A Note on Minimal Boolean Formula Size of One-Dimensional Cellular Automata

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In this note, we disprove 44 claims in [4] on minimal Boolean formula size of one-dimensional two-state nearest neighbor cellular automata as well as set a new upper bound.

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In [4] Wolfram asserts to have found minimal Boolean formulas for (what he denotes) rules of one-dimensional, two-state, nearest neighbor cellular automata (CA) or simply *elementary rules*. These formulas are minimal in the sense that they "use the minimum possible number of operators" over basis Ω_1 = {0,1,¬,∧,∨,⊕}.

Provided that elementary rules can be interpreted as 3-input Boolean functions and visualized via their respective truth table representation, we would like to draw attention to result (a) of [2], which states that the maximal formula size for 3-input Boolean functions over basis Ω_1 is 5. This result clearly disproves the minimalistic nature of Wolfram's 8 Boolean formulas in [4] of size 6 and sets a new upper bound on formula size.

We enumerate all 256 3-input Boolean functions via their respective truth table representation and their output column Boolean vector $\hat{\alpha}$ where $\alpha_i \in$ {0,1}. Each of the 256 functions represents one permutation of eight binary bits in the output column Boolean vector

$$\hat{\alpha} = [\alpha_0 \alpha_1 \cdots \alpha_7].$$

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We can now specify all 256 Boolean functions with a decimal number
$N = \{0,1,2,\ldots, 255\}$.

$$N = \sum_{i=0}^{7} \alpha_i 2^{7-i}$$

For sake of completeness, we provide an example of how Wolfram's enumeration scheme works using CA rule 110.

p	q	r	$f(p,q,r)$
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	1
0	0	0	0

$N = 0(2^7) + 1(2^6) + 1(2^5) + 0(2^4) + 1(2^3) + 1(2^2) + 1(2^1) + 0(2^0) = 110.$

A Boolean formula, ϕ, is an expression constructed from propositional variables with logical connectives. Properties are outlined next.

Symbol	Meaning	Size
$\neg$	Not	1
$\wedge$	And	1
$\vee$	Or	1
$\oplus$	Xor	1
1	True	0
0	False	0

Additionally, we define the size of a given Boolean formula ϕ, or simply size(ϕ), as the number of occurrences of logical connectives.
Note that, size(True) = size(False) = 0.

A Boolean formula is *minimal* if and only if for every formula ψ, shorter in size than ϕ, there exists an assignment of the variables such that ϕ and ψ evaluate to different values.

The following elementary rules or Boolean functions are represented by minimal Boolean formulas :

rule	minimal Boolean formula	rule	minimal Boolean formula
2	r∧ ¬(p∨q)	133	¬(p⊕r)∧(q∨¬p)
16	p⊕(p∧(r∨q))	138	r∧(¬p∨q)
18	¬q∧(p⊕r)	145	¬((r∧¬p)∨(q ⊕r))
22	p⊕((p∧q)∨(q⊕r))	151	(p∧q)⊕¬(r∧(q∨p))
25	¬((p∧q)∨(q⊕r))	155	¬((q⊕r)∧(q∨p))
33	¬(q∨(p⊕r))	157	¬ ((p∨r) ∧ (q⊕r))
37	¬((q∧r)∨(p⊕r))	167	¬((p⊕r)∧(q∨p)
61	¬(p∨r)∨(p⊕q)	181	¬((p⊕r)∧(r∨q))
67	¬((r∧p)∨(p⊕q))	183	¬(q ∧ (p⊕r))
72	q∧(p⊕r)	184	p⊕(q∧(p⊕r))
91	r⊕(p∨¬(r∨q))	188	(r∧q)∨(p⊕q)
101	r⊕(¬p∨q)	191	¬(q∧p)∨r
103	¬(r ∨p)∨(q⊕r)	199	¬((p⊕q)∧(r∨p))
104	(p∧q)⊕(r∧(q∨p))	207	¬p∨q
107	(p∧q)⊕(r ∨ ¬(q∨p))	211	¬((p⊕q) ∧ (r∨q))
109	r⊕¬((r∨p) ∧ (p⊕q))	218	(r∧q)∨(p⊕r)
110	(¬p ∧r) ∨ (q⊕r)	222	q∨(p⊕r)
121	r⊕¬((r∨q) ∧ (p⊕q))	223	¬(p∧r)∨q
122	(¬q∧r)∨(p⊕r)	226	r⊕(q∧(p⊕r))
123	¬q ∨(p⊕r)	230	(p∧q)∨(q⊕r)
124	(¬r ∧q) ∨ (p⊕q)	233	r⊕¬((r∧q)∨(p⊕q))
131	¬(p⊕q) ∧(r∨¬p)	247	¬(r∧q)∨p

Remark 0.1 *The formulas provided by Wolfram [4] for these rules are not minimal. Moreover for 8 of these cannot be minimal even by simple inspection since minimal formula sizes for 3-input Boolean functions over this basis never exceeds 5.*

Remark 0.2 *These formulas could be used for an efficient implementation of an elementary CA simulator. For more information, we refer the interested reader to [1].*

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REFERENCES

[1] F. Bagnoli, P. Palmerini and R. Rechtman, Mapping criticality into self criticality, Phys. Rev. E. 55, 3970, 1997.

[2] E. Georgiadis, I. Wegener, On maximal formula size of minimal boolean formulas, Unpublished, 2006.

[3] I. Wegener, *The Complexity of Boolean Functions*, John Wiley & Sons, 1987.

[4] S. Wolfram, *A New Kind of Science*,Wolfram Media, Inc., pp. 884-885, 2002.