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Riemann–Hilbert problems of a nonlocal reverse-time six-component AKNS system of fourth order and its exact soliton solutions

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We investigate the solvability of an integrable nonlinear nonlocal reverse-time six-component fourth-order AKNS system generated from a reduced coupled AKNS hierarchy under a reverse-time reduction. Riemann–Hilbert problems will be formulated by using the associated matrix spectral problems, and exact soliton solutions will be derived from the reflectionless case corresponding to an identity jump matrix.

Keywords: Riemann–Hilbert problem; inverse scattering; soliton solution; reverse-time, higher order nonlinear Schrödinger equation; soliton dynamics.

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1. Introduction

Integrable equations describe many physical phenomena such as magnetic fields, plasma physics, nonlinear optics and quantum fields, etc. Especially, nonlocal ones have been the center of interest of many studies nowadays.

Nonlocal equations describe PT symmetric dispersive waves. Interestingly, local integrable equations can be reduced to nonlocal ones under the time parity symmetry.[1–3]

There are many techniques to solve integrable equations such as Darboux transformation,[4] Hirota bilinear method[5] and the inverse scattering transform.[1,6–8] The Riemann–Hilbert technique is another powerful method to solve integrable equations and generate their soliton solutions.[9–12] In this paper, the associated Riemann–Hilbert problems are formulated for a six-component AKNS system of fourth-order equations in the AKNS multi-component hierarchy, and solutions to the Riemann–Hilbert problems are worked out while taking the identity jump matrix.[13–16]

The rest of the paper is structured as follows. In Sec. 2, we analyze the Riemann–Hilbert problems associated with the corresponding matrix spectral problems, which is closely related to the inverse scattering method. In Sec. 3, we will generate soliton solutions from the reflectionless problems, while in Sec. 4, we present a few examples of soliton solutions and look into their dynamics. Finally, the last section will be the conclusion, together with some remarks.

2. Riemann–Hilbert Problems

2.1. *Six-component AKNS hierarchy of coupled fourth-order integrable equations*

Let us consider the pair of spatial and temporal spectral problems for the six-component AKNS system[13]:

$$\psi_x = iU\psi, \tag{1}$$

$$\psi_t = iV^{[4]}\psi, \tag{2}$$

where ψ is the eigenfunction.

The spectral matrix is given by

$$U(u,\lambda) = \begin{pmatrix} \alpha_1\lambda & p_1 & p_2 & p_3 \\ r_1 & \alpha_2\lambda & 0 & 0 \\ r_2 & 0 & \alpha_2\lambda & 0 \\ r_3 & 0 & 0 & \alpha_2\lambda \end{pmatrix}, \tag{3}$$

where λ is a spectral parameter, α_1, α_2 are real constants, $p = (p_1, p_2, p_3)$ and $r = (r_1, r_2, r_3)^T$ are vector functions of (x, t) and $u = (p, r^T)^T$ is a vector of six potentials.

The Lax matrix operator $V^{[4]}$ is determined by

$$V^{[4]} = \begin{pmatrix} a^{[0]}\lambda^4 + a^{[2]}\lambda^2 & b_1^{[1]}\lambda^3 + b_1^{[2]}\lambda^2 & b_2^{[1]}\lambda^3 + b_2^{[2]}\lambda^2 & b_3^{[1]}\lambda^3 + b_3^{[2]}\lambda^2 \\ \quad + a^{[3]}\lambda + a^{[4]} & \quad + b_1^{[3]}\lambda + b_1^{[4]} & \quad + b_2^{[3]}\lambda + b_2^{[4]} & \quad + b_3^{[3]}\lambda + b_3^{[4]} \\ c_1^{[1]}\lambda^3 + c_1^{[2]}\lambda^2 & d_{11}^{[0]}\lambda^4 + d_{11}^{[2]}\lambda^2 & d_{12}^{[2]}\lambda^2 + d_{12}^{[3]}\lambda & d_{13}^{[2]}\lambda^2 + d_{13}^{[3]}\lambda \\ \quad + c_1^{[3]}\lambda + c_1^{[4]} & \quad + d_{11}^{[3]}\lambda + d_{11}^{[4]} & \quad + d_{12}^{[4]} & \quad + d_{13}^{[4]} \\ c_2^{[1]}\lambda^3 + c_2^{[2]}\lambda^2 & d_{21}^{[2]}\lambda^2 + d_{21}^{[3]}\lambda & d_{22}^{[0]}\lambda^4 + d_{22}^{[2]}\lambda^2 & d_{23}^{[2]}\lambda^2 + d_{23}^{[3]}\lambda \\ \quad + c_2^{[3]}\lambda + c_2^{[4]} & \quad + d_{21}^{[4]} & \quad + d_{22}^{[3]}\lambda + d_{22}^{[4]} & \quad + d_{23}^{[4]} \\ c_3^{[1]}\lambda^3 + c_3^{[2]}\lambda^2 & d_{31}^{[2]}\lambda^2 + d_{31}^{[3]}\lambda & d_{32}^{[2]}\lambda^2 + d_{32}^{[3]}\lambda & d_{33}^{[0]}\lambda^4 + d_{33}^{[2]}\lambda^2 \\ \quad + c_3^{[3]}\lambda + c_3^{[4]} & \quad + d_{31}^{[4]} & \quad + d_{32}^{[4]} & \quad + d_{33}^{[3]}\lambda + d_{33}^{[4]} \end{pmatrix}, \quad (4)$$

where all the involved functions are defined as follows:

$$\begin{cases} a^{[0]} = \beta_1, \\ a^{[1]} = 0, \\ a^{[2]} = -\dfrac{\beta}{\alpha^2} \displaystyle\sum_{i=1}^{3} p_i r_i, \\ a^{[3]} = -i\dfrac{\beta}{\alpha^3} \displaystyle\sum_{i=1}^{3} (p_i r_{i,x} - p_{i,x} r_i), \\ a^{[4]} = \dfrac{\beta}{\alpha^4} \left[3\left(\displaystyle\sum_{i=1}^{3} p_i r_i \right)^2 + \displaystyle\sum_{i=1}^{3} (p_i r_{i,xx} - p_{i,x} r_{i,x} + p_{i,xx} r_i) \right], \\ a^{[5]} = i\dfrac{\beta}{\alpha^5} \left[6\left(\displaystyle\sum_{i=1}^{3} p_i r_i \right) - \displaystyle\sum_{i=1}^{3} (p_i r_{i,x} - p_{i,x} r_i) \right. \\ \qquad \left. + \displaystyle\sum_{i=1}^{3} (p_i r_{i,xxx} - p_{i,xxx} r_i + p_{i,xx} r_{i,x} - p_{i,x} r_{i,xx}) \right], \end{cases}$$

$$
\begin{cases}
b_k^{[0]} = 0,\\
b_k^{[1]} = \dfrac{\beta}{\alpha} r_k,\\
b_k^{[2]} = -i\dfrac{\beta}{\alpha^2} p_{k,x},\\
b_k^{[3]} = -\dfrac{\beta}{\alpha^3}\left[p_{k,xx} + 2\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right)p_k\right],\\
b_k^{[4]} = i\dfrac{\beta}{\alpha^4}\left[p_{k,xxx} + 3\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right)p_{k,x} + 3\left(\displaystyle\sum_{i=1}^{3} p_{i,x} r_i\right)p_k\right],\\
b_k^{[5]} = \dfrac{\beta}{\alpha^5}\left[p_{k,xxxx} + 4\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right)p_{k,xx} + \left(6\displaystyle\sum_{i=1}^{3} p_{i,x} r_i + 2\displaystyle\sum_{i=1}^{3} p_i r_{i,x}\right)p_{k,x}\right.\\
\qquad\left. + \left(4\displaystyle\sum_{i=1}^{3} p_{i,xx} r_i + 2\displaystyle\sum_{i=1}^{3} p_{i,x} r_{i,x} + 2\displaystyle\sum_{i=1}^{3} p_i r_{i,xx} + 6\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right)^2\right)p_k\right],
\end{cases}
$$

$$
\begin{cases}
c_k^{[0]} = 0,\\
c_k^{[1]} = \dfrac{\beta}{\alpha} p_k,\\
c_k^{[2]} = i\dfrac{\beta}{\alpha^2} r_{k,x},\\
c_k^{[3]} = -\dfrac{\beta}{\alpha^3}\left[r_{k,xx} + 2\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right)r_k\right],\\
c_k^{[4]} = -i\dfrac{\beta}{\alpha^4}\left[r_{k,xxx} + 3\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right)r_{k,x} + 3\left(\displaystyle\sum_{i=1}^{3} p_{i,x} r_i\right)r_k\right],\\
c_k^{[5]} = \dfrac{\beta}{\alpha^5}\left[r_{k,xxxx} + 4\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right)r_{k,xx} + \left(6\displaystyle\sum_{i=1}^{3} p_i r_{i,x} + 2\displaystyle\sum_{i=1}^{3} p_{i,x} r_i\right)r_{k,x}\right.\\
\qquad\left. + \left(4\displaystyle\sum_{i=1}^{3} p_i r_{i,xx} + 2\displaystyle\sum_{i=1}^{3} p_{i,x} r_{i,x} + 2\displaystyle\sum_{i=1}^{3} p_{i,xx} r_i + 6\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right)^2\right)r_k\right]
\end{cases}
$$

and

$$\begin{cases} d_{kj}^{[0]} = \beta_2 I_3, \\ d_{kj}^{[1]} = 0, \\ d_{kj}^{[2]} = \dfrac{\beta}{\alpha^2} p_j r_k, \\ d_{kj}^{[3]} = -i\dfrac{\beta}{\alpha^3}(p_{j,x} r_k - p_j r_{k,x}), \\ d_{kj}^{[4]} = -\dfrac{\beta}{\alpha^4}\left[3p_j\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right) r_k + p_{j,xx} r_k - p_{j,x} r_{k,x} + p_j r_{k,xx}\right], \\ d_{kj}^{[5]} = \dfrac{\beta}{\alpha^5}\left[2p_j\left(\displaystyle\sum_{i=1}^{3} p_{i,x} r_i - p_i r_{i,x}\right) r_k + 4p_{j,x}\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right) r_k - p_j\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right) r_{k,x} \right. \\ \qquad\qquad \left. + p_{j,xxx} r_k - p_j r_{k,xxx} + p_{j,x} r_{k,xx} - p_{j,xx} r_{k,x}\right], \end{cases}$$

where $\alpha = \alpha_1 - \alpha_2$ and $\beta = \beta_1 - \beta_2$. We always assume that $b^{[i]} = (b_1^{[i]}, b_2^{[i]}, b_3^{[i]})$, $c^{[i]} = (c_1^{[i]}, c_2^{[i]}, c_3^{[i]})^T$ and $d^{[i]} = (d_{kl}^{[i]})_{3\times 3}$, for $i \in \{1,2,3,4,5\}$.

The compatibility condition $\psi_{xt} = \psi_{tx}$ will lead to the zero curvature equation:

$$U_t - V_x^{[4]} + i[U, V^{[4]}] = 0, \tag{5}$$

which gives the six-component system of soliton equations

$$u_t = \begin{pmatrix} p^T \\ r \end{pmatrix}_t = i\begin{pmatrix} \alpha b^{[5]T} \\ -\alpha c^{[5]} \end{pmatrix}, \tag{6}$$

where $b^{[5]}$ and $c^{[5]}$ are defined earlier. Thus, we deduce the coupled AKNS system of fourth-order equations[13]:

$$\begin{cases} p_{k,t} = i\dfrac{\beta}{\alpha^4}\left[p_{k,xxxx} + 4\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right) p_{k,xx} + \left(6\displaystyle\sum_{i=1}^{3} p_{i,x} r_i + 2\sum_{i=1}^{3} p_i r_{i,x}\right) p_{k,x} \right. \\ \qquad \left. + \left(4\displaystyle\sum_{i=1}^{3} p_{i,xx} r_i + 2\sum_{i=1}^{3} p_{i,x} r_{i,x} + 2\sum_{i=1}^{3} p_i r_{i,xx} + 6\left(\sum_{i=1}^{3} p_i r_i\right)^2\right) p_k\right], \\ r_{k,t} = -i\dfrac{\beta}{\alpha^4}\left[r_{k,xxxx} + 4\left(\displaystyle\sum_{i=1}^{3} p_i r_i\right) r_{k,xx} + \left(6\displaystyle\sum_{i=1}^{3} p_i r_{i,x} + 2\sum_{i=1}^{3} p_{i,x} r_i\right) r_{k,x} \right. \\ \qquad \left. + \left(4\displaystyle\sum_{i=1}^{3} p_i r_{i,xx} + 2\sum_{i=1}^{3} p_{i,x} r_{i,x} + 2\sum_{i=1}^{3} p_{i,xx} r_i + 6\left(\sum_{i=1}^{3} p_i r_i\right)^2\right) r_k\right], \end{cases} \tag{7}$$

where $k \in \{1,2,3\}$.

2.2. *Nonlocal reverse-time six-component AKNS system*

Let us consider a class of specific nonlocal reverse-time reductions for the spectral matrix

$$U^T(x,-t,-\lambda) = -CU(x,t,\lambda)C^{-1}, \tag{8}$$

where $C = \begin{pmatrix} 1 & 0 \\ 0 & \Sigma \end{pmatrix}$ and Σ is a constant invertible symmetric 3×3 matrix, i.e., $\Sigma^T = \Sigma$ and $\det \Sigma \neq 0$.

As $U(x,t,\lambda) = \lambda\Lambda + P(x,t)$, for $P = \begin{pmatrix} 0 & p \\ r & 0 \end{pmatrix}$ and $\Lambda = \mathrm{diag}(\alpha_1, \alpha_2 I_3)$, then we have

$$P^T(x,-t) = -CP(x,t)C^{-1}. \tag{9}$$

From the above (9), we get

$$p^T(x,-t) = -\Sigma r(x,t) \quad \text{i.e.,} \quad r(x,t) = -\Sigma^{-1} p^T(x,-t). \tag{10}$$

As $V^{[4]}(x,t,\lambda) = \lambda^4\Omega + Q(x,t,\lambda)$ and from (9), we prove that

$$\begin{aligned} {V^{[4]}}^T(x,-t,-\lambda) &= CV^{[4]}(x,t,\lambda)C^{-1} \quad \text{and} \\ Q^T(x,-t,-\lambda) &= CQ(x,t,\lambda)C^{-1}, \end{aligned} \tag{11}$$

where $\Omega = \mathrm{diag}(\beta_1, \beta_2 I_3)$.

Importantly, the two Lax pair matrices $U^T(x,-t,-\lambda)$ and ${V^{[4]}}^T(x,-t,-\lambda)$ satisfy an equivalent zero curvature equation.

From this specific nonlocal reduction, the coupled six-component fourth-order AKNS equations can be reduced to the nonlocal reverse-time six-component fourth-order equations.

As Σ is invertible and symmetric so diagonalizable, then we can take $\Sigma = \mathrm{diag}(\rho_1^{-1}, \rho_2^{-1}, \rho_3^{-1})$, for ρ_1, ρ_2, ρ_3 nonzero real. Thus, $\Sigma^{-1} = \mathrm{diag}(\rho_1, \rho_2, \rho_3)$ leads (10) to

$$r_i(x,t) = -\rho_i p_i(x,-t) \quad \text{for} \quad i \in \{1,2,3\}. \tag{12}$$

Therefore, the coupled equations (7) reduce to the nonlocal reverse-time fourth-order equation

$$\begin{aligned} p_{k,t}(x,t) = i\frac{\beta}{\alpha^4}\Bigg[& p_{k,xxxx}(x,t) - 4\left(\sum_{i=1}^{3} \rho_i p_i(x,t) p_i(x,-t)\right) p_{k,xx}(x,t) \\ & - \left(6\sum_{i=1}^{3} \rho_i p_{i,x}(x,t) p_i(x,-t) + 2\sum_{i=1}^{3} \rho_i p_i(x,t) p_{i,x}(x,-t)\right) p_{k,x}(x,t) \end{aligned}$$

$$
-\left(4\sum_{i=1}^{3}\rho_i p_{i,xx}(x,t)p_i(x,-t)+2\sum_{i=1}^{3}\rho_i p_{i,x}(x,t)p_{i,x}(x,-t)\right.
$$
$$
\left.\left.+2\sum_{i=1}^{3}\rho_i p_i(x,t)p_{i,xx}(x,-t)-6\left(\sum_{i=1}^{3}\rho_i p_i(x,t)p_i(x,-t)\right)^2\right)p_k(x,t)\right] \tag{13}
$$

for $k \in \{1,2,3\}$.

We should note that if Σ is negative definite i.e., each $\rho_i < 0$ for $i \in \{1,2,3\}$, then we obtain the focusing nonlocal reverse-time six-component fourth-order equation due to the fact that the dispersive term and nonlinear terms attract. If ρ_i's are not all the same sign for $i \in \{1,2,3\}$, we obtain combined focusing and defocusing cases.

2.3. *Riemann–Hilbert problems*

The Lax pair of the six-component fourth-order AKNS equations can be written as follows:

$$
\psi_x = iU\psi = i(\lambda\Lambda + P)\psi, \tag{14}
$$
$$
\psi_t = iV^{[4]}\psi = i(\lambda^4\Omega + Q)\psi, \tag{15}
$$

where $\Omega = \mathrm{diag}(\beta_1,\beta_2,\beta_2,\beta_2)$, $\Lambda = \mathrm{diag}(\alpha_1,\alpha_2,\alpha_2,\alpha_2)$ and

$$
P = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ r_1 & 0 & 0 & 0 \\ r_2 & 0 & 0 & 0 \\ r_3 & 0 & 0 & 0 \end{pmatrix},
$$

$$
Q = \begin{pmatrix}
a^{[2]}\lambda^2 + a^{[3]}\lambda + a^{[4]} & b_1^{[1]}\lambda^3 + b_1^{[2]}\lambda^2 + b_1^{[3]}\lambda + b_1^{[4]} & b_2^{[1]}\lambda^3 + b_2^{[2]}\lambda^2 + b_2^{[3]}\lambda + b_2^{[4]} & b_3^{[1]}\lambda^3 + b_3^{[2]}\lambda^2 + b_3^{[3]}\lambda + b_3^{[4]} \\
c_1^{[1]}\lambda^3 + c_1^{[2]}\lambda^2 + c_1^{[3]}\lambda + c_1^{[4]} & d_{11}^{[2]}\lambda^2 + d_{11}^{[3]}\lambda + d_{11}^{[4]} & d_{12}^{[2]}\lambda^2 + d_{12}^{[3]}\lambda + d_{12}^{[4]} & d_{13}^{[2]}\lambda^2 + d_{13}^{[3]}\lambda + d_{13}^{[4]} \\
c_2^{[1]}\lambda^3 + c_2^{[2]}\lambda^2 + c_2^{[3]}\lambda + c_2^{[4]} & d_{21}^{[2]}\lambda^2 + d_{21}^{[3]}\lambda + d_{21}^{[4]} & d_{22}^{[2]}\lambda^2 + d_{22}^{[3]}\lambda + d_{22}^{[4]} & d_{23}^{[2]}\lambda^2 + d_{23}^{[3]}\lambda + d_{23}^{[4]} \\
c_3^{[1]}\lambda^3 + c_3^{[2]}\lambda^2 + c_3^{[3]}\lambda + c_3^{[4]} & d_{31}^{[2]}\lambda^2 + d_{31}^{[3]}\lambda + d_{31}^{[4]} & d_{32}^{[2]}\lambda^2 + d_{32}^{[3]}\lambda + d_{32}^{[4]} & d_{33}^{[2]}\lambda^2 + d_{33}^{[3]}\lambda + d_{33}^{[4]}
\end{pmatrix}. \tag{16}
$$

Our purpose is to find soliton solutions from an initial condition $(p(x,0), r^T(x,0))^T$ to $(p(x,t), r^T(x,t))^T$ at any time t. We assume that any p_i and r_i decay exponentially, i.e., $p_i \to 0$ and $r_i \to 0$ as $x,t \to \pm\infty$ for $i \in \{1,2,3\}$. Therefore, from the spectral problems (14), (15), ψ will behave asymptotically $\psi(x,t) \rightsquigarrow e^{i\lambda\Lambda x + i\lambda^4\Omega t}$.

We can then expect the solution for the spectral problems to be

$$\psi(x,t) = \phi(x,t)e^{i\lambda\Lambda x + i\lambda^4\Omega t}. \tag{17}$$

For the Jost solution,[9,17] we require that

$$\phi(x,t) \to I_4, \quad \text{as} \quad x,t \to \pm\infty, \tag{18}$$

where I_4 is the 4×4 identity matrix. Substituting (17) into the Lax pair, (14) and (15), will result in the equivalent expression of the spectral problems

$$\phi_x = i\lambda[\Lambda, \phi] + iP\phi, \tag{19}$$

$$\phi_t = i\lambda^4[\Omega, \phi] + iQ\phi. \tag{20}$$

Now, we are going to work with the spatial spectral problem (19), assuming that the time is $t = 0$ for the direct scattering process.

By Liouville's formula,[9] as $\mathrm{tr}(iP) = 0$ and $\mathrm{tr}(iQ) = 0$, so $\det(\phi)$ is a constant, and using the boundary condition (18), we get

$$\det(\phi) = 1. \tag{21}$$

To construct the Riemann–Hilbert problems and their solutions in the reflectionless case, we are going to use the adjoint scattering equations of the spectral problems $\psi_x = iU\psi$ and $\psi_t = iV^{[4]}\psi$. Their adjoints are

$$\tilde{\psi}_x = -i\tilde{\psi}U, \tag{22}$$

$$\tilde{\psi}_t = -i\tilde{\psi}V^{[4]}, \tag{23}$$

and the equivalent spectral adjoint equations read

$$\tilde{\phi}_x = -i\lambda[\tilde{\phi}, \Lambda] - i\tilde{\phi}P, \tag{24}$$

$$\tilde{\phi}_t = -i\lambda^4[\tilde{\phi}, \Omega] - i\tilde{\phi}Q. \tag{25}$$

As $\phi_x^{-1} = -\phi^{-1}\phi_x\phi^{-1}$, we have from (19),

$$\phi_x^{-1} = -i\lambda[\phi^{-1}, \Lambda] - i\phi^{-1}P. \tag{26}$$

Therefore, we deduce that $(\phi^{\pm})^{-1}$ satisfies the adjoint equation (24). Similarly, we can show that $(\phi^{\pm})^{-1}$ satisfies (25) as well.

Now, if the eigenfunction $\phi(x,t,\lambda)$ is a solution of the spectral problem (19), then $C\phi^{-1}(x,t,\lambda)$ is a solution of the spectral adjoint problem (24) with the same eigenvalue because $\phi_x^{-1} = -\phi^{-1}\phi_x\phi^{-1}$. Also $\phi^T(x,-t,-\lambda)C$ is a solution of the spectral adjoint problem (24). As both solutions have the same boundary condition as $x \to \pm\infty$ which guarantees the uniqueness of the solution, so

$$\phi^T(x,-t,-\lambda)C = C\phi^{-1}(x,t,\lambda) \quad \text{or} \quad \phi^T(x,-t,-\lambda) = C\phi^{-1}(x,t,\lambda)C^{-1}. \tag{27}$$

This tells us that if λ is an eigenvalue of the spectral problems, then $-\lambda$ is also an eigenvalue.

For the rest of the problem, we assume that $\alpha < 0$ and $\beta < 0$ and $Y^\pm$ tell us at which end of the x-axis, the boundary conditions are set. We know that

$$\phi^\pm \to I_4 \quad \text{when} \quad x \to \pm\infty. \tag{28}$$

We can then write

$$\psi^\pm = \phi^\pm e^{i\lambda\Lambda x}. \tag{29}$$

As ψ^+ and ψ^- are two solutions of the spectral spatial differential equation of first-order (14), they are then linearly dependent, and so they are related by a scattering matrix $S(\lambda)$. As a result,

$$\psi^- = \psi^+ S(\lambda), \tag{30}$$

using (29), we have

$$\phi^- = \phi^+ e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}, \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{31}$$

where

$$S(\lambda) = (s_{ij})_{4\times 4} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}. \tag{32}$$

Because $\det(\phi^\pm) = 1$, one has

$$\det(S(\lambda)) = 1. \tag{33}$$

From (27) and (31), we have this involution relation

$$S^T(-\lambda) = CS^{-1}(\lambda)C^{-1}. \tag{34}$$

From (34), we deduce that

$$\hat{s}_{11}(\lambda) = s_{11}(-\lambda), \tag{35}$$

where the inverse scattering data matrix $S^{-1} = (\hat{s}_{ij})_{4\times 4}$ for $i, j \in \{1,2,3,4\}$.

We can see that the recovery of the potentials will depend on the information of the scattering data from the scattering matrix $S(\lambda)$. As $\phi^\pm \to I_4$ when $x \to \pm\infty$, we need to analyze the analyticity of the Jost matrix $\phi^\pm$ in order to formulate the Riemann–Hilbert problems.

One can write the solution $\phi^{\pm}$ in a unique manner by the Volterra integral equations using (14):

$$\phi^-(x,\lambda) = I_4 + i\int_{-\infty}^{x} e^{i\lambda\Lambda(x-y)}P(y)\phi^-(y,\lambda)e^{i\lambda\Lambda(y-x)}dy, \tag{36}$$

$$\phi^+(x,\lambda) = I_4 - i\int_{x}^{+\infty} e^{i\lambda\Lambda(x-y)}P(y)\phi^+(y,\lambda)e^{i\lambda\Lambda(y-x)}dy. \tag{37}$$

If the matrix ϕ^- is

$$\phi^- = \begin{pmatrix} \phi_{11}^- & \phi_{12}^- & \phi_{13}^- & \phi_{14}^- \\ \phi_{21}^- & \phi_{22}^- & \phi_{23}^- & \phi_{24}^- \\ \phi_{31}^- & \phi_{32}^- & \phi_{33}^- & \phi_{34}^- \\ \phi_{41}^- & \phi_{42}^- & \phi_{43}^- & \phi_{44}^- \end{pmatrix}, \tag{38}$$

then the components of the first column of ϕ^- are

$$\phi_{11}^- = 1 + i\int_{-\infty}^{x}(p_1(y)\phi_{21}^-(y,\lambda) + p_2(y)\phi_{31}^-(y,\lambda) + p_3(y)\phi_{41}^-(y,\lambda))dy, \tag{39}$$

$$\phi_{21}^- = i\int_{-\infty}^{x} r_1(y)\phi_{11}^-(y,\lambda)e^{-i\lambda\alpha(x-y)}dy, \tag{40}$$

$$\phi_{31}^- = i\int_{-\infty}^{x} r_2(y)\phi_{11}^-(y,\lambda)e^{-i\lambda\alpha(x-y)}dy, \tag{41}$$

$$\phi_{41}^- = i\int_{-\infty}^{x} r_3(y)\phi_{11}^-(y,\lambda)e^{-i\lambda\alpha(x-y)}dy. \tag{42}$$

Similarly, the components of the second column of ϕ^- are

$$\begin{aligned}\phi_{12}^- = i\int_{-\infty}^{x}&(p_1(y)\phi_{22}^-(y,\lambda) \\ &+ p_2(y)\phi_{32}^-(y,\lambda) + p_3(y)\phi_{42}^-(y,\lambda))e^{i\lambda\alpha(x-y)}dy,\end{aligned} \tag{43}$$

$$\phi_{22}^- = 1 + i\int_{-\infty}^{x} r_1(y)\phi_{12}^-(y,\lambda)dy, \tag{44}$$

$$\phi_{32}^- = i\int_{-\infty}^{x} r_2(y)\phi_{12}^-(y,\lambda)dy, \tag{45}$$

$$\phi_{42}^- = i\int_{-\infty}^{x} r_3(y)\phi_{12}^-(y,\lambda)dy, \tag{46}$$

and the components of the third column of ϕ^- are

$$\begin{aligned}\phi_{13}^- = i\int_{-\infty}^{x}&(p_1(y)\phi_{23}^-(y,\lambda) \\ &+ p_2(y)\phi_{33}^-(y,\lambda) + p_3(y)\phi_{43}^-(y,\lambda))e^{i\lambda\alpha(x-y)}dy,\end{aligned} \tag{47}$$

$$\phi_{23}^{-} = i\int_{-\infty}^{x} r_1(y)\phi_{13}^{-}(y,\lambda)dy, \tag{48}$$

$$\phi_{33}^{-} = 1 + i\int_{-\infty}^{x} r_2(y)\phi_{13}^{-}(y,\lambda)dy, \tag{49}$$

$$\phi_{43}^{-} = i\int_{-\infty}^{x} r_3(y)\phi_{13}^{-}(y,\lambda)dy, \tag{50}$$

and finally, the components of the fourth column of ϕ^- are

$$\begin{aligned}\phi_{14}^{-} = i\int_{-\infty}^{x} &(p_1(y)\phi_{24}^{-}(y,\lambda)\\ &+ p_2(y)\phi_{34}^{-}(y,\lambda) + p_3(y)\phi_{44}^{-}(y,\lambda))e^{i\lambda\alpha(x-y)}dy,\end{aligned} \tag{51}$$

$$\phi_{24}^{-} = i\int_{-\infty}^{x} r_1(y)\phi_{14}^{-}(y,\lambda)dy, \tag{52}$$

$$\phi_{34}^{-} = i\int_{-\infty}^{x} r_2(y)\phi_{14}^{-}(y,\lambda)dy, \tag{53}$$

$$\phi_{44}^{-} = 1 + i\int_{-\infty}^{x} r_3(y)\phi_{14}^{-}(y,\lambda)dy. \tag{54}$$

We can see that as $\alpha = \alpha_1 - \alpha_2 < 0$, if $\mathrm{Im}(\lambda) > 0$, then $\mathrm{Re}(e^{-i\lambda\alpha(x-y)})$ decays exponentially when $y < x$, and so each integral of the first column of ϕ^- converges. As a result, the components of the first column of ϕ^-, i.e., $\phi_{11}^-, \phi_{21}^-, \phi_{31}^-, \phi_{41}^-$ are analytic in the upper half complex plane for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$. But, if $\mathrm{Im}(\lambda) < 0$, $\mathrm{Re}(e^{i\lambda\alpha(x-y)})$ also decays, then the components of the last three columns of ϕ^- converge, and thus, they are analytic in the lower half plane for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

In the same way, for $y > x$, the components of the last three columns of ϕ^+ are analytic in the upper half plane for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$, and the components of the first column of ϕ^+ are analytic in the lower half plane for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Now let us construct the Riemann–Hilbert problems. Note that

$$\psi^{\pm} = \phi^{\pm}e^{i\lambda\Lambda x}, \quad \text{so} \quad \phi^{\pm} = \psi^{\pm}e^{-i\lambda\Lambda x}. \tag{55}$$

Let $\phi_j^{\pm}$ be the jth column of $\phi^{\pm}$ for $j \in \{1,2,3,4\}$, and so the first Jost matrix solution can be taken as

$$P^{+}(x,\lambda) = (\phi_1^{-}, \phi_2^{+}, \phi_3^{+}, \phi_4^{+}) = \phi^{-}H_1 + \phi^{+}H_2, \tag{56}$$

where $\phi_1^- = (\phi_{11}^-, \phi_{21}^-, \phi_{31}^-, \phi_{41}^-)^T$, $\phi_2^+ = (\phi_{12}^+, \phi_{22}^+, \phi_{32}^+, \phi_{42}^+)^T$, $\phi_3^+ = (\phi_{13}^+, \phi_{23}^+, \phi_{33}^+, \phi_{43}^+)^T$, $\phi_4^+ = (\phi_{14}^+, \phi_{24}^+, \phi_{34}^+, \phi_{44}^+)^T$ and $H_1 = \mathrm{diag}(1,0,0,0)$ and $H_2 = \mathrm{diag}(0,1,1,1)$. P^+ is then analytic for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

To construct the analytic counterpart of $P^+ \in \mathbb{C}_+$, it is going to be simpler to use the equivalent spectral adjoint equation (26). Because $\tilde{\phi}^{\pm} = (\phi^{\pm})^{-1}$ and

$\psi^{\pm} = \phi^{\pm} e^{i\lambda\Lambda x}$, we have

$$(\phi^{\pm})^{-1} = e^{i\lambda\Lambda x}(\psi^{\pm})^{-1}. \tag{57}$$

Now, let $\tilde{\phi}_j^{\pm}$ be the jth row of $\tilde{\phi}^{\pm}$ for $j \in \{1,2,3,4\}$. In the same way, we proved for P^+ above, we can get

$$P^-(x,\lambda) = (\tilde{\phi}_1^-, \tilde{\phi}_2^+, \tilde{\phi}_3^+, \tilde{\phi}_4^+)^T = H_1(\phi^-)^{-1} + H_2(\phi^+)^{-1}. \tag{58}$$

P^- is analytic for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

From (56), (58) and (55) along with $\phi^T(x,-t,-\lambda) = C\phi^{-1}(x,t,\lambda)C^{-1}$, we have the nonlocal involution property

$$(P^+)^T(x,-t,-\lambda) = CP^-(x,t,\lambda)C^{-1}. \tag{59}$$

Through what have been done above, we have been able to construct the matrix of eigenfunctions P^+ and P^- that are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, and continuous in $\mathbb{C}_+ \cup \mathbb{R}$ and $\mathbb{C}_- \cup \mathbb{R}$, respectively.

From (56) and (58), we have

$$P^-(x,\lambda)P^+(x,\lambda) = e^{i\lambda\Lambda x}(H_1 + H_2 S)(H_1 + S^{-1}H_2)e^{-i\lambda\Lambda x}, \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{60}$$

where the inverse scattering data matrix $S^{-1} = (\hat{s}_{ij})_{4\times 4}$ for $i,j \in \{1,2,3,4\}$.

Using (31) in (56), we have

$$P^+(x,\lambda) = \phi^+(e^{i\lambda\Lambda x} S e^{-i\lambda\Lambda x} H_1 + H_2), \tag{61}$$

as $\phi^+(x,\lambda) \to I_4$ when $x \to +\infty$, then

$$\lim_{x\to+\infty} P^+ = \begin{pmatrix} s_{11}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for} \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R}. \tag{62}$$

In the same way, we have as well

$$\lim_{x\to-\infty} P^- = \begin{pmatrix} \hat{s}_{11}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for} \quad \lambda \in \mathbb{C}_- \cup \mathbb{R}. \tag{63}$$

Thus, if we choose

$$G^+(x,\lambda) = P^+(x,\lambda) \begin{pmatrix} s_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$(G^-)^{-1}(x,\lambda) = \begin{pmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^-(x,\lambda), \tag{64}$$

then on the real line, the two generalized matrices generate the matrix Riemann–Hilbert problems for the six-component AKNS system of fourth-order given by

$$G^+(x,\lambda) = G^-(x,\lambda)G_0(x,\lambda), \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{65}$$

where the jump matrix $G_0(x,\lambda)$ can be cast as

$$\begin{aligned} G_0(x,\lambda) = e^{i\lambda\Lambda x} &\begin{pmatrix} \hat{s}_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (H_1 + H_2 S)(H_1 + S^{-1}H_2) \\ &\times \begin{pmatrix} s_{11}^{-1}(\lambda) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} e^{-i\lambda\Lambda x}, \end{aligned} \tag{66}$$

which can be explicitly written as

$$\begin{aligned} &G_0(x,\lambda) \\ &= \begin{pmatrix} s_{11}^{-1}\hat{s}_{11}^{-1} & \hat{s}_{12}\hat{s}_{11}^{-1}e^{i\lambda x(\alpha_1-\alpha_2)} & \hat{s}_{13}\hat{s}_{11}^{-1}e^{i\lambda x(\alpha_1-\alpha_2)} & \hat{s}_{14}\hat{s}_{11}^{-1}e^{i\lambda x(\alpha_1-\alpha_2)} \\ s_{21}s_{11}^{-1}e^{-i\lambda x(\alpha_1-\alpha_2)} & 1 & 0 & 0 \\ s_{31}s_{11}^{-1}e^{-i\lambda x(\alpha_1-\alpha_2)} & 0 & 1 & 0 \\ s_{41}s_{11}^{-1}e^{-i\lambda x(\alpha_1-\alpha_2)} & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{67}$$

with its canonical normalization conditions given by

$$G^+(x,\lambda) \to I_4 \quad \text{as} \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R} \to \infty, \tag{68}$$

$$G^-(x,\lambda) \to I_4 \quad \text{as} \quad \lambda \in \mathbb{C}_- \cup \mathbb{R} \to \infty. \tag{69}$$

From (59) along with (64) and (35), we obtain

$$(G^+)^T(x,-t,-\lambda) = C(G^-)^{-1}(x,t,\lambda)C^{-1}. \tag{70}$$

Also, from (66) and (35), we have this involution property

$$G_0^T(x,-t,-\lambda) = CG_0(x,t,\lambda)C^{-1}. \tag{71}$$

2.4. *Time evolution of the scattering data*

The process of the inverse scattering transform requires the time evolution of the scattering data. Differentiating Eq. (31) with respect to time t and applying (20) give

$$S_t = i\lambda^4[\Omega, S], \tag{72}$$

and thus,

$$S_t = \begin{pmatrix} 0 & i\beta\lambda^4 s_{12} & i\beta\lambda^4 s_{13} & i\beta\lambda^4 s_{14} \\ -i\beta\lambda^4 s_{21} & 0 & 0 & 0 \\ -i\beta\lambda^4 s_{31} & 0 & 0 & 0 \\ -i\beta\lambda^4 s_{41} & 0 & 0 & 0 \end{pmatrix}. \tag{73}$$

As a result, we have

$$\begin{cases} s_{12}(t,\lambda) = s_{12}(0,\lambda)e^{i\beta\lambda^4 t}, \\ s_{13}(t,\lambda) = s_{13}(0,\lambda)e^{i\beta\lambda^4 t}, \\ s_{14}(t,\lambda) = s_{14}(0,\lambda)e^{i\beta\lambda^4 t}, \\ s_{21}(t,\lambda) = s_{21}(0,\lambda)e^{-i\beta\lambda^4 t}, \\ s_{31}(t,\lambda) = s_{31}(0,\lambda)e^{-i\beta\lambda^4 t}, \\ s_{41}(t,\lambda) = s_{41}(0,\lambda)e^{-i\beta\lambda^4 t}, \end{cases} \tag{74}$$

and $s_{11}, s_{22}, s_{23}, s_{24}, s_{32}, s_{33}, s_{34}, s_{42}, s_{43}, s_{44}$ are constants.

3. Soliton Solutions

3.1. *General case*

In this section, we are going to write explicitly the one- and two-soliton solutions from the N-soliton solution based on the Riemann–Hilbert problems. In fact, the Riemann–Hilbert problems generate a unique solution in the regular case, i.e., the $\det(G^{\pm}) \neq 0$ when $G^{\pm} \to I_4$ as $\lambda \to \infty$. However, there are possible contingencies that $\det(G^{\pm})$ could be zero for some discrete $\lambda \in \mathbb{C}_{\pm}$ when nonregular. In that case, it is opportune to transform the nonregular case to a regular in order to guarantee a solution.

From (56) and (58) with (31), as $\det(\phi^{\pm}) = 1$, we prove that

$$\det(P^+(x,\lambda)) = s_{11}(\lambda) \tag{75}$$

and

$$\det(P^-(x,\lambda)) = \hat{s}_{11}(\lambda). \tag{76}$$

Because $\det(S(\lambda)) = 1$, so $S^{-1}(\lambda) = (\mathrm{cof}(S(\lambda)))^T$; thus,

$$\hat{s}_{11} = \begin{vmatrix} s_{22} & s_{23} & s_{24} \\ s_{32} & s_{33} & s_{34} \\ s_{42} & s_{43} & s_{44} \end{vmatrix}. \tag{77}$$

In order to get soliton solutions, the solutions of $\det(P^{\pm}(x,\lambda)) = 0$ are assumed to be simple. Let us suppose that $s_{11}(\lambda)$ has simple zeros $\lambda_k \in \mathbb{C}_+$ for $k \in \{1,2,\ldots,N\}$ and $\hat{s}_{11}(\lambda)$ has simple zeros $\hat{\lambda}_k \in \mathbb{C}_-$ for $k \in \{1,2,\ldots,N\}$, which are the poles of the transmission coefficients.[17]

From (35), we know that $\hat{s}_{11}(\lambda) = s_{11}(-\lambda)$. Hence, we have the involution relation

$$\hat{\lambda} = -\lambda. \tag{78}$$

Each $\mathrm{Ker}(P^+(x,\lambda_k))$ contains a singleton column vector v_k, and also $\mathrm{Ker}(P^-(x,\hat{\lambda}_k))$ contains a singleton row vector $\hat{v}_k$ for $k \in \{1,2,\ldots,N\}$ such that

$$P^+(x,\lambda_k)v_k = 0 \quad \text{for} \quad k \in \{1,2,\ldots,N\} \tag{79}$$

and

$$\hat{v}_k P^-(x,\hat{\lambda}_k) = 0 \quad \text{for} \quad k \in \{1,2,\ldots,N\}. \tag{80}$$

The Riemann–Hilbert problems can be solved explicitly when $G_0 = I_4$. This will force the reflection coefficients $s_{21} = s_{31} = s_{41} = 0$ and $\hat{s}_{12} = \hat{s}_{13} = \hat{s}_{14} = 0$.

In that case, we can present the solutions to special Riemann–Hilbert problems as follows[16,18]:

$$G^+(x,\lambda) = I_4 - \sum_{k,j=1}^{N} \frac{v_k (M^{-1})_{kj} \hat{v}_j}{\lambda - \hat{\lambda}_j} \tag{81}$$

and

$$(G^-)^{-1}(x,\lambda) = I_4 + \sum_{k,j=1}^{N} \frac{v_k (M^{-1})_{kj} \hat{v}_j}{\lambda - \lambda_k}, \tag{82}$$

where $M = (m_{kj})_{4\times 4}$ is a matrix defined as follows:

$$m_{kj} = \begin{cases} \dfrac{\hat{v}_k v_j}{\lambda_j - \hat{\lambda}_k} & \text{if} \quad \lambda_j \neq \hat{\lambda}_k \\ 0 & \text{if} \quad \lambda_j = \hat{\lambda}_k \end{cases}, \quad k,j \in \{1,2,\ldots,N\}. \tag{83}$$

The scattering vectors v_k and $\hat{v}_k$ are functions of (x,t), but λ_k and $\hat{\lambda}_k$ are constants, and so differentiating both sides of $P^+(x,\lambda_k)v_k = 0$ with respect to x and knowing that P^+ satisfies the spectral spatial equivalent equation (19) along with (79) give

$$P^+(x,\lambda_k)\left(\frac{dv_k}{dx} - i\lambda_k \Lambda v_k\right) = 0 \quad \text{for} \quad k,j \in \{1,2,\ldots,N\}, \tag{84}$$

and also differentiating it with respect to t and using the temporal equation (20) along with (79) give

$$P^+(x,\lambda_k)\left(\frac{dv_k}{dt} - i\lambda_k^4\Omega v_k\right) = 0 \quad \text{for} \quad k,j \in \{1,2,\ldots,N\}. \tag{85}$$

In the same way, by using (80) and the adjoint spectral equations (24) and (25), one can prove that

$$\left(\frac{d\hat{v}_k}{dx} + i\hat{\lambda}_k\hat{v}_k\Lambda\right)P^-(x,\hat{\lambda}_k) = 0 \tag{86}$$

and

$$\left(\frac{d\hat{v}_k}{dt} + i\hat{\lambda}_k^4\hat{v}_k\Omega\right)P^-(x,\hat{\lambda}_k) = 0. \tag{87}$$

As v_k is a single vector in the kernel of P^+, so $\frac{dv_k}{dx} - i\lambda_k\Lambda v_k$ and $\frac{dv_k}{dt} - i\lambda_k^4\Omega v_k$ are scalar multiples of v_k.

This permits one to obtain

$$v_k(x,t) = e^{i\lambda_k\Lambda x + i\lambda_k^4\Omega t}w_k \quad \text{for} \quad k \in \{1,2,\ldots,N\}. \tag{88}$$

In the same way, we will have for P^-,

$$\hat{v}_k(x,t) = \hat{w}_k e^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^4\Omega t} \quad \text{for} \quad k \in \{1,2,\ldots,N\}, \tag{89}$$

where the column vector w_k and the row vector $\hat{w}_k$ are constants.

Now from (79) and using (59), we get

$$\begin{aligned} &v_k^T(x,-t,-\lambda_k)(P^+)^T(x,-t,-\lambda_k) \\ &\quad = v_k^T(x,-t,-\lambda_k)CP^-(x,t,\lambda_k)C^{-1} = 0 \quad \text{for} \quad k \in \{1,2,\ldots,N\}. \end{aligned} \tag{90}$$

Because $v_k^T(x,-t,-\lambda_k)CP^-(x,t,\lambda_k)$ could be zero and using (80) leads to

$$\begin{aligned} v_k^T(x,-t,-\lambda_k)CP^-(x,t,\lambda_k) &= \hat{v}_k(x,t,\hat{\lambda}_k)P^-(x,t,\hat{\lambda}_k) \\ &= \hat{v}_k(x,t,-\hat{\lambda}_k)P^-(x,t,-\hat{\lambda}_k) = 0. \end{aligned} \tag{91}$$

As $\hat{\lambda}_k = -\lambda_k$ from (78), then we can take

$$\hat{v}_k(x,t,-\hat{\lambda}_k) = v_k^T(x,-t,-\lambda_k)C \quad \text{for} \quad k \in \{1,2,\ldots,N\}. \tag{92}$$

These involution relations will give then

$$v_k(x,t) = e^{i\lambda_k\Lambda x + i\lambda_k^4\Omega t}w_k \quad \text{for} \quad k \in \{1,2,\ldots,N\}, \tag{93}$$

$$\hat{v}_k(x,t) = w_k^T e^{-i\hat{\lambda}_k\Lambda x - i\hat{\lambda}_k^4\Omega t}C \quad \text{for} \quad k \in \{1,2,\ldots,N\}. \tag{94}$$

The jump matrix being $G = I_4$ allows to recover the potential P from the generalized matrix Jost eigenfunctions. Because P^+ is analytic, we can expand G^+ as $\lambda \to \infty$ in this form at order 3,

$$G^+(x,\lambda) = I_4 + \frac{1}{\lambda}G_1^+(x) + O\left(\frac{1}{\lambda^2}\right) \quad \text{when} \quad \lambda \to \infty. \tag{95}$$

Because G^+ satisfies the spectral problem, substituting it in (19) and matching the coefficients of the same power of $\frac{1}{\lambda}$, at order $O(1)$, we get

$$P = -[\Lambda, G_1^+]. \tag{96}$$

If

$$G_1^+ = \begin{pmatrix} (G_1^+)_{11} & (G_1^+)_{12} & (G_1^+)_{13} & (G_1^+)_{14} \\ (G_1^+)_{21} & (G_1^+)_{22} & (G_1^+)_{23} & (G_1^+)_{24} \\ (G_1^+)_{31} & (G_1^+)_{32} & (G_1^+)_{33} & (G_1^+)_{34} \\ (G_1^+)_{41} & (G_1^+)_{42} & (G_1^+)_{43} & (G_1^+)_{44} \end{pmatrix}, \tag{97}$$

then

$$\begin{aligned} P &= -[\Lambda, G_1^+] \\ &= \begin{pmatrix} 0 & -\alpha(G_1^+)_{12} & -\alpha(G_1^+)_{13} & -\alpha(G_1^+)_{14} \\ \alpha(G_1^+)_{21} & 0 & 0 & 0 \\ \alpha(G_1^+)_{31} & 0 & 0 & 0 \\ \alpha(G_1^+)_{41} & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{98}$$

As a result, we can now recover the potentials p_i and r_i for $i \in \{1,2,3\}$ as follows:

$$\begin{aligned} p_1 &= -\alpha(G_1^+)_{12}, \quad r_1 = \alpha(G_1^+)_{21}, \\ p_2 &= -\alpha(G_1^+)_{13}, \quad r_2 = \alpha(G_1^+)_{31}, \\ p_3 &= -\alpha(G_1^+)_{14}, \quad r_3 = \alpha(G_1^+)_{41}. \end{aligned} \tag{99}$$

Also, from (95), we have

$$G_1^+ = \lambda \lim_{\lambda \to \infty} (G^+(x, \lambda) - I_4); \tag{100}$$

so from (81), we prove

$$G_1^+ = -\sum_{k,j=1}^{N} v_k (M^{-1})_{k,j} \hat{v}_j. \tag{101}$$

From (9) and (96), we easily prove the nonlocal involution property

$$(G_1^+)^T(x, -t) = C G_1^+(x, t) C^{-1}. \tag{102}$$

Using (99) along with (93), (94) and (101) will generate the N-soliton solution to the nonlocal reverse-time six-component AKNS system of fourth-order

$$p_i = \alpha \sum_{k,j=1}^{N} v_{k1} (M^{-1})_{kj} \hat{v}_{j,i+1} \quad \text{for} \quad i \in \{1,2,3\}, \tag{103}$$

where w_k is an arbitrary constant column vector in $\mathbb{C}^4$, and

$$v_k = (v_{k1}, v_{k2}, v_{k3}, \ldots, v_{kn+1})^T, \quad \hat{v}_k = (\hat{v}_{k1}, \hat{v}_{k2}, \hat{v}_{k3}, \ldots, \hat{v}_{kn+1}).$$

4. Exact Soliton Solutions and Dynamics

4.1. *One-soliton solution*

A general explicit solution for a single-soliton in the reverse-time case when $N = 1$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $\lambda_1 \in \mathbb{C}$ is arbitrary and $\hat{\lambda}_1 = -\lambda_1$ is given by

$$p_1(x,t) = \frac{2\rho_2\rho_3\lambda_1(\alpha_1-\alpha_2)w_{11}w_{12}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^4(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}, \tag{104}$$

$$p_2(x,t) = \frac{2\rho_1\rho_3\lambda_1(\alpha_1-\alpha_2)w_{11}w_{13}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^4(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}, \tag{105}$$

$$p_3(x,t) = \frac{2\rho_1\rho_2\lambda_1(\alpha_1-\alpha_2)w_{11}w_{14}e^{i\lambda_1(\alpha_1+\alpha_2)x+i\lambda_1^4(\beta_1-\beta_2)t}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}. \tag{106}$$

We can get the amplitude of p_1:

$$\begin{aligned}|p_1(x,t)| &= 2e^{-\mathrm{Im}(\lambda_1^4(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x)} \\ &\quad\times\left|\frac{\lambda_1\rho_2\rho_3(\alpha_1-\alpha_2)w_{11}w_{12}}{\rho_1\rho_2\rho_3 w_{11}^2 e^{2i\lambda_1\alpha_1 x} + (\rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2)e^{2i\lambda_1\alpha_2 x}}\right|.\end{aligned} \tag{107}$$

About the dynamics of the one-soliton, we can see from p_1 that there is no speed, i.e., the soliton is not a traveling wave. By choosing any arbitrary constant $x = x_0$, $\beta_1 - \beta_2 < 0$ and $\lambda_1 \notin i\mathbb{R}$ in $|p_1(x,t)|$, we see that the soliton's amplitude grows exponentially if $\mathrm{Im}(\lambda_1^4) > 0$, while it decays exponentially if $\mathrm{Im}(\lambda_1^4) < 0$, but when $\mathrm{Im}(\lambda_1^4) = 0$, the amplitude is constant over the time. If we choose $x = x_0$ and $\lambda_1 \in i\mathbb{R}$, we have a constant amplitude for the soliton indeed.

In this reverse-time case, any one-soliton does not collapse, either it strictly increases, decreases or stays constant.

From the spectral plane, let $\lambda_1 = \xi + i\eta = re^{i\theta}$, where $r > 0$ and $0 < \theta < 2\pi$ then:

$$\text{if}\begin{cases}\theta \in \left(0,\dfrac{\pi}{4}\right)\cup\left(\dfrac{\pi}{2},\dfrac{3\pi}{4}\right)\cup\left(\pi,\dfrac{5\pi}{4}\right)\cup\left(\dfrac{3\pi}{2},\dfrac{7\pi}{4}\right), \\ \quad\text{then the amplitude of the soliton is increasing,} \\[2ex] \theta \in \left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right)\cup\left(\dfrac{3\pi}{4},\pi\right)\cup\left(\dfrac{5\pi}{4},\dfrac{3\pi}{2}\right)\cup\left(\dfrac{7\pi}{4},2\pi\right), \\ \quad\text{the amplitude of the soliton is decreasing,} \\[2ex] \theta \in \left\{\dfrac{\pi}{4},\dfrac{\pi}{2},\dfrac{3\pi}{4},\dfrac{5\pi}{4},\dfrac{3\pi}{2},\dfrac{7\pi}{4}\right\}, \quad \text{the amplitude of the soliton is constant,} \\ \theta \in \{0,\pi,2\pi\}, \quad \text{we obtain one breather with constant amplitude.}\end{cases} \tag{108}$$

This illustration is shown in Fig. 1.

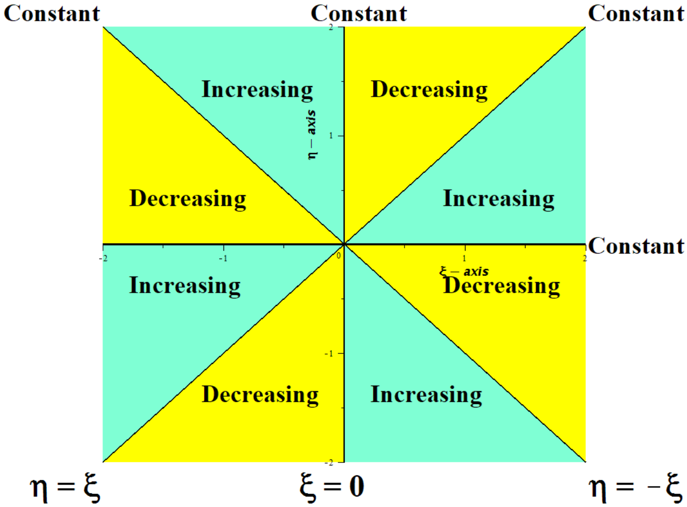

Fig. 1. (Color online) Spectral plane of eigenvalues.

Let us graph the one-soliton solution. When λ_1 does not lie on the real axis ($\eta = 0$), imaginary axis ($\xi = 0$) or the bisectors ($\eta = \pm\xi$), the amplitude of the potential grows or decays exponentially if $\mathrm{Im}(\lambda_1^4) > 0$ or $\mathrm{Im}(\lambda_1^4) < 0$, respectively. Two examples are illustrated in Figs. 2 and 3, where we have growing and decaying amplitudes.

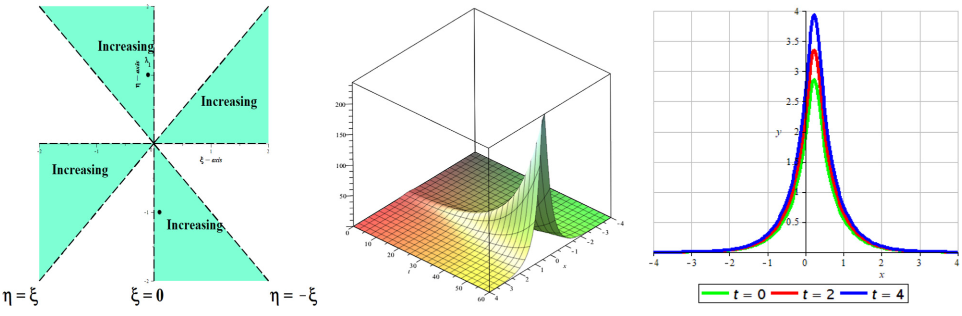

Fig. 2. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton in the focusing case with parameter values $\rho_1 = -1$, $\rho_2 = -2$, $\rho_3 = -1$, $\lambda_1 = -0.01 + i$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (1, i, 2 + i, 1)$. The 2D plot is for time values, $t = 0$, 2 and 4.

When $\mathrm{Im}(\lambda_1^4) = 0$, the amplitude does not change. In that case, λ_1 lies on the imaginary axis, the bisectors or the real axis. If λ_1 lies on the imaginary axis or on the bisectors, then we have a fundamental soliton (Fig. 4), whereas if $\lambda_1 \in \mathbb{R}$, then we have a periodic one-soliton with period $\frac{\pi}{\lambda_1(\alpha_1 - \alpha_2)}$ which is a breather (Fig. 5).

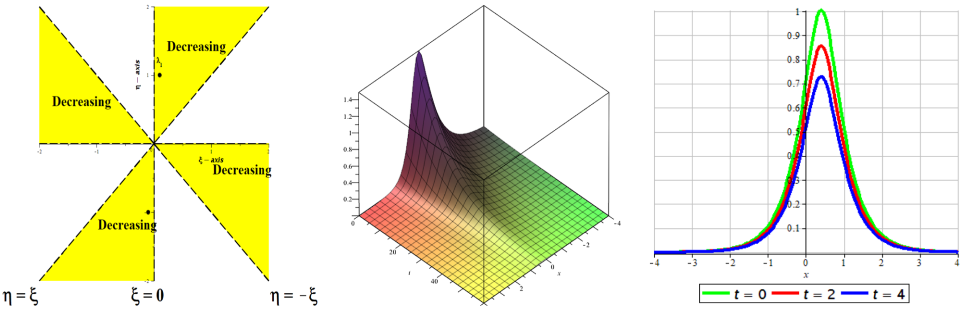

Fig. 3. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 0.01 + i$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (1, i, 2+i, 1)$. The 2D plot is for time values, $t = 0$, 2 and 4.

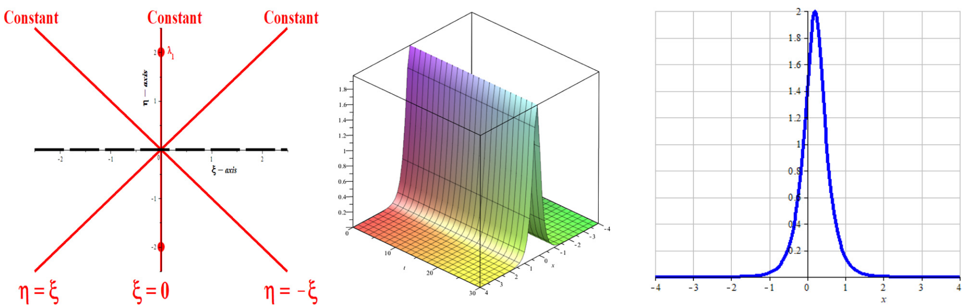

Fig. 4. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 2i$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $w_1 = (1, i, 2+i, 1)$. The 2D plot is for any time value.

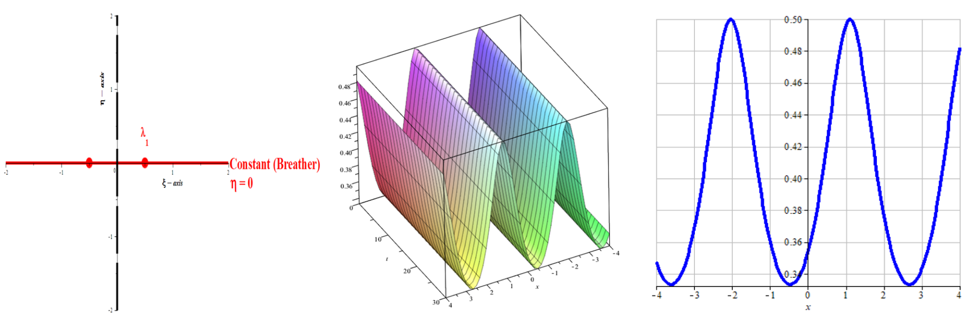

Fig. 5. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the one-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 0.5$, $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -1$, $\beta_2 = 1$, $\rho = 1$, $w_1 = (1, i, 2+i, 1)$. The 2D plot is for any time value.

4.2. *Two-soliton solution*

A general explicit two-soliton solution in the reverse-time case when $N = 2$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $w_2 = (w_{21}, w_{22}, w_{23}, w_{24})^T$, $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ are arbitrary, and

$\hat{\lambda}_1 = -\lambda_1$, $\hat{\lambda}_2 = -\lambda_2$, is given if $\lambda_1 \neq -\lambda_2$ by

$$p_1(x,t) = 2\rho_2\rho_3(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{A(x,t)}{B(x,t)}, \tag{109}$$

$$p_2(x,t) = 2\rho_1\rho_3(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{C(x,t)}{B(x,t)}, \tag{110}$$

$$p_3(x,t) = 2\rho_1\rho_2(\lambda_1+\lambda_2)(\alpha_1-\alpha_2)\frac{D(x,t)}{B(x,t)}, \tag{111}$$

where

$$\begin{aligned}
A(x,t) &= e^{i[\lambda_2^4(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]} \\
&\quad \cdot [(w_{22}M(\lambda_1+\lambda_2) - 2w_{12}K\lambda_1)w_{21}\lambda_2 e^{2\alpha_2\lambda_1 x} \\
&\quad - \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{22}\lambda_2 e^{2\alpha_1\lambda_1 x}] \\
&\quad + e^{i[\lambda_1^4(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]} \cdot [(w_{12}N(\lambda_1+\lambda_2) - 2w_{22}K\lambda_2)w_{11}\lambda_1 e^{2\alpha_2\lambda_2 x} \\
&\quad + \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{12}w_{21}^2\lambda_1 e^{2\alpha_1\lambda_2 x}], \\
B(x,t) &= -4\rho_1\rho_2\rho_3\lambda_1\lambda_2 w_{11}w_{21}K e^{i(\lambda_1+\lambda_2)(\alpha_1+\alpha_2)x} \\
&\quad \cdot [e^{i(\lambda_1^4-\lambda_2^4)(\beta_1-\beta_2)t} + e^{-i(\lambda_1^4-\lambda_2^4)(\beta_1-\beta_2)t}] + \rho_1\rho_2\rho_3 w_{21}^2 M(\lambda_1 \\
&\quad + \lambda_2)^2 e^{i2(\alpha_1\lambda_2+\alpha_2\lambda_1)x} + \rho_1\rho_2\rho_3 w_{11}^2 N(\lambda_1+\lambda_2)^2 e^{i2(\alpha_1\lambda_1+\alpha_2\lambda_2)x} \\
&\quad + \rho_1^2\rho_2^2\rho_3^2 w_{11}^2 w_{21}^2(\lambda_1-\lambda_2)^2 e^{i2\alpha_1(\lambda_1+\lambda_2)x} + [(\lambda_1^2+\lambda_2^2)MN \\
&\quad + (2MN-4K^2)\lambda_1\lambda_2]e^{i2\alpha_2(\lambda_1+\lambda_2)x}, \\
C(x,t) &= e^{i[\lambda_2^4(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]} \cdot [(w_{23}M(\lambda_1+\lambda_2) - 2w_{13}K\lambda_1)w_{21}\lambda_2 e^{2\alpha_2\lambda_1 x} \\
&\quad - \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{23}\lambda_2 e^{2\alpha_1\lambda_1 x}] + e^{i[\lambda_1^4(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]} \\
&\quad \cdot [(w_{13}N(\lambda_1+\lambda_2) - 2w_{23}K\lambda_2)w_{11}\lambda_1 e^{2\alpha_2\lambda_2 x} \\
&\quad + \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{13}w_{21}^2\lambda_1 e^{2\alpha_1\lambda_2 x}], \\
D(x,t) &= e^{i[\lambda_2^4(\beta_1-\beta_2)t+\lambda_2(\alpha_1+\alpha_2)x]} \cdot [(w_{24}M(\lambda_1+\lambda_2) - 2w_{14}K\lambda_1)w_{21}\lambda_2 e^{2\alpha_2\lambda_1 x} \\
&\quad - \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}^2 w_{21}w_{24}\lambda_2 e^{2\alpha_1\lambda_1 x}] \\
&\quad + e^{i[\lambda_1^4(\beta_1-\beta_2)t+\lambda_1(\alpha_1+\alpha_2)x]} \cdot [(w_{14}N(\lambda_1+\lambda_2) - 2w_{24}K\lambda_2)w_{11}\lambda_1 e^{2\alpha_2\lambda_2 x} \\
&\quad + \rho_1\rho_2\rho_3(\lambda_1-\lambda_2)w_{11}w_{14}w_{21}^2\lambda_1 e^{2\alpha_1\lambda_2 x}],
\end{aligned}$$

and $M = \rho_2\rho_3 w_{12}^2 + \rho_1\rho_3 w_{13}^2 + \rho_1\rho_2 w_{14}^2$, $N = \rho_2\rho_3 w_{22}^2 + \rho_1\rho_3 w_{23}^2 + \rho_1\rho_2 w_{24}^2$ and $K = \rho_2\rho_3 w_{12}w_{22} + \rho_1\rho_3 w_{13}w_{23} + \rho_1\rho_2 w_{14}w_{24}$.

About the dynamics of two-soliton solution, many phenomena could occur. Either the two solitons move (repeatedly or not) in opposite directions or one moves while the other stays stationary or both are stationary.

Now, since $\lambda_1 \neq -\lambda_2$, let $\lambda_1 = \xi + i\eta$ and $\lambda_2 = \xi' + i\eta'$. If $\lambda_1 = \pm a \pm ib$, $\lambda_2 = \pm a \pm ib$ where $a \neq b$ and $a \neq 0, b \neq 0$ that means both λ_1 and λ_2 are symmetric with respect to the real axis or the imaginary axis, then the two solitons will be collapsing repeatedly or noncollapsing while moving in opposite directions. Each keeping the same amplitude before and after interaction (see Fig. 6), or both keep their amplitude before interaction, but the amplitude changes after the collision to a new constant amplitude (as illustrated in Fig. 7), depending on the choice of w_1, w_2.

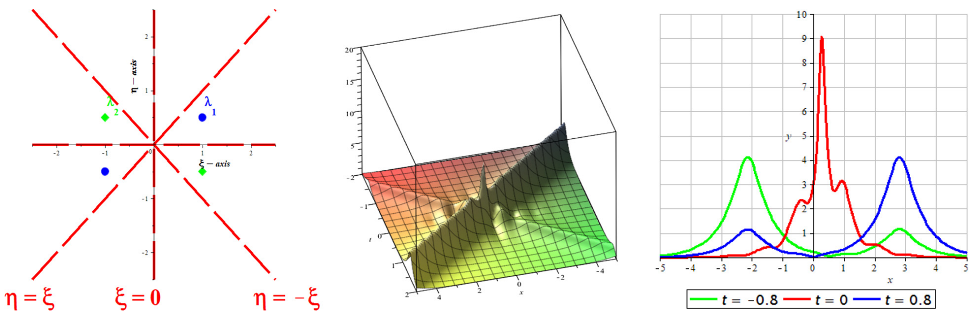

Fig. 6. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two traveling waves in the focussing case with parameter values $\rho_1 = -1$, $\rho_2 = -2$, $\rho_3 = -3$, $\lambda_1 = 1 + 0.5i$, $\lambda_2 = -1 + 0.5i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1 - 0.5i, 1 + 3i, -i, 1 + i)$ and $w_2 = (-1 + 2i, 1 - 1.5i, i, 1 - i)$. The 2D plot is for time values $t = -0.8, 0, 0.8$.

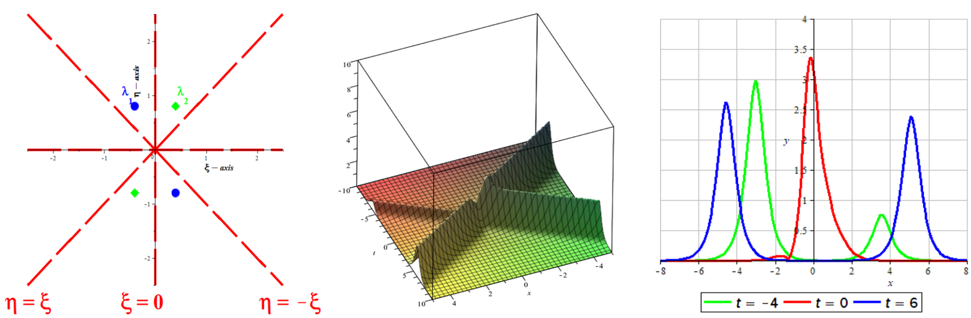

Fig. 7. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = 1$, $\rho_3 = -1$, $\lambda_1 = -0.4 + 0.8i$, $\lambda_2 = 0.4 + 0.8i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 1 - i, -0.1 + i, 1 + i)$ and $w_2 = (-1 + 2i, 1 - 0.1i, 3 + i, 0)$. The 2D plot is for time values $t = -4, 0, 6$.

We may have the case of two soliton waves moving in opposite directions, and after interaction, they get embedded into a single wave (Fig. 8). Also, we can have the case where one soliton unfolds to two soliton waves[19] (Fig. 9). The choice of those eigenvalues may be helpful in explaining some physical phenomena.[20]

Remark 1. We note that Figs. 7, 8 and 9 resemble the collision of two Manakov solitons.[9,21]

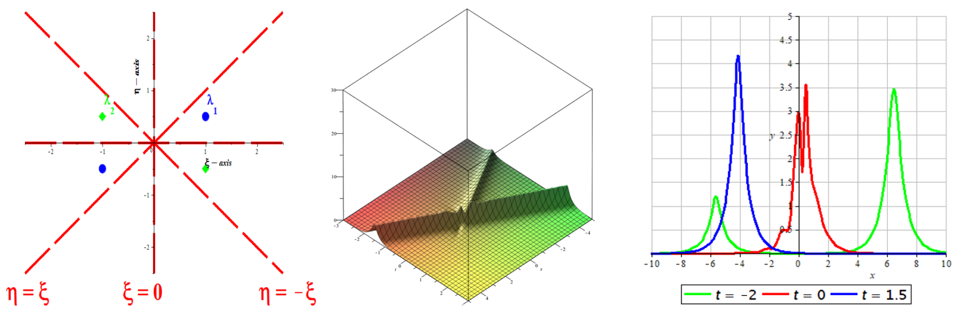

Fig. 8. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = -1$, $\lambda_1 = 1 + 0.5i$, $\lambda_2 = -1 + 0.5i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, -0.1 + i, 1 + i)$ and $w_2 = (-1, 1 - 2i, 3 + i, 0)$. The 2D plot is for time values $t = -2, 0, 1.5$.

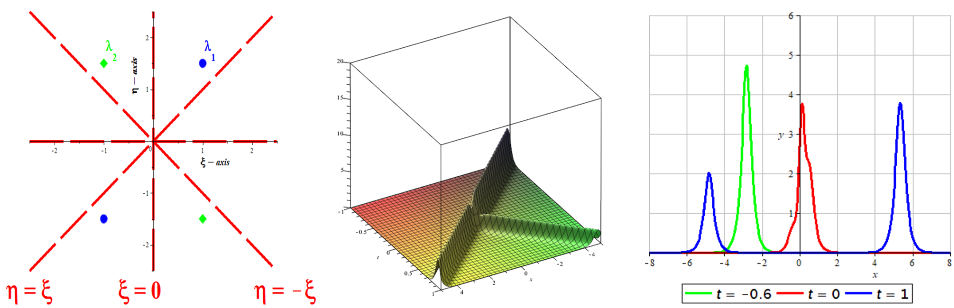

Fig. 9. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 1 + 1.5i$, $\lambda_2 = -1 + 1.5i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, -2 - i, 1 - i)$ and $w_2 = (-1, 1 - 2i, 3 - i, 0)$. The 2D plot is for time values $t = -0.6, 0, 1$.

If $b = a$, we have $\lambda_1 = \pm a \pm ia$, $\lambda_2 = \pm a \pm ia$, still λ_1, λ_2 are symmetric with respect to the real axis or imaginary axis and they lie on the bisectors, then the two solitons will be stationary and will have constant amplitudes.

For the other choices of λ_1, λ_2, if both lie anywhere on the real axis, imaginary axis or the bisectors where both are not real, that means

$$\begin{cases} \lambda_1 = \pm a + ia,\ \lambda_2 = \pm b + ib, \\ \quad \text{i.e., both lie on the same bisector or each on different bisector,} \\ \lambda_1 = \pm a + ia, \lambda_2 = ib, \\ \quad \text{i.e., one lies on a bisector and the other one on the imaginary axis,} \\ \lambda_1 = \pm a + ia, \lambda_2 = b, \\ \quad \text{i.e., one lies on a bisector and the other one on the real axis,} \\ \lambda_1 = ia, \lambda_2 = ib, \quad \text{i.e., both lie on the imaginary axis,} \\ \lambda_1 = a, \lambda_2 = ib, \\ \quad \text{i.e., one on the real axis while the other lies on the imaginary axis,} \end{cases} \tag{112}$$

then the two solitons could be noncollapsing or collapsing repeatedly and/or periodically creating a standing state wave (as shown in Fig. 10). Whereas if both λ_1, λ_2 are real, i.e., $\lambda_1 = a, \lambda_2 = b$, we have two breather periodic waves with period $\frac{2\pi}{(\lambda_1^4 - \lambda_2^4)(\beta_1 - \beta_2)}$ in standing state (see Fig. 11).

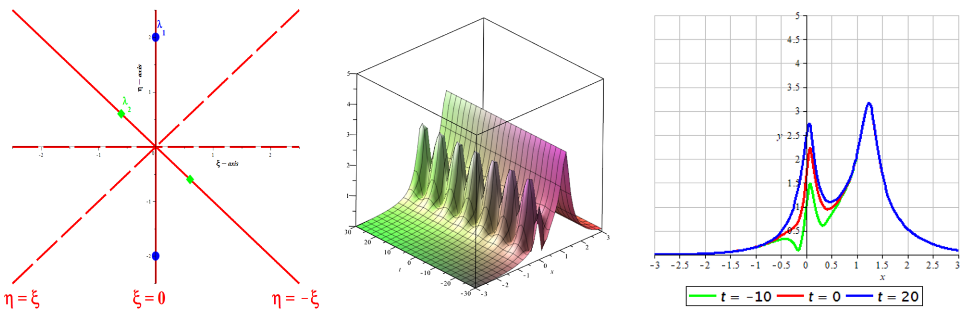

Fig. 10. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = 1$, $\rho_3 = -1$, $\lambda_1 = 2i$, $\lambda_2 = -0.6 + 0.6i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, -0.1 + i, 1 + i)$ and $w_2 = (-1, 1 - 2i, 3 + i, 0)$. The 2D plot is for time values $t = -10, 0, 20$.

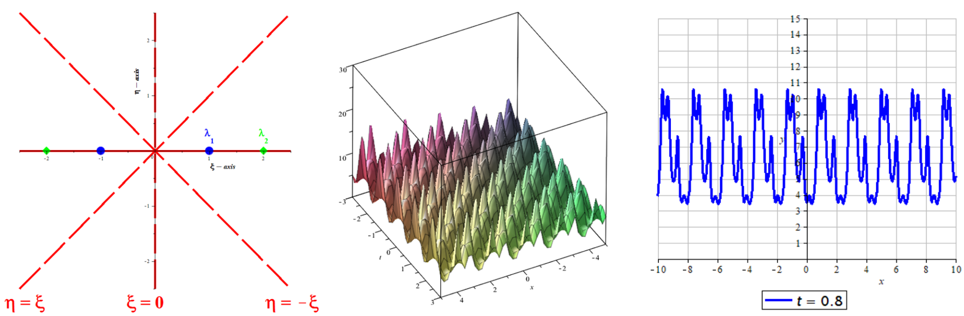

Fig. 11. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = -1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, 2 + i, 1 - i)$ and $w_2 = (-1, 1 - 2i, -i, 0)$. The 2D plot is for time values $t = 0.8$.

Remark 2. If $\lambda_1 \neq -\lambda_2$ and λ_1, λ_2 are symmetric about the real axis, imaginary axis or the bisectors or also if each lies anywhere on the real axis, imaginary axis or the bisectors, then $\mathrm{Im}(\lambda_1^4 + \lambda_2^4) = 0$.

Remark 3. If $\mathrm{Im}(\lambda_1^4 + \lambda_2^4) = 0$ and $|\lambda_1|^4 = |\lambda_2|^4$, then λ_1, λ_2 are symmetric about the real axis, imaginary axis or the bisectors.

If $\mathrm{Im}(\lambda_1^4) = 0$ and $\mathrm{Im}(\lambda_2^4) = 0$, then this means that each of λ_1 and λ_2 lies on one of the real axes, imaginary axes or the bisectors.

Remark 4. If λ_1, λ_2 are symmetric about the bisectors, then the dynamics of the two solitons is different from when the two eigenvalues are symmetric about the η-axis (or ξ-axis).

We can note that any λ_1 and λ_2 satisfying any condition mentioned previously will satisfy $\mathrm{Im}(\lambda_1^4 + \lambda_2^4) = 0$ as well. If λ_1 and λ_2 are symmetric with respect to the bisectors i.e., $\lambda_1 = a + bi$, $\lambda_2 = b + ai$ or $\lambda_1 = a + bi$, $\lambda_2 = -b - ai$, then they still satisfy $\mathrm{Im}(\lambda_1^4 + \lambda_2^4) = 0$, but the dynamics of the two solitons will be different from what was discussed previously.

Now, if λ_1 and λ_2 do not satisfy any of the above conditions, then the two solitons move in opposite directions and could collapse repeatedly, where they will be decreasing or increasing over the time (see Fig. 12).

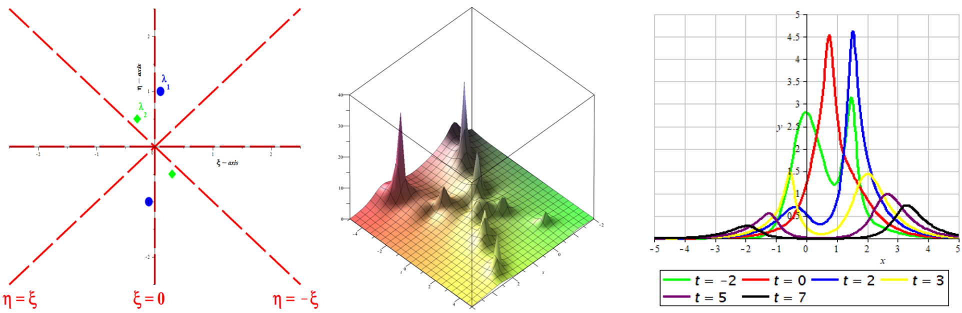

Fig. 12. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the two-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = -1$, $\lambda_1 = 0.1 + i$, $\lambda_2 = -0.3 + 0.5i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1 + i, 1 - 3i, 2 + i, 1 - i)$ and $w_2 = (-1 + i, 1 + 3i, 2 - i, 1 + i)$. The 2D plot is for time values $t = -2, 0, 2, 3, 5, 7$.

4.3. *Three-soliton solution*

The three-soliton solution is given, for which $N = 3$, $w_1 = (w_{11}, w_{12}, w_{13}, w_{14})^T$, $w_2 = (w_{21}, w_{22}, w_{23}, w_{24})^T$, $w_3 = (w_{31}, w_{32}, w_{33}, w_{34})^T$, $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$, and $\hat{\lambda}_1 = -\lambda_1$, $\hat{\lambda}_2 = -\lambda_2$, $\hat{\lambda}_3 = -\lambda_3$, by

$$p_1 = \alpha \sum_{k,j=1}^{3} v_{k1} (M^{-1})_{kj} \hat{v}_{j,2}, \tag{113}$$

$$p_2 = \alpha \sum_{k,j=1}^{3} v_{k1} (M^{-1})_{kj} \hat{v}_{j,3}, \tag{114}$$

$$p_3 = \alpha \sum_{k,j=1}^{3} v_{k1} (M^{-1})_{kj} \hat{v}_{j,4}. \tag{115}$$

For the three-soliton, if $\lambda_i = -\lambda_j$ for $i \neq j$, $i, j \in \{1, 2, 3\}$, then we have the one-soliton dynamics. Also, if two of $\{\lambda_1, \lambda_2, \lambda_3\}$ are equal, then we have the dynamics of the two-soliton.

Let $\lambda_1 = \xi + i\eta$, $\lambda_2 = \xi' + i\eta'$ and $\lambda_3 = \xi'' + i\eta''$. If two of the eigenvalues are symmetric about the real axis, imaginary axis and the other eigenvalue lie on the real axis, imaginary axis or on the bisectors, then we have two solitons collapsing repeatedly or noncollapsing moving in opposite directions, while the third one stays stationary. Either each keeps the same amplitude before and after interaction (see Fig. 13), or they keep their amplitudes before interaction, but their amplitudes change at the collision moment to new constant amplitudes (illustration in Fig. 14), depending on the choice of w_1, w_2.

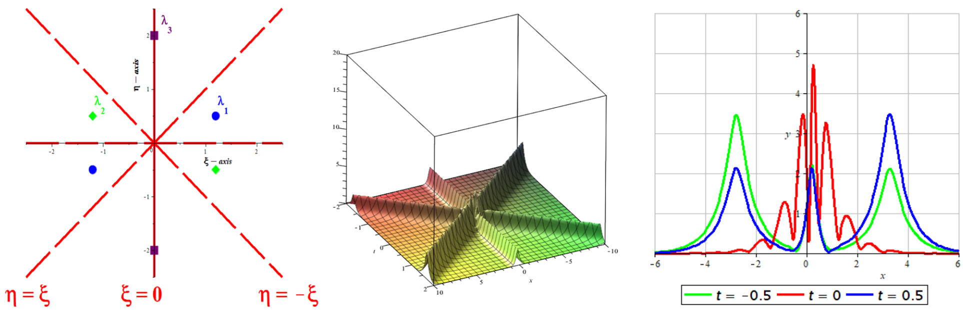

Fig. 13. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ for two traveling waves and a constant-amplitude stationary wave with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$ $\lambda_1 = 1.2 + 0.5i$, $\lambda_2 = -1.2 + 0.5i$, $\lambda_3 = 2i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (-1.5 + 2i, 2 - 3i, i, 1 - i)$ and $w_2 = (3 + 2i, -1 + 3i, -i, 1 + i)$, $w_3 = (1, 1, 2, 1)$. The 2D plot is for time values, $t = -0.5$, 0, 0.5.

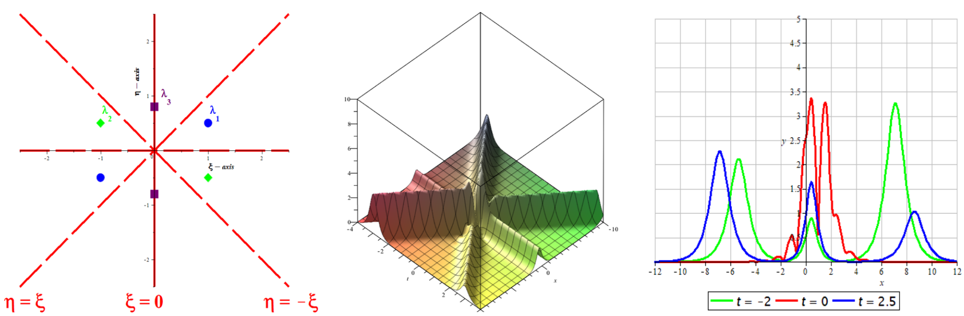

Fig. 14. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = -0.5$, $\rho_2 = -0.5$, $\rho_3 = -0.5$, $\lambda_1 = 1 + 0.5i$, $\lambda_2 = -1 + 0.5i$, $\lambda_3 = 0.8i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, i, 3 + i, 1 - i)$, $w_2 = (-1, 1 - 3i, -i, 0)$, $w_3 = (2 + i, 1 + 2i, 1, 2i)$. The 2D plot is for time values, $t = -2$, 0, 2.5.

We can also have two other different cases of interaction. The first case is where the three-soliton after interaction is embedded into two-soliton (Fig. 15). The second case happens when the two solitons after interaction unfold to three-soliton (Fig. 16). As said before, those phenomena may be relevant to some nonlinear problems in applied physics.

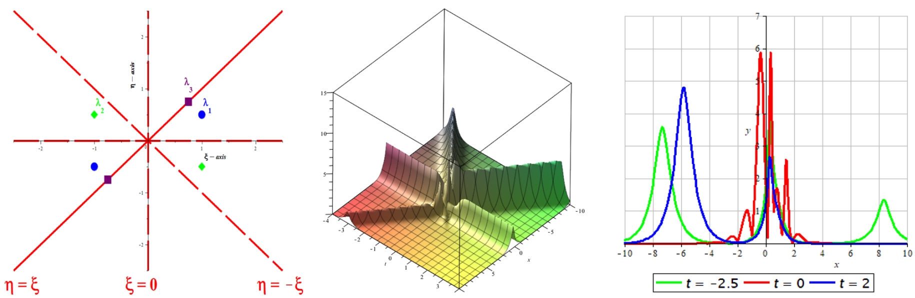

Fig. 15. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 1 + 0.5i$, $\lambda_2 = -1 + 0.5i$, $\lambda_3 = 0.75 + 0.75i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, 2 + i, 1 - i)$, $w_2 = (-1, 1 - 2i, -i, 0)$, $w_3 = (2 + i, 1 + 2i, 1, 2i)$. The 2D plot is for time values, $t = -2.5$, 0,2.

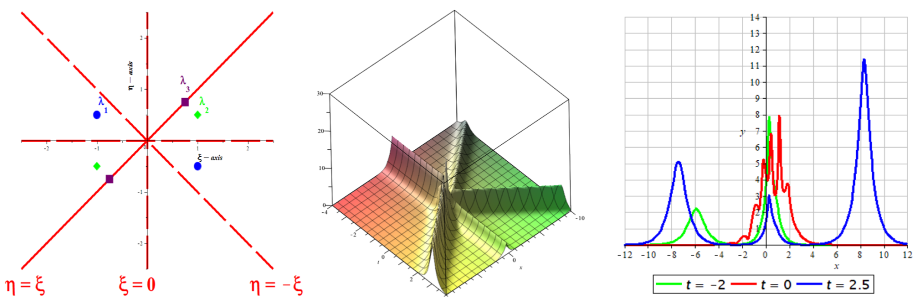

Fig. 16. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = -1 + 0.5i$, $\lambda_2 = 1 + 0.5i$, $\lambda_3 = 0.75 + 0.75i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, 2 + i, 1 - i)$, $w_2 = (-1, 1 - 2i, -i, 0)$, $w_3 = (2 + i, 1 + 2i, 1, 2i)$. The 2D plot is for time values, $t = -2$, 0, 2.5.

If λ_1, λ_2, λ_3 are all real, then we have breather solitons as shown in Fig. 17. Otherwise, if λ_1, λ_2, λ_3 are not all real lying on the real axis, imaginary axis or the bisectors, we will have three solitons collapsing repeatedly in standing state.

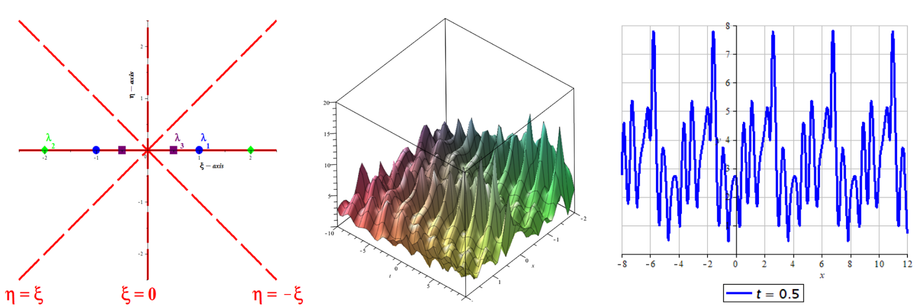

Fig. 17. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = 1$, $\rho_2 = 1$, $\rho_3 = 1$, $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 0.5$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1, 0, 2 + i, 1 - i)$, $w_2 = (-1, 1 - 2i, -i, 0)$, $w_3 = (1 + i, 1 + 2i, 0, 2i)$. The 2D plot is for time values, $t = 0.5$.

If two of the λ_1, λ_2, λ_3 are symmetric (but not real) about the η-axis (or ξ-axis) and the third one lies off of the real axis, imaginary axis, bisectors, then we can have three solitons interacting, with two solitons moving in opposite directions with constant amplitudes. After collision, their amplitudes change, but still stay constant, while the third soliton is stationary and its amplitude is either increasing or decreasing, as shown in Fig. 18.

If λ_1, λ_2, λ_3 all lie off the real axis, imaginary axis, bisectors, or one of them is real or two of them are real, then we have two solitons that could repeatedly collapse or noncollapse decreasingly or increasingly in their motion while the third one stays stationary, as in Fig. 19.

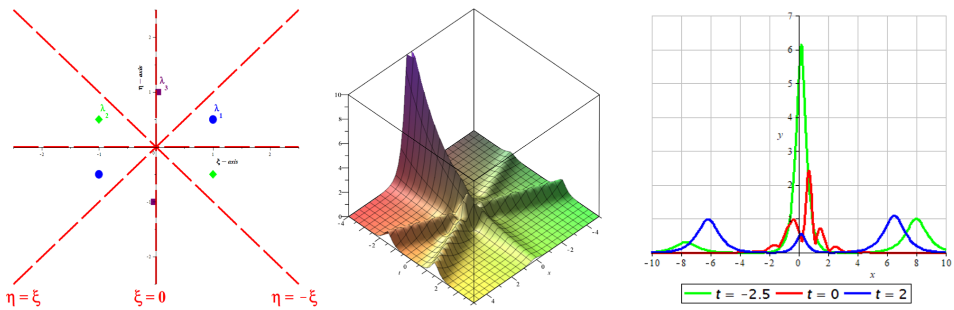

Fig. 18. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = 2$, $\rho_2 = 2$, $\rho_3 = 2$, $\lambda_1 = 1 + 0.5i$, $\lambda_2 = -1 + 0.5i$, $\lambda_3 = 0.04 + i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (1 - 2i, 1 + 3i, -i, 1 + i)$, $w_2 = (-1 + 2i, 1 - 3i, i, 1 - i)$, $w_3 = (1 + i, 1 + 2i, 0, 2i)$. The 2D plot is for time values, $t = -2.5$, 0, 2.

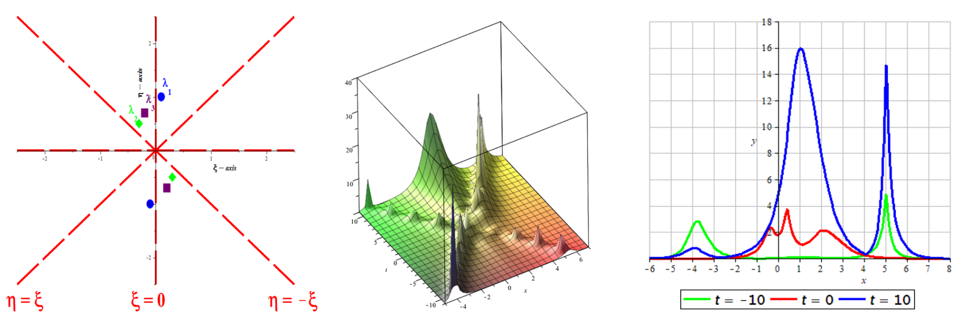

Fig. 19. (Color online) Spectral plane along with 3D plot and 2D plots of $|p_1|$ of the three-soliton with parameter values $\rho_1 = -1$, $\rho_2 = -1$, $\rho_3 = -1$, $\lambda_1 = 0.1 + i$, $\lambda_2 = -0.3 + 0.5i$, $\lambda_3 = -0.2+0.7i$, $\alpha_1 = -2$, $\alpha_2 = 1$, $\beta_1 = -2$, $\beta_2 = 1$, $w_1 = (2, 2i, 2+i, 1-i)$, $w_2 = (-1, 1-2i, -i, 0)$, $w_3 = (-1 + i, -1 + 2i, 0, -2i)$. The 2D plot is for time values, $t = -10$, 0, 10.

5. Conclusion

In this paper, by using the Riemann–Hilbert technique, we have obtained the N-soliton solution of a nonlocal nonlinear six-component fourth-order AKNS system under a reverse-time reduction, and particularly, the one- and two-soliton solutions have been presented explicitly. What is interesting in this reverse-time case is that the symmetry involution guarantees a pair of eigenvalues, one being in the upper half complex plane and its symmetric partner being in the lower half plane. Therefore, the Riemann–Hilbert problem becomes easier to solve than in the reverse-space or in the reverse-space–time cases.[22] Also, we have noted that in comparison to classical solitons which keep their shapes and amplitudes over the time, in the reverse-time case, the amplitude of the soliton potential changes, and sometimes, the solution itself collapses while moving. Such solutions show that they have singularities at a finite time. Moreover, at least two nonlocal solitons do not always collide elastically in a nonlinear superposition manner like classical solitons. Besides the Riemann–Hilbert approach, one could investigate the solvability of those nonlocal integrable equations by the Hirota bilinear method or Darboux transformation.

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