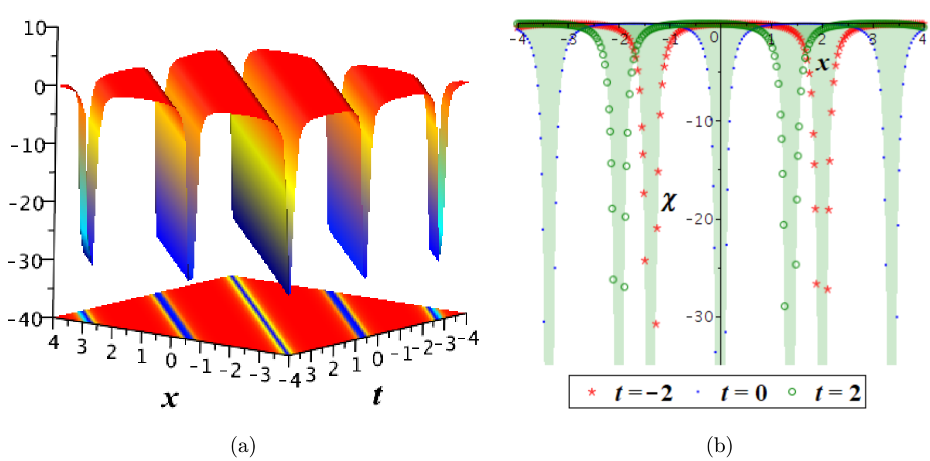


Fig. 4. (Color online) Profile of χ_{13}: (a) 3D graphs and (b) equivalent 2D curves.

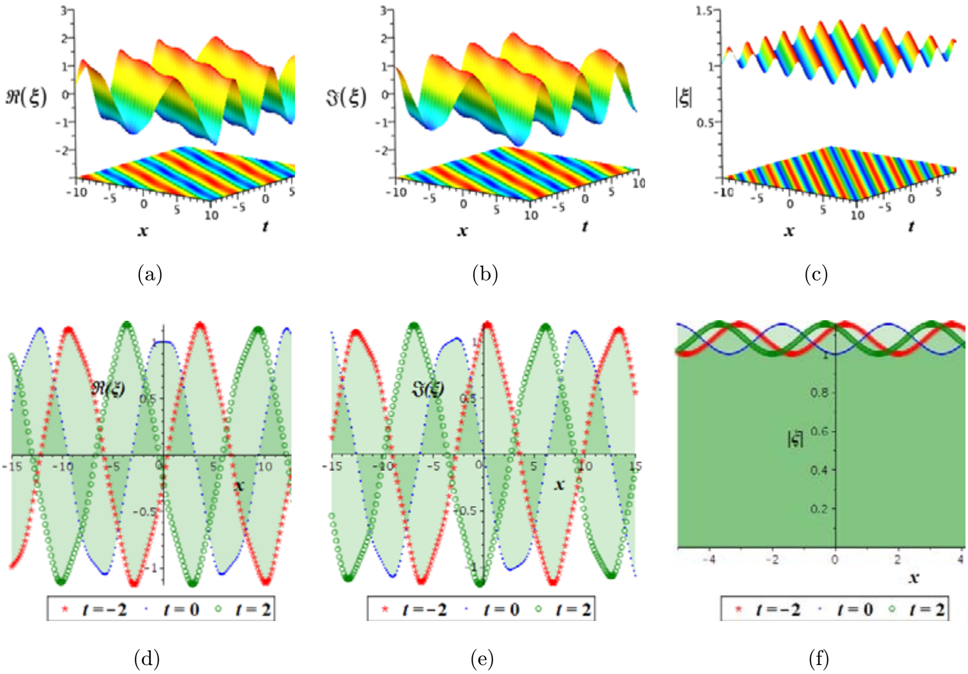


Fig. 5. (Color online) Behavior of ξ_{11}: (a–c) 3D graphs and (d–f) equivalent 2D graphs.

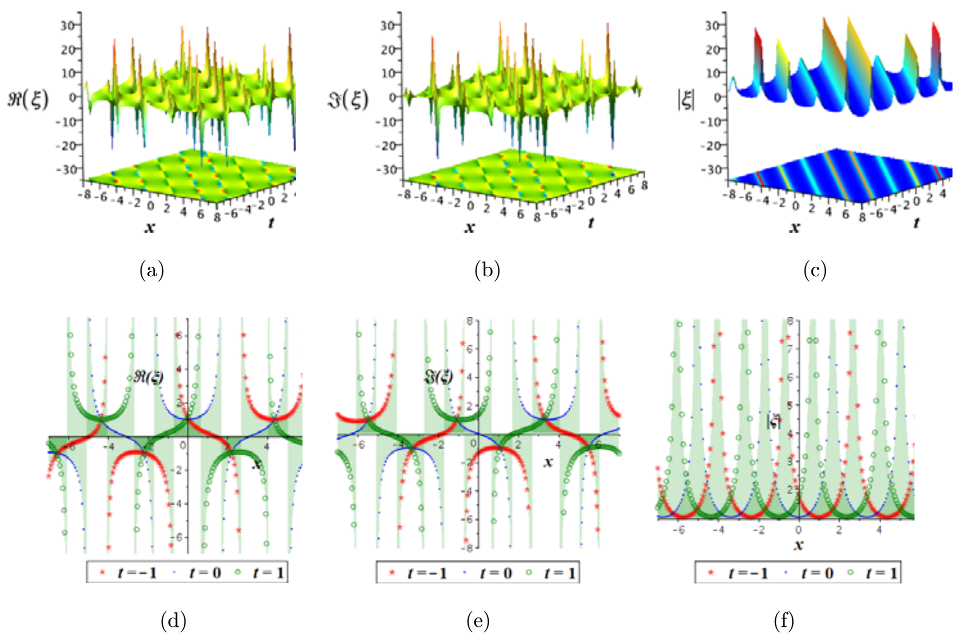


Fig. 6. (Color online) Wave profile of ξ_{14}: (a–c) 3D curves and (d–f) equivalent 2D plots.

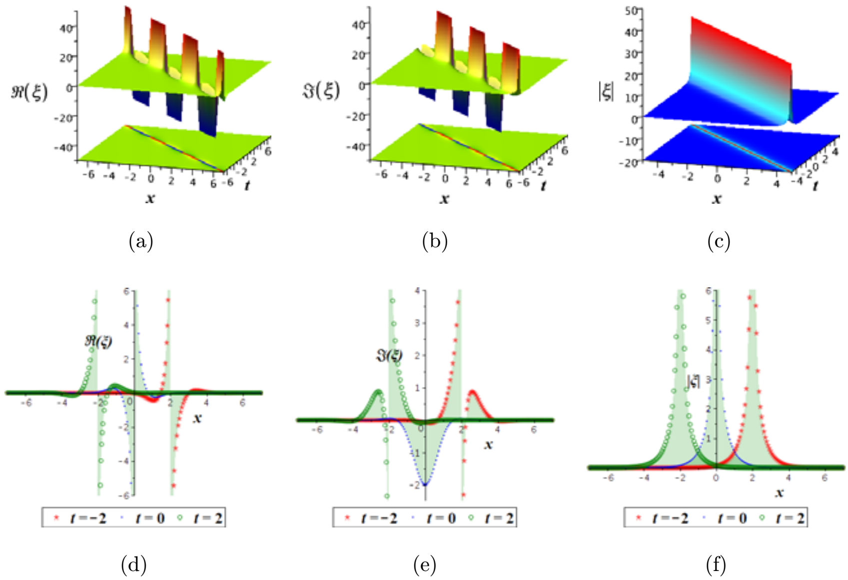


Fig. 7. (Color online) Profile of $\xi_{13,1}$: (a–c) 3D curves and (d–f) equivalent 2D plots.

Lastly, we get singular bright dark breather wave, periodic breather wave with singularity, and lump type breather wave from the outcomes (ξ_3, $\xi_{13,1}$, $\xi_{16,1}$), ($\xi_{10,1}$, $\xi_{12,1}$), and $\xi_{15,1}$ respectively which are displayed in Figs. 7–9 by $\xi_{13,1}$, $\xi_{10,1}$, and $\xi_{15,1}$, respectively for $\rho = 1$, $p = 0.5$, $a_{-1} = 1$, $y = 0$.

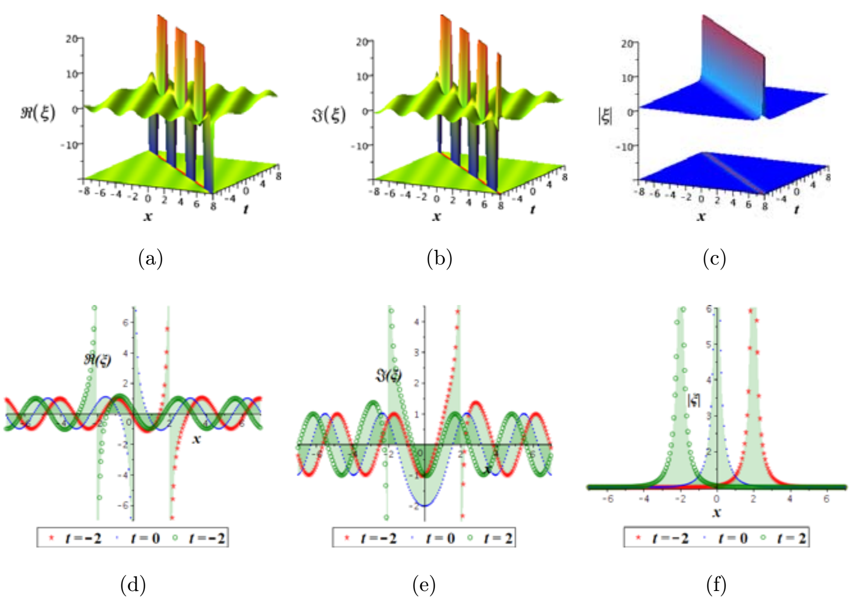


Fig. 8. (Color online) Behavior of $\xi_{10,1}$: (a–c) 3D graphs and (d–f) corresponding 2D graphs.

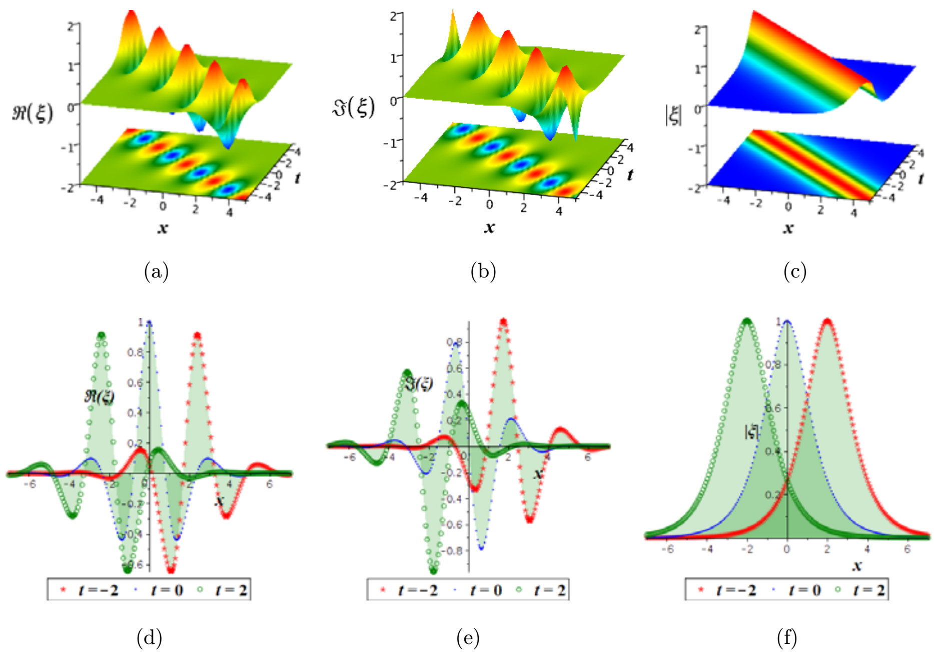


Fig. 9. (Color online) Wave profile of $\xi_{15.1}$: (a–c) 3D graphs and (d–f) corresponding 2D graphs.

7. Results Comparison

This section focuses on the novelty of the results and the achievements of this work. To judge the uniqueness, we compared our results with recently published works.[40–43] Mixed solitons can be found within the mentioned model through the multiplier method by Ghosh and Maitra.[40] Zafar and co-authors obtained bright soliton, periodic waves, and dark solitons for the proposed system through the modified Kudryashov's technique.[41] For the suggested nonlinear model, Raza and others derived explicit solutions via the general projective Riccati equation technique and improved $\tan(\frac{\Phi(\rho)}{2})$-expansion process.[42] Tarla and his co-workers studied bright-dark patterns for the recommended model using the JEF expansion technique.[43]

Table 1. Comparison of the obtained solutions and existing paper.[44]

Ma *et al.* solution	Our solution
If $P = \pm\sqrt{6}$, $Q = 2$, $G = x + y + \omega\tau$, $s(\tau, x, y) = \xi(x, y, t)$, $r(\tau, x, y) = \chi(x, y, t)$, and $\theta = px + qy + r\tau$, then the solution Eq. (49) of Ref. 44 becomes	If $P = \pm\sqrt{-6b}$, $Q = 2b$, $G = \sqrt{-b}(x + y + st)$, $\theta = -(px + qy + rt)$, $\xi_1(x, y, t) = \xi(x, y, t)$, and $\chi_1(x, y, t) = \chi(x, y, t)$ with $b < 0$, then our solutions $\xi_1(x, y, t)$ and $\chi_1(x, y, t)$ becomes
$\xi(x, y, t) = P\tanh(G)e^{i\theta}$, $\chi(x, y, t) = Q\tanh^2(G)$.	$\xi(x, y, t) = P\tanh(G)e^{i\theta}$, $\chi(x, y, t) = Q\tanh^2(G)$.

Table 2. Comparison of the obtained solutions and existing paper.[45]

Wazwaz's solution	Our solution
If $P=\pm\sqrt{6}k$, $Q=-2ks$, $G=kx+sy-\frac{7k^3-2ks-r}{k}t$, $\theta=kx+cy+rt$, $u(x,y,t)=\xi(x,y,t)$, and $v(x,y,t)=\chi(x,y,t)$, then the solution Eq. (95) of Ref. 45 becomes $\xi(x,y,t)=P\coth(\zeta)e^{i\theta}$, $\chi(x,y,t)=Q\coth^2(\zeta)$.	If $P=\pm\sqrt{-6b}$, $Q=2b$, $G=\sqrt{-b}(x+y+st)$, $\theta=-(px+qy+rt)$, $\xi_2(x,y,t)=\xi(x,y,t)$, and $\chi_2(x,y,t)=\chi(x,y,t)$ with $b<0$, then our solutions $\xi_2(x,y,t)$ and $\chi_2(x,y,t)$ becomes $\xi(x,y,t)=P\coth(G)e^{i\theta}$, $\chi(x,y,t)=Q\coth^2(G)$.

In our study, it can be found various outcomes: bright soliton, dark soliton, different types of periodic and breather waves, and lump-type breather waves. In addition, stability analysis is studied in this work. There are some different finer results than recently published works[40–43] which can be found in our work. In Tables 1 and 2, we also compare our results with those in Refs. 44 and 45.

8. Conclusion

For the first time, the METF technique and the EJEFE technique based on the HM model have been discussed in this paper. Some exact and analytical soliton outcomes can be found. We obtain a bright soliton, a dark soliton, periodic waves, dark periodic waves, double periodic waves, multiple bright dark breather waves, periodic breather waves, and bright dark breather waves. We can also find a special lump-breather wave. Additionally, the stability analysis of the obtained solutions is addressed by employing the Hamiltonian technique. 2D and 3D plotting techniques are considered to draw the graph of the results. We may conclude that the innovative approaches evaluated here apply to a wide range of nonlinear models, providing valuable insights into their dynamics. These findings illuminate nonlinear system behavior and can help identify relationships between variables.

ORCID

Noor Alam https://orcid.org/0009-0000-0411-7087
Wen-Xiu Ma https://orcid.org/0000-0001-5309-1493
Mohammad Safi Ullah https://orcid.org/0000-0002-8889-7865
Aly R. Seadawy https://orcid.org/0000-0002-7412-4773
Mahinur Akter https://orcid.org/0009-0005-2308-7756

References

1. M. S. Ullah, H. O. Roshid, W. X. Ma, M. Z. Ali and Z. Rahman, *Chin. J. Phys.* **68** (2020) 699.
2. N. Alam, S. Poddar, M. E. Karim, M. S. Hasan and G. Lorenzini, *Math. Mod. Eng. Prob.* **8** (2021) 739.
3. S. C. Anco and A. F. Cheviakov, *Int. J. Nonlin. Mech.* **126** (2020) 103569.
4. M. S. Ullah, M. Z. Ali and H. O. Roshid, *Part. Diff. Equ. Appl. Math.* **10** (2024) 100716.
5. W. X. Ma, *Opt. Quantum Electron.* **52** (2020) 511.

6. W. X. Ma, *Int. J. Nonlin. Sci. Numer. Simul.* **23** (2022) 123.
7. W. X. Ma, *Mathematics* **11** (2023) 4664.
8. M. S. Uddin, M. Begum, H. O. Roshid, M. S. Ullah and A. Abdeljabbar, *Part. Diff. Equ. Appl. Math.* **8** (2023) 100591.
9. N. Zhao, J. Manafian, O. A. Ilhan, G. Singh and R. Zulfugarova, *Int. J. Mod. Phys.* **35** (2021) 215017.
10. Y. Gu, S. Malmir, J. Manafian, O. A. Ilhan, A. Alizadeh and A. J. Othman, *Results Phys.* **43** (2022) 10603.
11. H. Zhang, J. Manafian, G. Singh, O. A. Ilhan and A. O. Zekiy, *Results Phys.* **25** (2021) 104168.
12. S. Duran, A. Yokus and G. Kilinc, *Phys. Scr.* **98** (2023) 125265.
13. A. Yokus and H. M. Baskonus, *Soft. Comput.* **26** (2022) 13605.
14. W. X. Ma, *Commun. Theor. Phys.* **73** (2021) 065001.
15. W. X. Ma and Y. You, *Trans. Amer. Math. Soc.* **357** (2005) 1753.
16. M. S. Ullah, M. Z. Ali and H. O. Roshid, *Sci. Rep.* **14** (2024) 11907.
17. M. Zhang, X. Xie, J. Manafian, O. A. Ilhan and G. Singh, *J. Adv. Res.* **38** (2022) 131.
18. R. Hirota, *J. Math. Phys.* **14** (1973) 805.
19. A. Maccari, *J. Math. Phys.* **39** (1998) 6547.
20. N. Cheemaa and M. Younis, *Nonlinear Dyn.* **83** (2016) 1395.
21. M. S. Ullah, D. Baleanu, M. Z. Ali and H. O. Roshid, *Chaos Solitons Fract.* **174** (2023) 3856.
22. M. S. Ullah, H. O. Roshid, M. Z. Ali and H. Rezazadeh, *Results Phys.* **49** (2023) 106535.
23. M. S. Ullah, *Part. Diff. Equ. Appl. Math.* **8** (2023) 100566.
24. Z. Rahman, M. Z. Ali, H. O. Roshid, M. S. Ullah and X. Y. Wen, *Pramana J. Phys.* **95** (2021) 134.
25. S. T. M. Din, A. Irshad, N. Ahmed and U. Khan, *Results Phys.* **7** (2017) 3901.
26. A. H. Ganie, A. M. Wazwaz, A. R. Seadawy, M. S. Ullah, H. O. Roshid, H. D. Afroz and R. Akter, *Pramana J. Phys.* **98** (2024) 46.
27. M. Zhang, X. Xie, J. Manafian, O. A. Ilhan and G. Singh, *J. Adv. Res.* **38** (2022) 131.
28. A. Rani, M. Ashraf and J. Ahmad, *Opt. Quantum Electron.* **54** (2022) 158.
29. M. S. Ullah, H. O. Roshid and M. Z. Ali, *PLoS ONE* **18** (2023) e0291071.
30. I. Onder, A. Secer, M. Ozisik and M. Bayram, *Optik* **257** (2022) 168761.
31. S. Kumar, K. S. Nisar and A. Kumar, *Results Phys.* **28**, 104621 (2021).
32. M. Alquran, *Opt. Quantum Electron.* **54** (2022) 666.
33. E. H. M. Zahran and M. M. A. Khater, *App. Math. Model.* **40** (2016) 1769.
34. W. Zhang, *Int. J. Diff. Equ.* **2011** (2011) 451420.
35. S. T. R. Rizvi, A. R. Seadawy, S. Ahmed, M. Younis and K. Ali, *Chaos Solitons Fract.* **151** (2021) 111251.
36. H. Alotaibi, *Symmetry* **13** (2021) 2126.
37. S. A. Elwakil, S. K. El-labany, M. A. Zahran and R. Sabry, *Phys. Lett. A* **299** (2002) 179.
38. M. S. Ullah, H. O. Roshid and M. Z. Ali, *Opt. Quantum Electron.* **56** (2024) 240.
39. M. Akter, M. S. Ullah, A. M. Wazwaz and A. R. Seadawy, *Opt. Quant. Electron.* **56** (2024) 1127.
40. A. Ghosh and S. Maitra, *Phys. Scr.* **98** (2023) 035219.
41. A. Zafar, M. Ijaz, S. M. Eldin, S. Anwar and I. Siddique, *Results Phys.* **43** (2022) 106049.
42. N. Raza, A. Jhangeer, H. Rezazadeh and A. Bekir, *Int. J. Mod. Phys. B* **33** (2019) 1950360.
43. S. Tarla, K. K. Ali and R. Yilmazer, *Opt. Quantum Electron.* **54** (2022) 613.
44. W. X. Ma, M. S. Osman, S. Arshed, N. Raza and H. M. Srivastava, *Chin. J. Phys.* **72** (2021) 475.
45. A. M. Wazwaz, *Phys. Scr.* **85** (2012) 0650.