DOMINATION NUMBERS OF GRID GRAPHS

by

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A dissertation submitted in partial fulfillment of the requirements
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Major Professor: W. Edwin Clark, Ph.D.

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CERTIFICATE OF APPROVAL

Ph. D. Dissertation

This is to certify that the Ph. D. Dissertation of

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An Abstract

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A $k \times n$ grid graph is the product $P_k \times P_n$ of a path P_k of length k and a path P_n of length n. $\gamma_{k,n}$ is the domination number of the graph $P_k \times P_n$. The domination numbers of the grid graphs have been studied for several years. Exact valus for $\gamma_{k,n}$, $k = 1, 2, 3, 4$, were found early on. But before now only a finite number of values were known for each $k \geq 5$, and these were found by computer.

We establish the domination numbers $\gamma_{5,n}$ and $\gamma_{6,n}$, $n \geq 1$, and find upper bounds for $\gamma_{k,n}$ for $7 \leq k \leq 10$ by constructing dominating sets. We also establish a general upper bound (called the *standard upper bound*) for $\gamma_{k,n}$, $n \geq k \geq 8$, which we conjecture is also a lower bound for k sufficiently large.

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CHAPTER 1

INTRODUCTION

A *dominating set* in a graph G is a set S of vertices having the property that every vertex is either in S or adjacent to a vertex in S. The *domination number* $\gamma(G)$ of G is the cardinality of a smallest dominating set in G. For an extensive survey of domination problems and a comprehensive bibliography we refer the reader to the recent survey volume edited by Hedetniemi and Laskar [1].

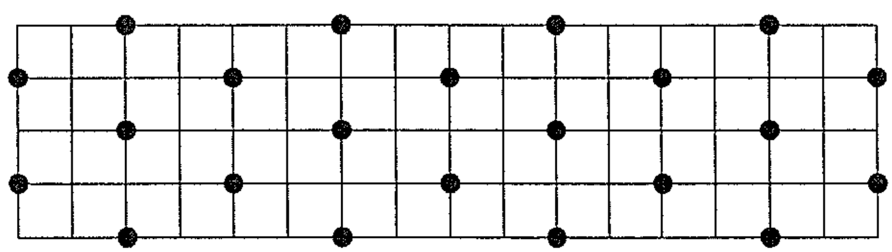


Figure 1.1: A dominating set for $P_5 \times P_{17}$

A *path* P_m of length m is a graph having vertex set $\{v_1, v_2, \ldots, v_m\}$ and edge set $\{v_i v_{i+1} : i = 1, 2, \ldots, m-1\}$. Here ab denotes the unordered pair $\{a, b\}$. A $k \times n$ *(complete) grid graph* $P_k \times P_n$ is the product graph of a path of length k and a path of length n. We take the vertex set of the grid graph $P_k \times P_n$ to be the set of ordered pairs of positive integers (i, j) where $1 \leq i \leq k$ and $1 \leq j \leq n$. By the definition of the product graph, there is an edge from vertex (i, j) to vertex (p, q) if and only if $|i - p| + |j - q| = 1$. To illustrate, in Figure 1.1 we represent a dominating set for the 5×17 grid graph. The elements of the dominating set are indicated by black dots. The intersections of the vertical and horizontal lines are the vertices. Using the terminology of the oriental game Go we call the elements of a dominating set *stones.* The dominating set in Figure 1.1 has 22 stones. This shows that the domination number $\gamma_{5,17}$ of the grid $P_k \times P_n$ is less than or equal to 22. In fact $\gamma_{5,17} = 22$.

One of the applications of the domination problem is resource allocation and placement in parallel computers, Livingston and Stout. [2]

A Go player might be interested in the question, "How many stones are required to dominate the Go board, a 19×19 grid?" In Figure 1.2 we give a dominating set for $P_{19} \times P_{19}$ of cardinality 84. This shows $\gamma(P_{19} \times P_{19}) \leq 84$. We suspect but cannot prove that in fact $\gamma(P_{19} \times P_{19}) = 84$. In Figure 1.2, *the slanted lines with slopes* ± 2 *or* $\pm\frac{1}{2}$ *are not part of the grid graph. They are drawn in just to bring out the pattern formed by the vertices of the dominating set.*

Jacobson and Kinch [3] established $\gamma_{k,n} = \gamma(P_k \times P_n)$ for $k = 1, 2, 3$ and 4. Beyond $k = 4$ the problem becomes much more difficult. E. O. Hare [5][6] developed a dynamic algorithm to compute $\gamma_{k,n}$ and using the output of an implementation of her algorithm she was able to find expressions for $\gamma_{k,n}$ for a number of different values of k and n. In [7] Cockayne, *et al*, establish some upper and lower bounds for $\gamma_{n,n}$ (the square case).

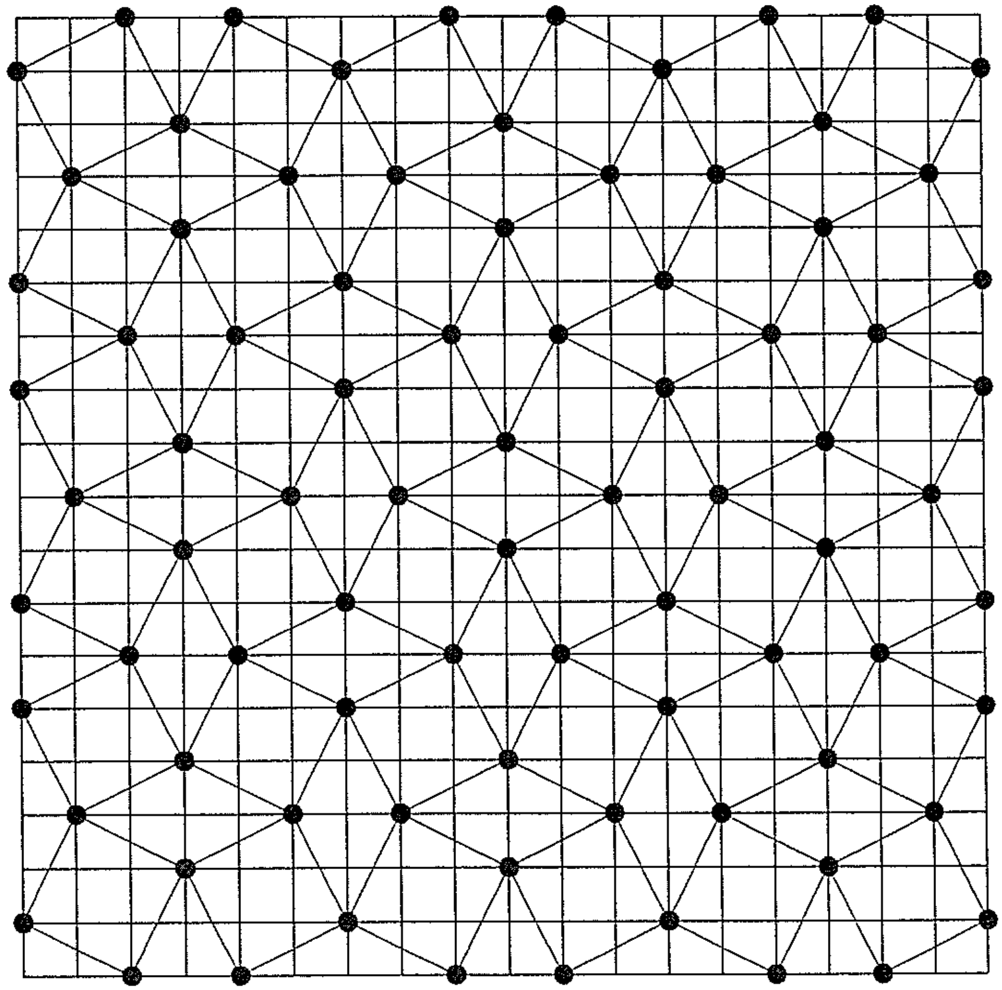


Figure 1.2: A "butterfly" dominating set for $P_{19} \times P_{19}$.

For ease of reference we list below the results of Jacobson and Kinch [3] for $k = 1, 2, 3, 4$. The results for $k = 1, 2, 3$, were also established by Cockayne, *et al*, [7].

$$\gamma_{1,n} = \lceil \frac{n}{3} \rceil \tag{1.1}$$

$$\gamma_{2,n} = \lceil \frac{n+1}{2} \rceil \tag{1.2}$$

$$\gamma_{3,n} = n - \lfloor \frac{n-1}{4} \rfloor \tag{1.3}$$

$$\gamma_{4,n} = \begin{cases} n+1 & \text{for n=1,2,3,5,6 or 9} \\ n & \text{otherwise} \end{cases} \tag{1.4}$$

The following results were obtained by E. O. Hare [5].

For $8 \le n \le 500$:

$$\gamma_{5,n} = n + \lceil \frac{n+4}{5} \rceil \tag{1.5}$$

For $4 \le n \le 500$:

$$\gamma_{6,n} = \begin{cases} n + \lceil \frac{3n-2}{7} \rceil & \text{if } n \equiv 1 \bmod 7 \\ n + 2 + \lceil \frac{3n-2}{7} \rceil & \text{if } n \equiv 3 \bmod 7 \\ n + 1 + \lceil \frac{3n-2}{7} \rceil & \text{otherwise} \end{cases} \tag{1.6}$$

Hare also obtained the values of $\gamma_{k,n}$ for $k = 7, 8, 9, 10, 11, 12$ for limited range of n. (See chapter 3 below)

In Chapter 2, we establish the following upper bound:

$$\gamma_{k,n} \le \lfloor \frac{(k+2)(n+2)}{5} \rfloor - 4, \text{ for } n \ge k \ge 8. \tag{1.7}$$

For small k, this bound is not tight. In Chapter 3, we give improved bounds for $k = 5, 6, \ldots, 10$, and all n by constructing dominating sets. We prove, in Chapter 4, that Hare's formulas for $\gamma_{5,n}$ and $\gamma_{6,n}$ extend to all positive integers n. It is our hope that the methods used here will eventually lead to a determination of $\gamma_{k,n}$ for all k and n. We conjecture that (1.7) holds with equality for k and n sufficiently large. In chapter 5 we discuss recent results due to David Fisher.

The domination problem has also been studied in coding theory; there it is called the *covering problem.* For a given finite metric space (X, d), a *p-cover* of X is a subset S of X such that for every $x \in X$, the distance from x to S

$$d(x, S) := \min_{y \in S} d(x, y) \leq p.$$

The *p-covering number* $\gamma_p(X)$ is

$$\gamma_p(X) = \min \{|S| : S \text{ is a p-cover}\}.$$

The Hamming metric [8] in coding theory is the same as the path metric of K_q^n, where K_q is the complete graph with q vertices. The Lee metric [8] is the same as the path metric of C_q^n, where C_q is the cycle with q vertices. Both the Hamming metric and Lee metric have been studied extensively. Now we define a metric space on the product of n paths. Let P_{q_i} be paths, $i = 1, \ldots, n$, then the set

$$\prod_{i=1}^{n} P_{q_i}$$

with the metric

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \ x_i, y_i \in P_{q_i}$$

is a metric space. If we take $q_i = q$ for all i we get the class of graphs P_q^n analogous to K_q^n and C_q^n. Note that both K_q^n and C_q^n are vertex transitive, but for $q \geq 2$ P_q^n is clearly not vertex transitive; in fact, not even regular. In this dissertation we concentrate on $P_k \times P_n$. In this case

$$d((i,j),(p,q)) = |i-p| + |j-q|.$$

This metric is the same as the path metric on the $k \times n$ grid graph. The 1-covering number is the same as the domination number.

For convenience we list some basic notation:

1. $P_k \times P_n$ is the $k \times n$ grid graph. Sometimes we use $P_k \times P_n$ itself as the vertex set of the grid graph. P_q is a set of q consecutive integers, but in most cases $P_q = \{1, \ldots, q\}$.

2. $\gamma_{k,n}$ is the domination number of $P_k \times P_n$. Since $\gamma_{k,n} = \gamma_{n,k}$, we always suppose $n \geq k$.

3. C_j is the set of vertices in the j-th column of $P_k \times P_n$.

4. R_i is the set of vertices in the i-th row of $P_k \times P_n$.

5. If S is a dominating set of $P_k \times P_n$ then $S_j = S \cap C_j$. And s_j is the number of elements in S_j.

6. S is always a dominating set or "would be" dominating set of $P_k \times P_n$. A *stone* is an element of S.

7. The distance between $u = (x_1, y_1)$ and $v = (x_2, y_2)$ is

$$d(u, v) = |x_2 - x_1| + |y_2 - y_1|.$$

8. A vertex v is said to be *covered by* a vertex u if $v = u$ or v is adjacent to u, equivalently $d(u, v) \leq 1$.

9. The *boundary* of $P_k \times P_n$ is the set of vertices having at least one coordinate equal to 1 or k or n.

CHAPTER 2

BOUNDS FOR THE DOMINATION NUMBERS OF $P_k \times P_n$

Upper bounds and lower bounds for the square grid graphs $P_n \times P_n$ were given by Cockayne, Hare, Hedetniemi and Wimer [7]. They gave the formulas: for $q \geq 2$,

$$\gamma_{n,n} \leq \begin{cases} \frac{1}{5}(n^2+4n-16) & n = 5q-2 \\ \frac{1}{5}(n^2+4n-17) & n = 5q-1 \\ \frac{1}{5}(n^2+4n-20) & n = 5q \\ \frac{1}{5}(n^2+4n-20) & n = 5q+1 \\ \frac{1}{5}(n^2+4n-17) & n = 5q+2 \end{cases}$$

In this chapter we will establish the following "standard upper bound":

$$\gamma_{k,n} \leq \lfloor \frac{(k+2)(n+2)}{5} \rfloor - 4 \quad \text{for } n \geq k \geq 8. \tag{2.1}$$

When k is small, say $k < 16$, this bound is not tight. We will show this for $k \leq 10$ by constructing dominating sets in next chapter.

2.1 Standard upper bound

Let Z denote the additive group of integers, $Z^2 = Z \times Z$, the product of Z with itself and Z_5, the group of integers modulo 5. Let $f : Z^2 \longrightarrow Z_5$ be the homomorphism given by $f(x,y) = x + 2y$, for $(x,y) \in Z^2$. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Then

$$f(u \pm e_1) = f(u) \pm 1 \textit{ and } f(u \pm e_2) = f(u) \pm 2, \text{ for all } u \in Z^2.$$

The *unit ball* $B(u)$ about a vertex $u \in Z^2$ is defined as the set

$$B(u) = \{v : d(v,u) \leq 1\}.$$

So

$$B(u) = \{u, u \pm e_1, u \pm e_2\}.$$

Hence

$$f(B(u)) = \{f(u), f(u) \pm 1, f(u) \pm 2\} = f(u) + Z_5 = Z_5$$

So f restricted to $B(u)$ is a bijection from $B(u)$ to Z_5. Hence if $v, w \in B(u)$ and $f(v) = f(w)$, we must have $w = v$. In fact it is easy to verify that

$$B(u) \cap B(v) = \phi, \text{ if } u, v \in f^{-1}(t) \text{ and } u \neq v.$$

A *perfect dominating set* S of a graph G is a dominating set such that every vertex is covered by one and only one element in S. For fixed t, there is exactly one element in $B(u) \cap f^{-1}(t)$ for all $u \in Z^2$. Thus $f^{-1}(t)$ is a perfect dominating set of Z^2, which may be consider as an infinite grid graph with no boundaries.

In a finite grid graph, we define

$$V(t) = f^{-1}(t) \cap (P_k \times P_n) \qquad t = 0, 1, 2, 3, 4.$$

In general $V(t)$ is not a perfect dominating set since there always exist vertices on the boundary of $P_k \times P_n$ left uncovered. The only grid graphs for which a perfect dominating set exists are $P_4 \times P_4$, $P_1 \times P_n$ and $P_2 \times P_n$ (n is odd)[2]. But $|V(t)|$ always gives a lower bound for the domination number.

Lemma 2.1 *Let S be a dominating set of $P_k \times P_n$ then $|S| \geq |V(t)|$ for all t.*

Proof Since S is a dominating set, every $u \in V(t)$ must be covered by an element $s \in S$. If we chose one $s \in S$, which covers u, as the image of u, a mapping from $V(t)$ to S is well defined. If $v \neq u$, $v \in V(t)$ is also covered by s we have $s \in B(u) \cap B(v)$, a contradiction. Hence $u \mapsto s$ is 1-1 so $|S| \geq |V(t)|$. ∎

The number of vertices in $|V(t)|$ can be easily counted as in the following Lemma.

Lemma 2.2

$$\lfloor \frac{kn}{5} \rfloor \leq |V(t)| \leq \lceil \frac{kn}{5} \rceil \quad \textit{for all } t,$$

and there exist t_0, t_1 such that

$$|V(t_0)| = \lfloor \frac{kn}{5} \rfloor \textit{ and } |V(t_1| = \lceil \frac{kn}{5} \rceil$$

Proof It is easy to verify that every five consecutive vertices u, $u + e_1$, $u + 2e_1$, $u + 3e_1$, $u + 4e_1$ in the same column (or row) will have all five different values under the mapping f and $f(u) = f(u + 5e_1)$. Therefore in the same column (resp. row), the number of vertices in $V(t)$ is either $\lfloor \frac{k}{5} \rfloor$ or $\lfloor \frac{k}{5} \rfloor + 1$ (resp. $\lfloor \frac{n}{5} \rfloor$ or $\lfloor \frac{n}{5} \rfloor + 1$). In particular if $kn = 0 \bmod 5$, then $|V(t)| = \frac{kn}{5} = \lfloor \frac{kn}{5} \rfloor = \lceil \frac{kn}{5} \rceil$, for all t. So the lemma is true for this special case. The rest of proof is for the case $kn \neq 0 \bmod 5$.

For fixed t, $f(u) = t \iff f(u + e_1) = t + 1$. Therefore in two consecutive rows R_i and R_{i+1}, we have

$$|R_{i+1} \cap V(t+1)| = |R_i \cap V(t)| \quad \text{for } i = 1, 2, \ldots, k-1.$$

But

$$|\,|R_1 \cap V(t+1)| - |R_k \cap V(t)|\,| \leq 1,$$

since both of $|R_1 \cap V(t+1)|$ and $|R_k \cap V(t)|$ are between $\lfloor \frac{n}{5} \rfloor$ and $\lfloor \frac{n}{5} \rfloor + 1$. Thus the number of vertices in $V(t+1)$ differs from the number of vertices in $V(t)$ at most one. The same argument is true for $|V(t-1)|$ and $|V(t)|$. Since $f(u+e_2) = f(u)+2$ we apply the above argument to the columns and obtain $|V(t+2)|$ differs from $|V(t)|$ at most one and $|V(t-2)|$ differs from $|V(t)|$ at most one.

Now, let $V(t_0)$ be the class having smallest number of vertices. Then each of the other classes has either $|V(t_0)|$ or $|V(t_0)|+1$ vertices. Since

$$V(t_0) \cup V(t_0-1) \cup V(t_0-2) \cup V(t_0+1) \cup V(t_0+2) = P_k \times P_n,$$

we have that

$$5|V(t_0)| \leq |V(t_0-1)| + |V(t_0-2)| + |V(t_0)| + |V(t_0+1)| + |V(t_0+2)| = kn,$$

hence $|V(t_0)| \leq \frac{kn}{5}$ and $|V(t_0)| \leq \lfloor \frac{kn}{5} \rfloor$. Also

$$5|V(t_0)| + 4 \geq |V(t_0-1)| + |V(t_0-2)| + |V(t_0)| + |V(t_0+1)| + |V(t_0+2)| = kn$$

implies $|V(t_0)| \geq \frac{kn}{5} - 1$. In the case $kn \neq 0 \bmod 5$,

$$|V(t_0)| \geq \lceil \frac{kn}{5} \rceil - 1 = \lfloor \frac{kn}{5} \rfloor.$$

Thus $|V(t_0)| = \lfloor \frac{kn}{5} \rfloor$. Since any class has number of vertices at most one more than $|V(t_0)|$, we have

$$\lfloor \frac{kn}{5} \rfloor \leq |V(t)| \leq \lfloor \frac{kn}{5} \rfloor + 1 = \lceil \frac{kn}{5} \rceil, \text{ for all } t.$$

Since kn is not divisible by 5, it can not happen that $|V(t)| = \lfloor \frac{kn}{5} \rfloor$, for all t. So there must be a class $V(t_1)$ having $\lceil \frac{kn}{5} \rceil$ vertices. ∎

By Lemma 2.1 and Lemma 2.2, a lower bound for $\gamma_{k,n}$ in the next theorem follows.

Theorem 2.1 *If S is a dominating set of $P_k \times P_n$ then $|S| \geq \lceil \frac{kn}{5} \rceil$.*

Theorem 2.2

$$\gamma_{k,n} \leq \lfloor \frac{(k+2)(n+2)}{5} \rfloor - 4, \quad \textit{for } k \leq n \leq 8.$$

Proof Consider the induced subgraph of Z^2 with the vertex set

$$V = \{0, 1, \ldots, k+1\} \times \{0, 1, \ldots, n+1\},$$

this is a graph isomorphic to $P_{k+2} \times P_{n+2}$. Let $V(t) = V \cap f^{-1}(t)$, then by Lemma 2, we can choose t so that

$$|V(t)| = \lfloor \frac{(k+2)(n+2)}{5} \rfloor.$$

Every vertex in $P_k \times P_n = \{1, \ldots, k\} \times \{1, \ldots, n\}$ is covered by $V(t)$. If we move every stone in $V(t)$, which is on the boundary of $P_{k+2} \times P_{n+2}$ and not a corner vertex, one step inside, we can get a dominating set of $P_k \times P_n$. And morever we'll show that one stone can always be eliminated near each of the four corners for any $V(t)$. By constructing such a dominating set, the inequality in the theorem follows.

We discuss the construction in detail for any t and for the upper left corner. This construction is also shown in Figure 2.1. In the figure we number the rows from the top down beginning with 0 and we number the columns from left to right beginning with 0. In Figure 2.1, the vertices indicated by empty circle will be the stones deleted and the vertices indicated by framed dots will be the stones added in.

In Figure 2.1, the construction of a dominating set on the upper-left corner is described as following.

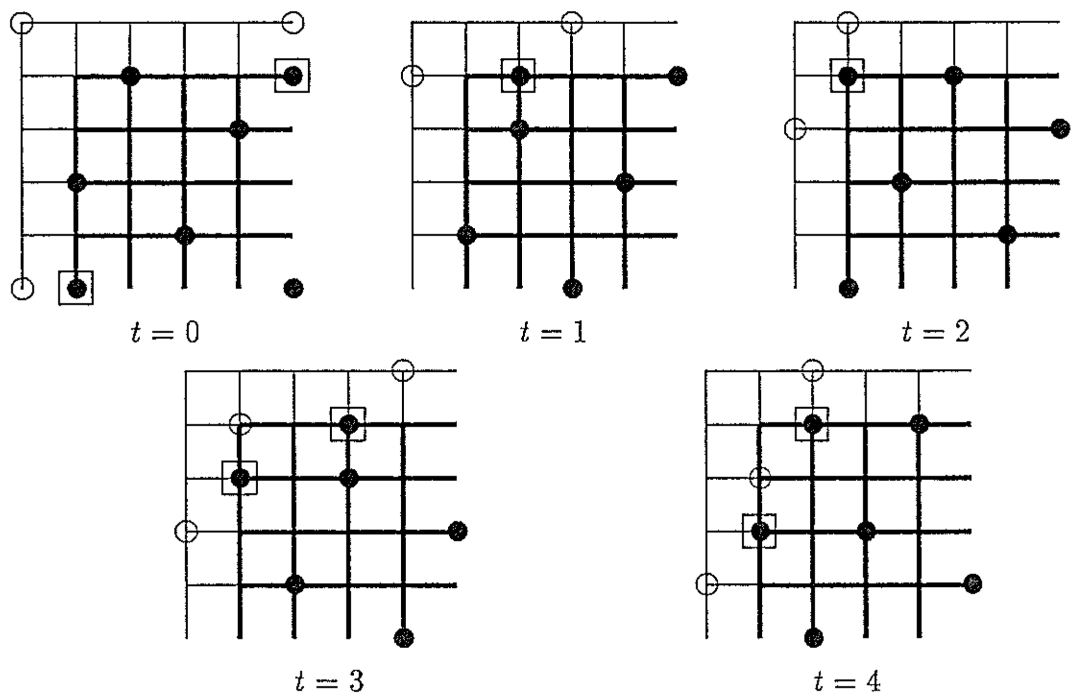


Figure 2.1: Eliminating a stone in the upper left corner.

t=0: The stone (0,0) does not cover any vertex of $P_k \times P_n$, it can be simply deleted.

t=1: Originally, the vertex (1,1) was covered by stone (1,0) and the vertex (1,3) was covered by stone (0,3). By deleting those two stones on the boundary of $V(1)$ and adding in a stone (1,2), we can get a dominating set with number of stones one less than $|V(1)|$.

t=2: By adding a stone (1,1), both of stones (2,0) and (0,1) can be deleted.

t=3: By moving the stone form (0,4) to (1,3) and moving the stone from (1,1) to (2,1), the stone (3,0) can be deleted.

t=4: By moving the stone form (0,2) to (1,2) and moving the stone from (2,1) to (3,1), the stone (4,0) can be deleted.

A similar discussing can be done for the other corners. We omit the details. Noticing the structure of $V(t)$, all the stones on the bounbary of $P_{k+2} \times P_{n+2}$ that have been deleted have the other coordinate less than 4. Hence, for other corners, all the stones that were deleted have the other coordinate greater than $n-3$ or $k-3$. So when $n \geq k \geq 8$, none of stone will be deleted for more than one corner. Therefore we can always construct a dominating set of $P_k \times P_n$ having cardinality

$$\lfloor \frac{(k+2)(n+2)}{5} \rfloor - 4. \blacksquare$$

We call a dominating set constructed as in the proof of Theorem 2.2 using any $V(t)$ a *standard dominating* set of $P_k \times P_n$. If a standard dominating set has cardinality

$$\lfloor \frac{(k+2)(n+2)}{5} \rfloor - 4,$$

then it is called a *minimal standard dominating* set. Formula (2.1) is call the *standard upper bound* for $\gamma_{k,n}$.

2.2 Butterfly upper bound

When $k = 1 \bmod 3$ and $n = 1 \bmod 3$, $k \geq 4$, $n \geq 4$, we can construct a dominating set called a *butterfly dominating* set. When k is small a butterfly dominating set gives an upper bound for $\gamma_{k,n}$ smaller than the standard upper bound. An example of a butterfly dominating set is given in Figure 1.2 for $P_{19} \times P_{19}$. (The 12 stones with the slanted lines in row 1 to 7 and column 4 to 10 form a picture resembling a butterfly.)

The set S_j of stones in the j-th column of a butterfly dominating set for $P_k \times P_n$ is defined by

$$\begin{aligned}
S_1 &= \{(2+6q,1) : q = 0,1,\ldots\} \cup \{(6+6q,1) : q = 0,1,\ldots\} \\
S_2 &= \{(4+6q,2) : q = 0,1,\ldots\} \\
S_3 &= \{(1+6q,3) : q = 0,1,\ldots\} \\
S_4 &= \{(3+6q,4) : q = 0,1,\ldots\} \cup \{(5+6q,4) : q = 0,1,\ldots\} \\
S_5 &= \{(1+6q,5) : q = 0,1,\ldots\} \\
S_6 &= \{(4+6q,6) : q = 0,1,\ldots\} \\
S_j &= \{(x,j) : (x,j-6) \in S_{j-6}\} \qquad \text{for } j \geq 7
\end{aligned}$$

The number of stones in a butterfly dominating set can be easily counted. Recall $s_j = |S_j|$ and note $S_n = S_1$ or S_4.

$$s_1 = s_4 = s_n = \frac{k-1}{3}, \; s_2 + s_3 = s_5 + s_6 = \frac{k-1}{3} + 1$$

Hence

$$\sum_{j=1}^{3} s_j = \sum_{j=4}^{6} s_j = 2\frac{k-1}{3} + 1.$$

Therefore

$$\begin{aligned}
|S| &= \sum_{j=1}^{3} s_j + \sum_{j=4}^{6} s_j + \cdots + \sum_{j=n-3}^{n-1} s_j + s_n \\
&= \frac{n-1}{3}\left(2\frac{k-1}{3} + 1\right) + \frac{k-1}{3} \\
&= 2 \cdot \frac{(n-1)(k-1)}{9} + \frac{n-1}{3} + \frac{k-1}{3}.
\end{aligned}$$

This gives an upper bound called *butterfly bound* for $\gamma_{k,n}$.

Theorem 2.3 *If $k = 1 \bmod 3$ and $n = 1 \bmod 3$, $k \geq 4$, $n \geq 4$, then*

$$\gamma_{k,n} \leq 2 \cdot \frac{(n-1)(k-1)}{9} + \frac{n-1}{3} + \frac{k-1}{3}. \tag{2.2}$$

We now compare the standard bound to the butterfly bound. Let $k = 3p+1$ and $n = 3q+1$, $q \geq p \geq 3$. The butterfly bound can be written as

$$\begin{aligned} \gamma_{k,n} &\leq 2 \cdot \frac{n-1}{3}\frac{k-1}{3} + \frac{n-1}{3} + \frac{k-1}{3} \\ &= 2pq + p + q. \end{aligned}$$

And the standard bound can be written as

$$\begin{aligned} \gamma_{k,n} &\leq \lfloor \frac{(k+2)(n+2)}{5} \rfloor - 4 \\ &= \lfloor \frac{(3p+1+2)(3q+1+2)}{5} \rfloor - 4 \\ &= \lfloor \frac{9(p+1)(q+1)}{5} \rfloor - 4. \end{aligned}$$

Observe that when $p > 4$ we have

$$\begin{aligned} & \frac{9(p+1)(q+1)}{5} - 4 < 2pq + p + q \\ \Longleftrightarrow\ & 9pq + 9p + 9q + 9 - 20 < 10pq + 5p + 5q \\ \Longleftrightarrow\ & 4p + 4q - pq - 11 < 0 \\ \Longleftrightarrow\ & q(4-p) + 4p - 11 < 0 \\ \Longleftrightarrow\ & q > \frac{4p-11}{p-4} = 4 + \frac{5}{p-4}. \end{aligned}$$

Therefore when $p > 4$,

$$q > 4 + \frac{5}{p-4} \Longrightarrow \lfloor \frac{9(p+1)(q+1)}{5} \rfloor - 4 < 2pg + p + q. \tag{2.3}$$

By (2.3) together with some simple calculations, we have the results:

1. When $q \geq p \geq 7$, i.e., $n \geq k \geq 22$, since the left side inequality in (2.3) is always satisfied, the standard bound has less stones.

2. When $p = 6$, i.e., $k = 19$:

   For $q \geq 7$, i.e., $n \geq 22$, the standard bound is better by (2.3).

   When $n = 19$, i.e., $q = 6$, they have the same number of stones which is 84.

3. When $k = 16$, i.e., $p = 5$:

   For $q \geq 9$, i.e., $n \geq 28$, the standard bound is better by (2.3).

   When $n = 16$, 19, 22 and 25, they have the same number of stones 60, 71, 82 and 93 respectively.

4. When $k = 13$, i.e., $p = 4$ the butterfly bound becomes $9q + 4$ and the standard bound becomes $9q+5$. The butterfly bound is smaller than the standard bound.

5. When $k = 10$, i.e., $p = 3$, then

$$
\begin{aligned}
& \lfloor \frac{9(p+1)(q+1)}{5} \rfloor - 4 - (2pq + p + q) \\
= & \lfloor \frac{36(q+1)}{5} \rfloor - 4 - 7q - 3 \\
= & \lfloor \frac{q+1}{5} \rfloor + 7q + 7 - 4 - 7q - 3 \\
= & \lfloor \frac{q+1}{5} \rfloor \geq 0.
\end{aligned}
$$

   This shows that the butterfly bound is better than the standard bound. In next chapter, we will show there an upper bound for $P_{10} \times P_n$ smaller than the butterfly bound.

2.3 Lower bounds for $\gamma_{k,n}$

The structure of $f^{-1}(t)$ and Hare's data lead to the conjucture that for k and n large enough the standard upper bound may be itself a lower bound. This has not been proved yet. But some small lower bound may still be helpful.

The lower bound $\lceil \frac{kn}{5} \rceil$ given in Theorem 2.1 seems too low. In [7] the lower bound $\frac{1}{5}(n^2 + n - 3) \leq \gamma_{n,n}$ is established. By a similar argument one may show

$$\gamma_{k,n} \geq \lfloor \frac{kn + \frac{1}{2}k + \frac{1}{2}n}{5} \rfloor.$$

If

$$\gamma_{k,n} \geq \lfloor \frac{kn + k + n}{5} \rfloor,$$

or

$$\gamma_{k,n} \geq \lfloor \frac{(k+2)(n+1)}{5} \rfloor,$$

can be proved, then of course they provide much better low bound. But there is still a big gap between this and the minimal standard upper bound.

CHAPTER 3

UPPER BOUNDS FOR $P_k \times P_n$, $1 \leq k \leq 10$

For ease of comparison we have converted the values of $\gamma_{k,n}$ found by Jacobson [3] and Hare [5] into the standard form:

$$\lfloor \frac{an+b}{p} \rfloor,$$

where a, b and p are positive integers depending on k and in some case b depends on the value of n mod p. The converted results are shown in the following formulas: Formulas (3.5) and (3.6) will be proved for all n in the next chapter,

$$\gamma_{1,n} = \lfloor \frac{n+2}{3} \rfloor \tag{3.1}$$

$$\gamma_{2,n} = \lfloor \frac{n+2}{2} \rfloor \tag{3.2}$$

$$\gamma_{3,n} = \lfloor \frac{3n+4}{4} \rfloor \tag{3.3}$$

$$\gamma_{4,n} = \begin{cases} n+1, & n = 1,2,3,5,6,9 \\ n, & \text{otherwise} \end{cases} \tag{3.4}$$

$$\gamma_{5,n} = \begin{cases} \lfloor \frac{6n+6}{5} \rfloor, & n = 2,3,7 \\ \lfloor \frac{6n+8}{5} \rfloor, & \text{otherwise} \end{cases} \tag{3.5}$$

$$\gamma_{6,n} = \begin{cases} \lfloor \frac{10n+10}{7} \rfloor, & n \geq 6 \text{ and } n \equiv 1 \bmod 7 \\ \lfloor \frac{10n+12}{7} \rfloor, & \text{otherwise if } n \geq 4 \end{cases} \tag{3.6}$$

$$\gamma_{7,n} = \lfloor \frac{5n+3}{3} \rfloor, \quad 2 \leq n \leq 500 \tag{3.7}$$

$$\gamma_{8,n} = \lfloor \frac{15n+14}{8} \rfloor, \quad 7 \leq n \leq 500 \tag{3.8}$$

$$\gamma_{9,n} = \lfloor \frac{23n+20}{11} \rfloor, \quad 4 \leq n \leq 233 \tag{3.9}$$

$$\gamma_{10,n} = \begin{cases} \lfloor \frac{30n+37}{13} \rfloor, & \text{for } n \equiv 0 \text{ or } 3 \bmod 13 \text{ and } n \neq 13, 16 \\ \lfloor \frac{30n+24}{13} \rfloor, & \text{otherwise for } 10 \leq n \leq 125 \end{cases} \tag{3.10}$$

In this chapter we will construct dominating sets for $k = 5, \ldots, 10$ and all $n \geq k$ of cardinality agreeing with the right hand side of the formulas above. Therefore we obtain upper bounds for $\gamma_{k,n}$, for $k = 5, \ldots, 10$. Now we compare the new upper bounds to the standard bounds and the butterfly bounds. Note that the standard bound applies only when $k, n \geq 8$ and the butterfly bounds, only when $k \equiv n \equiv 1 \pmod 3$. We have that

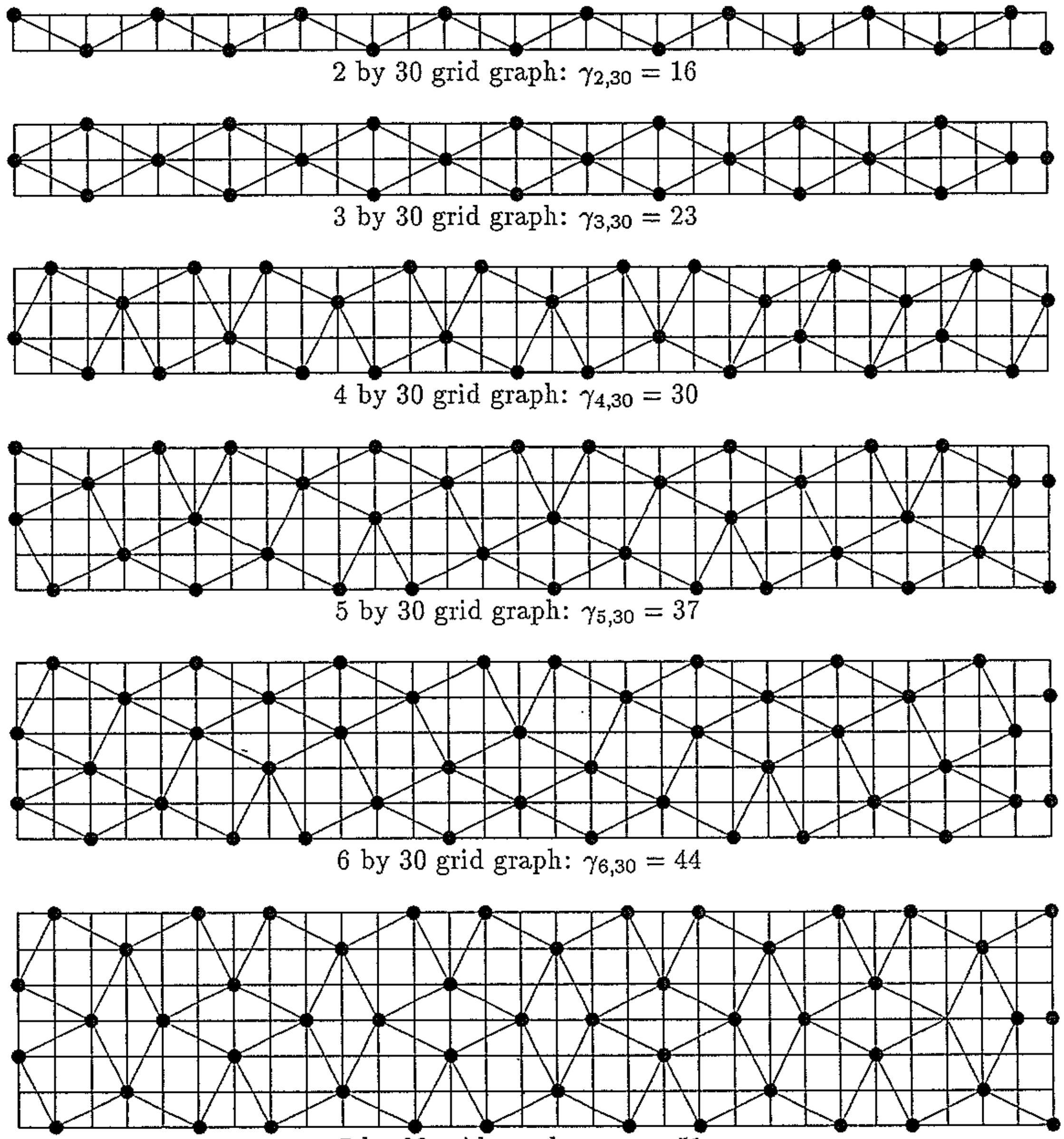
1. for $k = 8, 9, 10$, there exists an n such that the new bounds is strictly less than the standard upper bound;

Now for given k, first we construct blocks A and B of dimensions $k \times p$, where p is the integer appearing in the denominators of Formulas (3.5) to (3.10). Actually B is the symmetric image of A with respect to the horizontal line passing through the center point of A. Then we construct a few smaller blocks B_j and D_j. Thus we have a certain number of basic blocks. Finally we concatenate these basic blocks to construct dominating sets of the form

$$S = E(AB)^s F \text{ or } E(BA)^s F,$$

where s is a nonnegative integer, E and F are basic blocks. The number of stones of each basic block are given, hence we can easily calculate the number of stones in the dominating set S.

Using the above described representation we give in Figure 3.1 (a) and 3.1 (b) some sample dominating sets for the $k \times n$ grid graphs. We will show our concatenations of these dominating sets in later sections.



2 by 30 grid graph: $\gamma_{2,30} = 16$

3 by 30 grid graph: $\gamma_{3,30} = 23$

4 by 30 grid graph: $\gamma_{4,30} = 30$

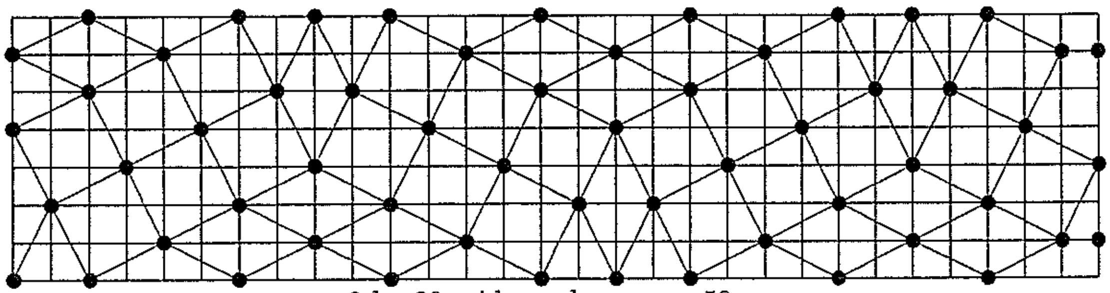
5 by 30 grid graph: $\gamma_{5,30} = 37$

6 by 30 grid graph: $\gamma_{6,30} = 44$

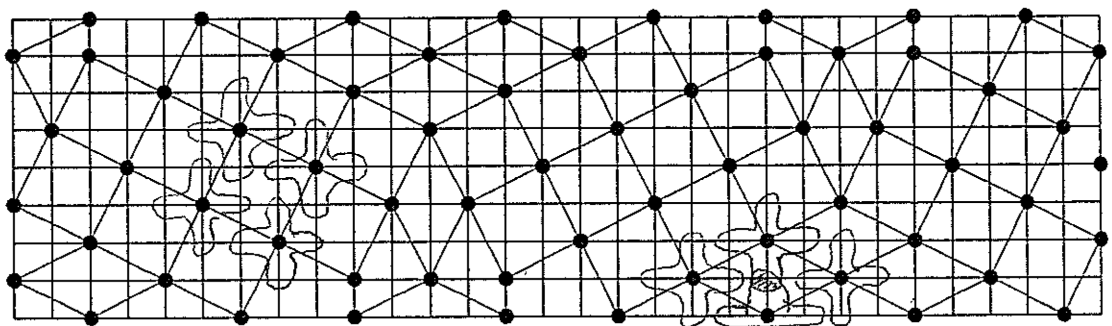
7 by 30 grid graph: $\gamma_{7,30} = 51$

Figure 3.1: (a) Some dominating sets of minimal cardinality

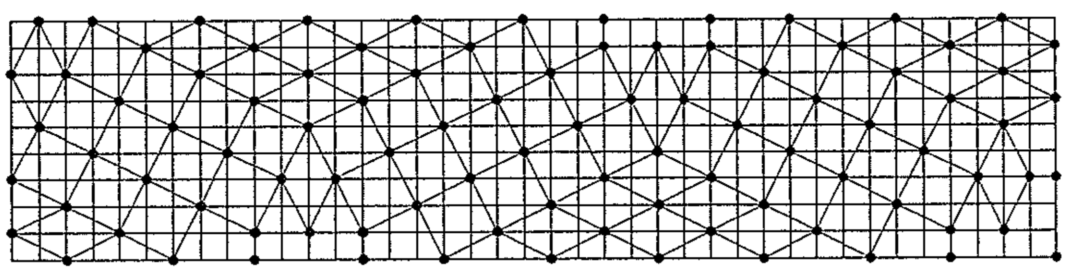
(The slanted lines with slopes ± 2 *or* $\pm\frac{1}{2}$ *are not part of the grid graph. They are drawn in just to bring out the pattern formed by the vertices of the dominating set.)*



8 by 30 grid graph: $\gamma_{8,30} = 58$



9 by 30 grid graph: $\gamma_{9,30} = 64$



10 by 40 grid graph: $\gamma_{10,40} = 94$

Figure 3.1: (b) Some dominating sets of minimal cardinality

(The slanted lines with slopes ± 2 *or* $\pm\frac{1}{2}$ *are not part of the grid graph. They are drawn in just to bring out the pattern formed by the vertices of the dominating set.)*

3.2 Dominating sets for $P_5 \times P_n$

We construct here dominating sets S for $P_5 \times P_n$ when $n \geq 5$ and $n \neq 7$ of cardinality

$$\phi_5(n) = \lfloor \frac{6n+8}{5} \rfloor.$$

Since $n \geq 5$, n may be written in the form

$$n = 5q + r \quad \text{where } 1 \leq r \leq 5 \text{ and } q \geq 0.$$

Then from (3.11)

$$\phi_5(n) = 6q + b_r,$$

where

$$b_1 = 2,\ b_2 = 4,\ b_3 = 5,\ b_4 = 6,\ b_5 = 7.$$

To construct our dominating set S for $P_5 \times P_n$ we use the blocks A, B, B_1, B_2, B_3, B_4 and B_5 of Figure 3.2.

We let $(BA)^s = BABA \dots BA$ denote the concatenation of BA with itself $s \geq 0$ times. Note that $(BA)^s$ is a $5 \times 10s$ block with $6 \cdot 2s$ stones with only two vertices that are not covered by these stones.

Now let $n = 5q + r$, $1 \leq r \leq 5$. If q is even then $q = 2s$, $s \geq 0$. One easily verifies that the stones in $(BA)^s B_r$ give a dominating set for $P_5 \times P_n$ with $6q + b_r = \phi_5(n)$ elements. If q is odd, write $q = 2s+1$; then the stones in $A(BA)^s B_r$ give a dominating set with $6q + b_r = \phi_5(n)$ elements.

3.3 Dominating sets for $P_6 \times P_n$

Here we will construct a dominating set S for $P_6 \times P_n$ of cardinality

$$\phi_6(n) = \begin{cases} \lfloor \frac{10n+10}{7} \rfloor & n \geq 6 \text{ and } n \equiv 1 \bmod 7 \\ \lfloor \frac{10n+12}{7} \rfloor & \text{otherwise if } n \geq 6 \end{cases}$$

Let $n = 7q + r$, $1 \leq r \leq 7$, then from (3.11)

$$\phi_6(n) = 10q + b_r,$$

where the values of b_r are given by the following table:

r	1	2	3	4	5	6	7
b_r	2	4	6	7	8	10	11

Table 3.1: b_r for constructing S in $P_6 \times P_n$.

Now use the blocks in Figure 3.4, to construct S. Note that A and B each have 10 stones and B_r has b_r stones. Again we have two cases:

$q = 2s$: $(BA)^s B_r$ gives a dominating set with the desired number $10q + b_r$ of stones.

$q = 2s + 1$: $A(BA)^s B_r$ gives a dominating set with the desired number $10q + b_r$ of stones.

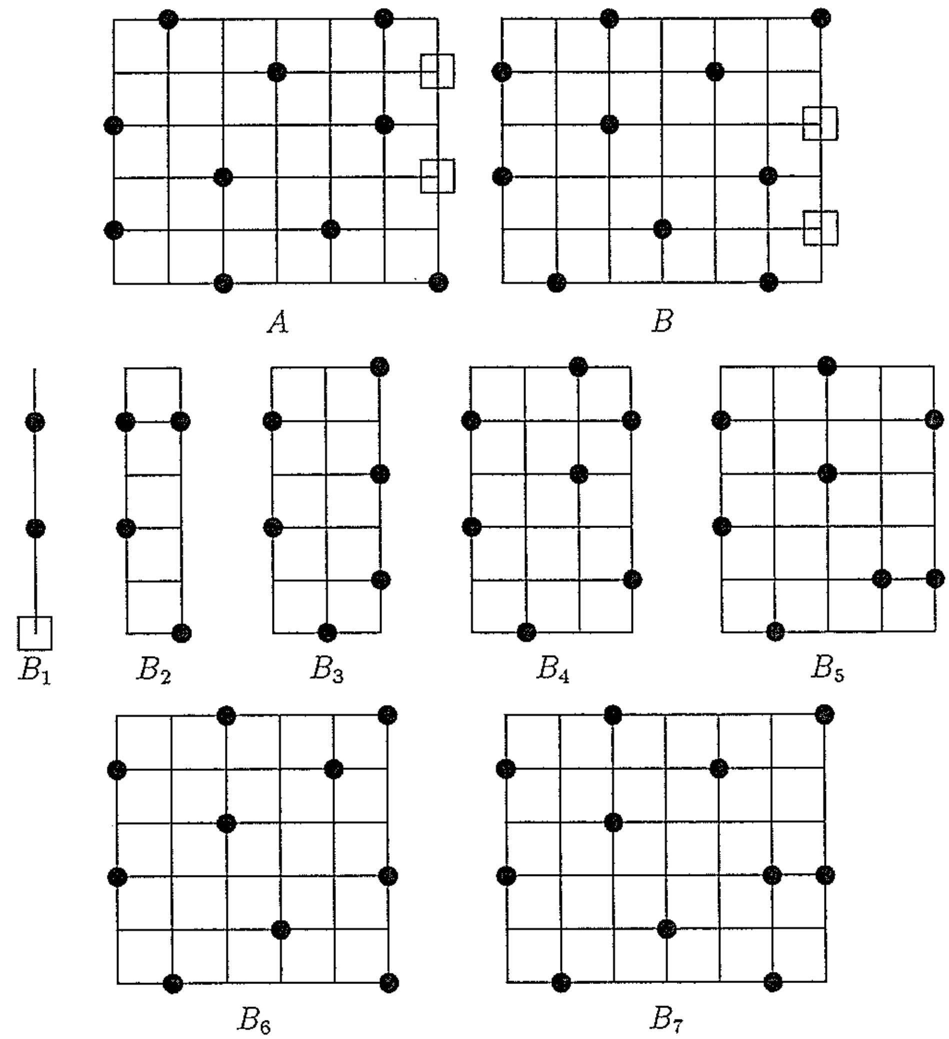


Figure 3.4: Blocks for constructing S in $P_6 \times P_n$.

3.4 Dominating sets for $P_7 \times P_n$

Here we will construct a dominating set S for $P_7 \times P_n$ of cardinality

$$\phi_7(n) = \lfloor \frac{5n+3}{3} \rfloor.$$

Let $n = 6q + r$, $1 \le r \le 6$, then from (3.11)

$$\phi_7(n) = 10q + b_r,$$

where the values of b_r are given by the following table:

r	1	2	3	4	5	6
b_r	2	4	6	7	9	11

Table 3.2: b_r for constructing S in $P_7 \times P_n$.

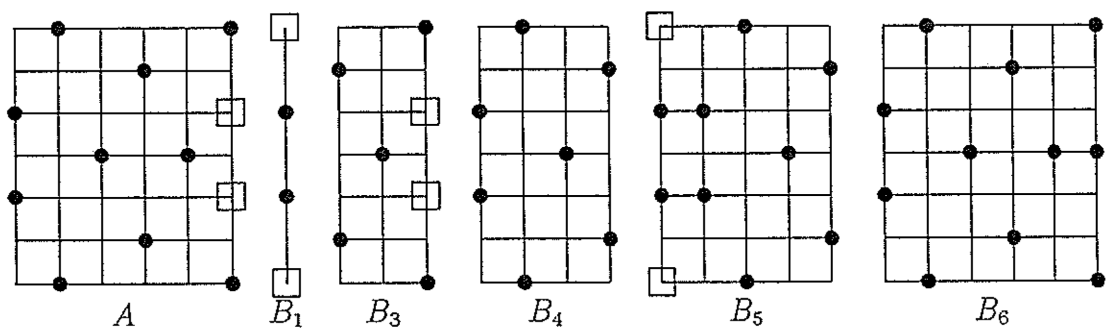


Figure 3.5: Blocks for constructing S in $P_7 \times P_n$.

We use the blocks in Figure 3.5 to construct S. Note that A and B each have 10 stones and B_r has b_r stones for $r = 1, 4, 5, 6$. We must handle the cases $r = 2$ and 3 separately from the other values of r. For each case we exhibit below a particular concatenation of the blocks in Figure 3.5 in which the stones form a dominating set of $P_7 \times P_n$ of cardinality $10q + b_r = \phi_7(n)$ where $n = 6q + r$, $1 \le r \le 6$. Note that since $n \ge 7$, we have $q \ge 1$.

r = 2: $A^{q-1}B_4B_4$ has $10(q-1)+14 = 10q+4$ stones.

r = 3: $B_3A^{q-1}B_6$ has $5+10(q-1)+11 = 10q+6$ stones.

r = 1,4,5,6: A^qB_r has $10q+b_r$ stones.

3.5 Dominating sets for $P_8 \times P_n$

Here we will construct a dominating set S for $P_8 \times P_n$ of cardinality

$$\phi_8(n) = \lfloor \frac{15n+14}{8} \rfloor.$$

If $n = 8q+r$, $1 \leq r \leq 8$, then from (3.11)

$$\phi_8(n) = 15q + b_r,$$

where the values of b_r are given by the following table:

r	1	2	3	4	5	6	7	8
b_r	3	5	7	9	11	13	14	16

Table 3.3: b_r for constructing S in $P_8 \times P_n$.

We use the blocks in Figure 3.6, to construct S. Note that b_r is the number of stones in the block B_r, the blocks A and B each have 15 stones and the blocks A' ans B' each have 16 stones. As in previous sections we simply exhibit an appropriate concatenation in each case which has $\phi_8(n) = 15q + b_r$ stones given $n = 8q+r$, $1 \leq n \leq 8$.

For $1 \leq r \leq 6$, we take

$$\begin{cases} (AB)^sB_r & \text{if } q = 2s \\ A(AB)^sB_r & \text{if } q = 2s+1 \end{cases}$$

For $r = 7$, we take

$$\begin{cases} B_7(BA)^s B' & \text{if } q = 2s+1 \\ B_7(BA)^s BA' & \text{if } q = 2s \end{cases}$$

For $r = 8$, we take

$$\begin{cases} A(BA)^s B' & \text{if } q = 2s+1 \\ (BA)^s B' & \text{if } q = 2s \end{cases}$$

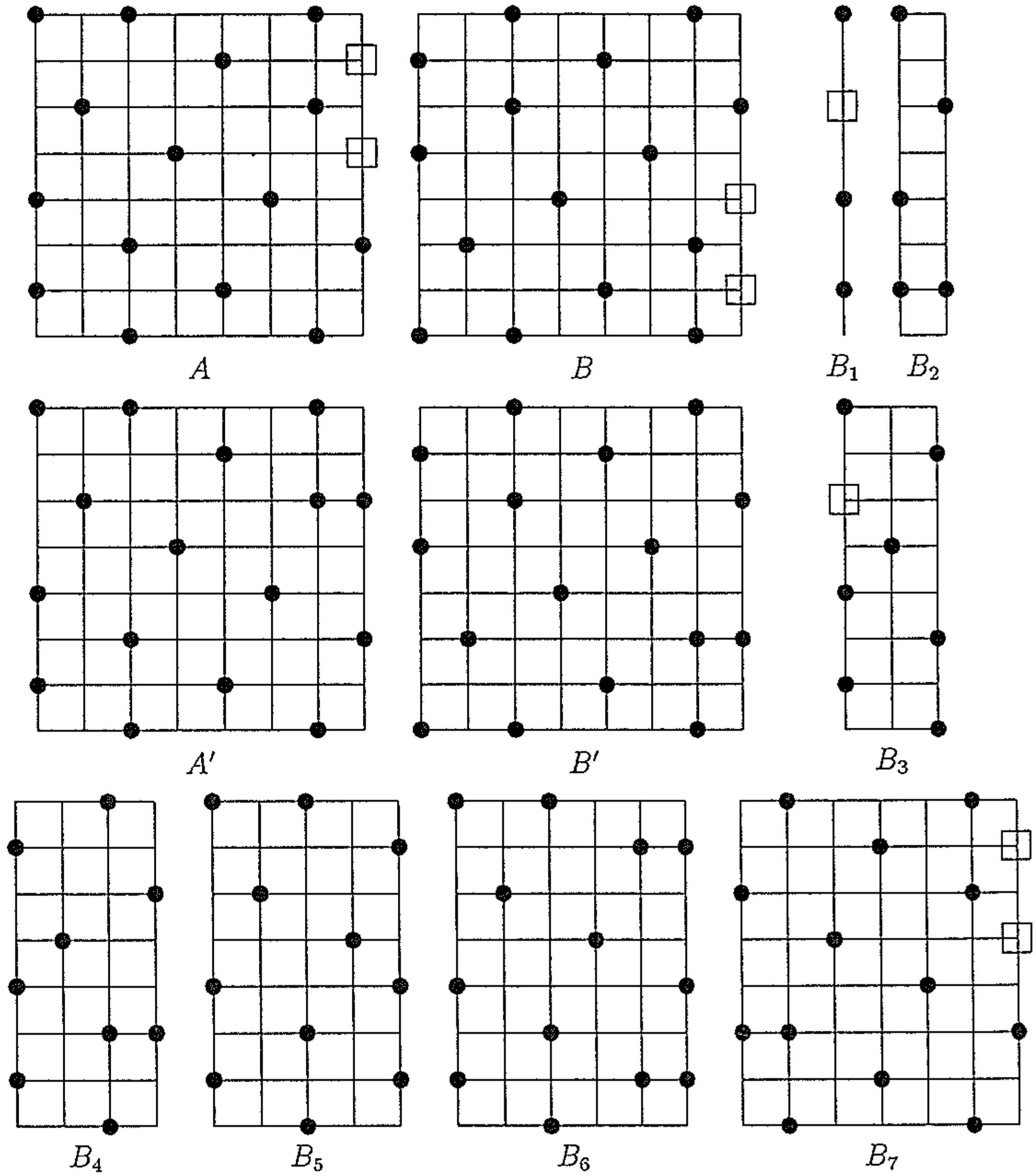


Figure 3.6: Blocks for constructing S in $P_8 \times P_n$.

3.6 Dominating sets for $P_9 \times P_n$

We now construct a dominating set S for $P_9 \times P_n$ of cardinality

$$\phi_9(n) = \lfloor \frac{23n + 20}{11} \rfloor.$$

If $n = 11q + r$, $1 \leq r \leq 11$, then

$$\phi_9(n) = 23q + b_r,$$

where the values of b_r are given by the following table:

r	1	2	3	4	5	6	7	8	9	10	11
b_r	3	6	8	10	12	14	16	18	20	22	24

Table 3.4: b_r for constructing S in $P_9 \times P_n$.

We use the blocks in Figure 3.7. For $r \neq 6, 7$, the blocks named B_r have b_r stones and the blocks A and B have 23 stones each. It follows that if $r \neq 6, 7$, the desired dominating sets are given by

$$\begin{cases} (AB)^s B_r & \text{if } q = 2s \\ B(AB)^s B_r & \text{if } q = 2s + 1. \end{cases}$$

To handle the cases $r = 6$ and 7 we use the remaining blocks P and Q. In the case of $r = 6$, we take

$$\begin{cases} P(AB)^s B_8 & \text{if } q = 2s + 1 \\ QB(AB)^{s-1} B_8 & \text{if } q = 2s; \end{cases}$$

and in the case of $r = 7$, we take

$$\begin{cases} P(AB)^s B_9 & \text{if } q = 2s + 1 \\ QB(AB)^{s-1} B_9 & \text{if } q = 2s. \end{cases}$$

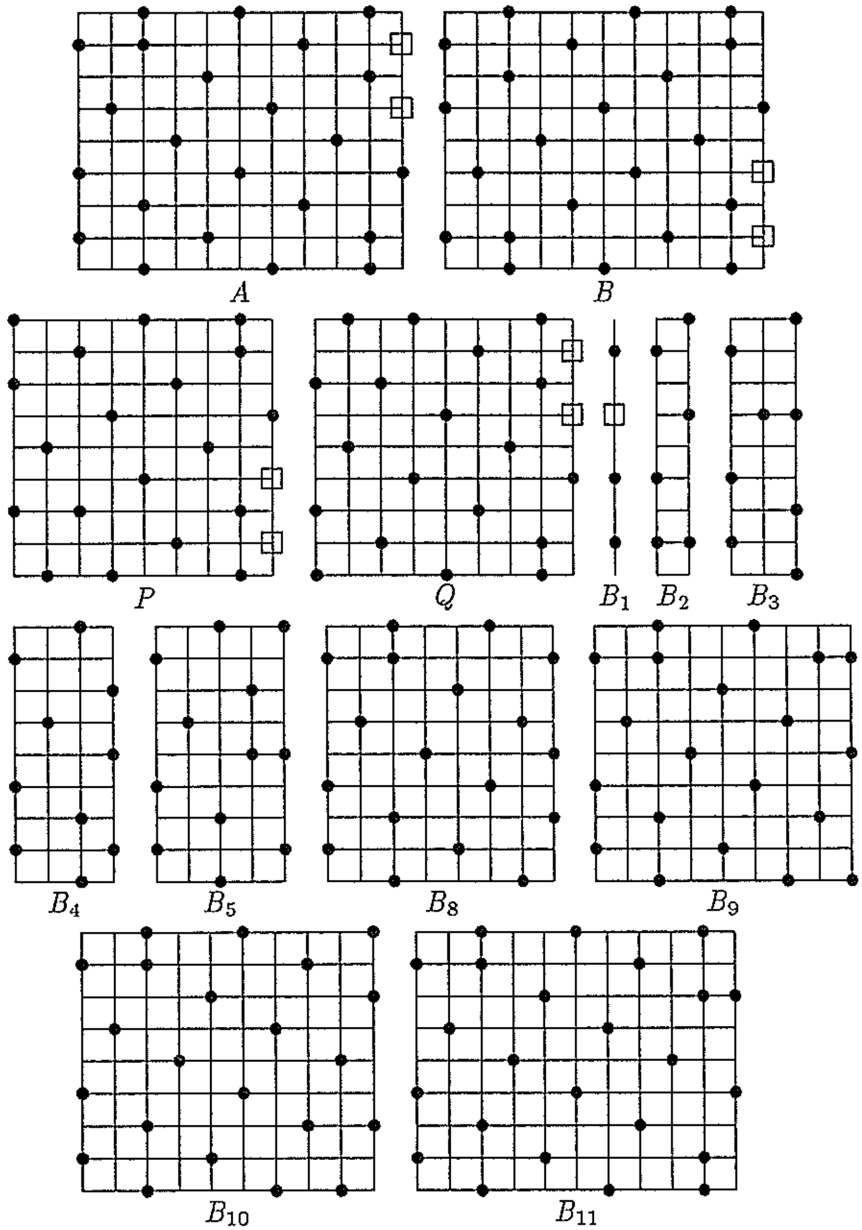


Figure 3.7: Blocks for constructing S in $P_9 \times P_n$.

3.7 Dominating sets for $P_{10} \times P_n$

We now construct dominating sets S for $P_{10} \times P_n$ of cardinality

$$\phi_{10}(n) = \begin{cases} \lfloor \frac{30n+37}{13} \rfloor & \text{for } n \equiv 0 \text{ or } 3 \bmod 13, \text{ but } n \neq 13, 16 \\ \lfloor \frac{30n+24}{13} \rfloor & \text{otherwise.} \end{cases}$$

If $n = 13q + r$, $1 \leq r \leq 13$, then except for $n = 13$ and 16 we have

$$\phi_{10}(n) = 30q + b_r,$$

where b_r is given by the following table:

r	1	2	3	4	5	6	7	8	9	10	11	12	13
b_r	4	6	9	11	13	15	18	20	22	24	27	29	32

Table 3.5: b_r for constructing S in $P_{10} \times P_n$.

We now refer to the blocks in Figure 3.8(a) and 3.8(b).

For the special cases $n = 13$ and 16 the blocks D_{13} and D_{16} in Figure 3.8 (b) give, respectively, dominating sets with $\phi_{10}(13) = 31$ and $\phi_{10}(16) = 38$ stones. If $q = 0$, we have only the remaining cases $n = 10$, 11 and 12. In these cases one may obtain a dominating set by replacing the first two columns of B_r, $r = 10$, 11 and 12, by E_2.

We may now assume that $q \geq 1$ and if $q = 1$ then $r \neq 3$ since we have already handled the exceptional case $n = 16$. The blocks A, A', B, B' have 30 stones each and the blocks B_r, for $r \neq 4$, 6, have b_r stones each. So for $r \neq 4$, 6 we obtain the required $30q + b_r$ stones by taking

$$\begin{cases} A'(BA)^s B_r & \text{if } q = 2s+1,\ s \geq 0 \\ B'A(BA)^{s-1} B_r & \text{if } q = 2s,\ s \geq 1. \end{cases}$$

For $r = 4$ note that D_2 has 5 stones which with the 6 stones in B_2 gives $11 = b_4$ stones. So we take

$$\begin{cases} D_2'(BA)^s B_2 & \text{if } q = 2s,\ s \geq 1 \\ D_2 A(BA)^{s-1} B_2 & \text{if } q = 2s+1,\ s \geq 0. \end{cases}$$

For $r = 6$, we note that B_{19} has $45 = 30 + 15 = 30 + b_6$ stones. So if $q > 1$ we take

$$\begin{cases} A'(BA)^{s-1} B_{19} & \text{if } q = 2s,\ s \geq 1 \\ B'A(BA)^{s-1} B_{19} & \text{if } q = 2s+1,\ s \geq 1. \end{cases}$$

If $q = 1$, then $n = 19$. In this case we replace the first two columns of B_{19} by E_2. This produces a block with a dominating set consisting of requisite $30 + b_6 = 45$ stones.

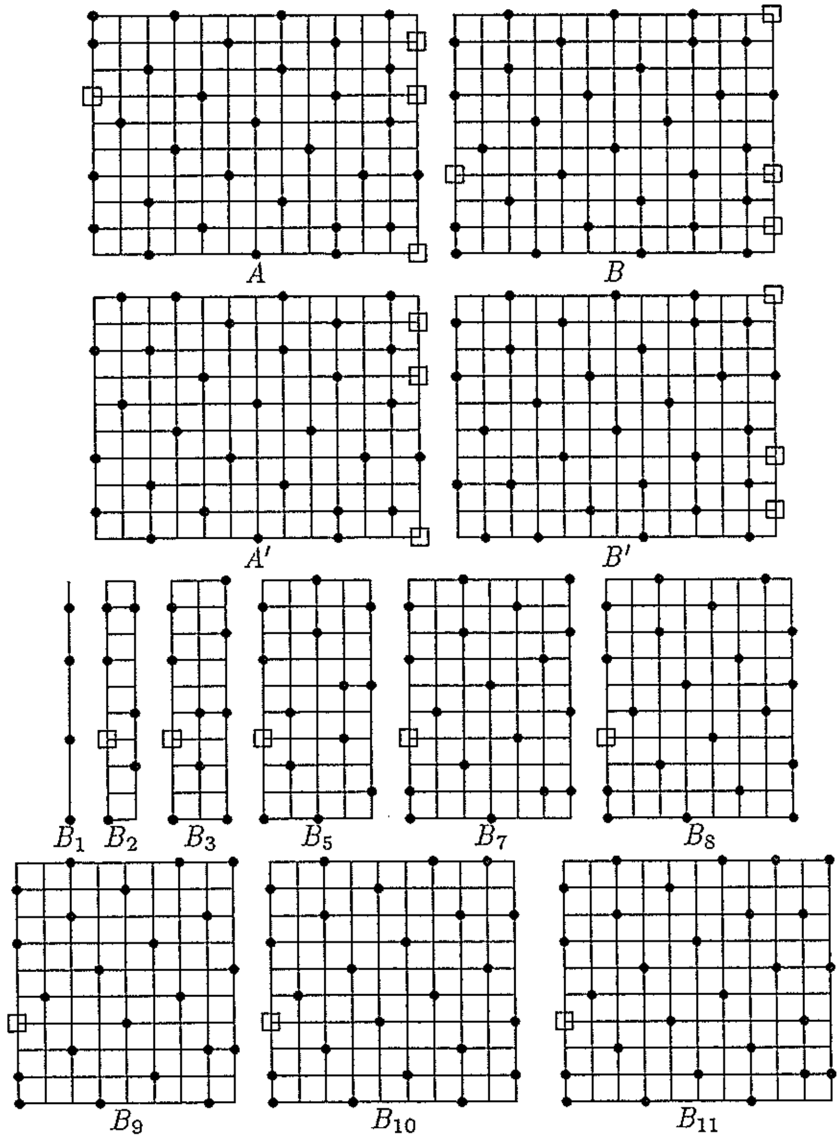


Figure 3.8: (a) Blocks for constructing S in $P_{10} \times P_n$.

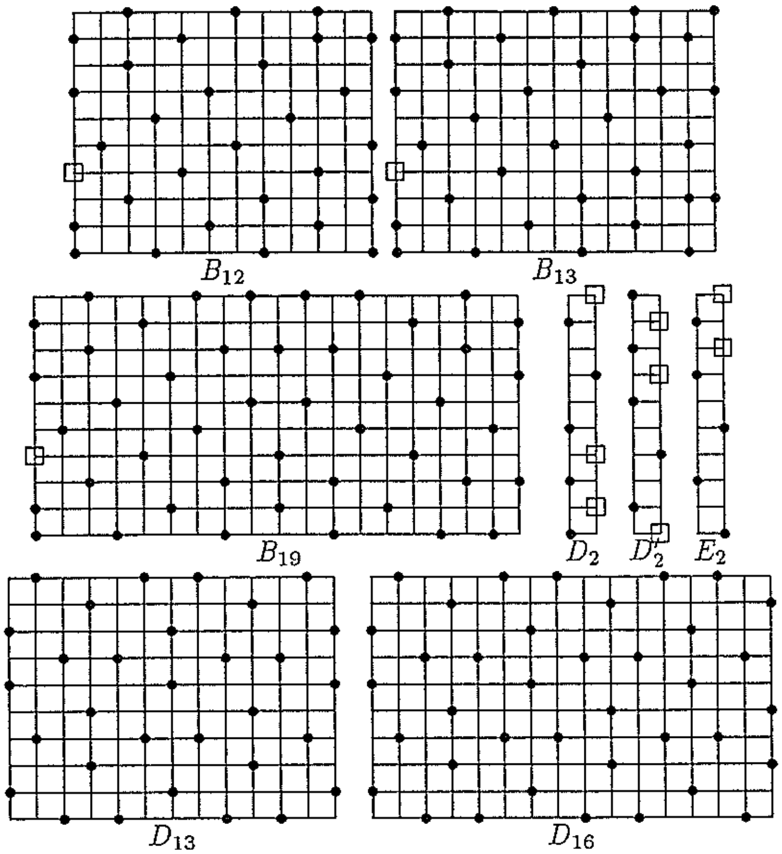


Figure 3.8: (b) Blocks for constructing S in $P_{10} \times P_n$.

Remark. Using her algorithm, Hare has also obtained the domination numbers: $\gamma_{11,n}$, for $n \leq 122$; $\gamma_{12,n}$, for $n \leq 84$; $\gamma_{13,n}$, for $n \leq 92$.

Using an enhanced version of Hare's algorithm, Fisher [10] calculated the domination numbers $\gamma_{k,n}$ for $k \leq 16$ and all n. We discuss his methods in chapter 5. We note however that at last communication Fisher's algorithm did not produce minimal dominating sets for all n. We emphasize that it is easier to find the domination numbers than it is to find the dominating sets of minimal cardinality.

CHAPTER 4

THE DOMINATION NUMBERS OF THE $5 \times n$ AND $6 \times n$ GRID GRAPHS

In section 3.2 and 3.3, we gave upper bounds for $\gamma_{5,n}$ and $\gamma_{6,n}$. In this chapter we will prove these upper bounds themselves are lower bounds, so they are actually the domination numbers. For convenience, we reformulate the formulas (3.5) and (3.6) as follows : (We denote them by ϕ and ψ respectively.)

$$\phi(n) = \begin{cases} n+1+\lfloor \frac{n+1}{5} \rfloor & \text{for n=2,3,or 7,} \\ n+1+\lfloor \frac{n+3}{5} \rfloor & \text{otherwise,} \end{cases} \tag{4.1}$$

and

$$\psi(n) = \begin{cases} n+1+\lfloor \frac{3n+3}{7} \rfloor & \text{if } n \equiv 1 \bmod 7 \text{ or } n=3 \\ n+1+\lfloor \frac{3n+5}{7} \rfloor & \text{otherwise} \end{cases} \tag{4.2}$$

Definition 4.1 *Let S be a dominating set of $P_k \times P_n$. The* dominating sequence *is a sequence of nonnegative integers $s_1, s_2, \ldots, s_n$, where*

$$s_j = |S \cap C_j| \qquad \textit{for } j = 1, 2, \ldots, n.$$

Clearly $\gamma_{k,n}$ is the minimum of $s_1 + s_2 + \cdots + s_n$ over all dominating sequences $(s_1, s_2, \ldots, s_n)$ for $P_k \times P_n$. We will use the following notation: for a dominating sequence for $P_k \times P_n$ we write

$$\Sigma_i^j = \sum_{t=i}^{j} s_t.$$

Our strategy for proving $\gamma_{k,n}$ is as follows: First we establish a number of special cases and inequalities which must hold for any dominating sequence for $P_k \times P_n$. Then using these special cases and inequalites we prove by induction on n (for fixed k) that $\gamma_{k,n} \geq \phi_k(n)$. However the details of the $6 \times n$ case are considerably more complex. As in any proof by induction we must have the answer in advance. Professor Hare's formulas were of utmost importance and we are grateful to her for sharing her results with us.

4.1 The domination number $\gamma_{5,n}$.

First we list the following statements P1-P9. The proofs are given below.

P1 $\gamma_{5,n} \leq \phi(n)$, $n \geq 1$.
P2.1

$$\gamma_{5,1} = 2, \gamma_{5,2} = 3, \gamma_{5,3} = 4, \gamma_{5,4} = 6,$$

$$\gamma_{5,5} = 7, \gamma_{5,6} = 8, \gamma_{5,7} = 9, \gamma_{5,8} = 11,$$

P2.2 For later convenience we restate P2.1 as follows:

$$\gamma_{5,n} = \begin{cases} n+1+\lfloor\frac{n+1}{5}\rfloor & \text{for } n = 3 \\ n+1+\lfloor\frac{n+2}{5}\rfloor & \text{for } n = 2 \text{ or } 7 \\ n+1+\lfloor\frac{n+3}{5}\rfloor & \text{for } n = 1, 4, 5, 6, 8 \end{cases}$$

P3 $0 \leq s_i \leq 5$

P4.1 $\Sigma_1^j \geq \gamma_{5,j-1}$ for $j \geq 2$.

P4.2 $\Sigma_j^n \geq \gamma_{5,n-j}$ for $j < n$.

P4.3 $\Sigma_i^j \geq \gamma_{5,j-i-1}$ for $j \geq i+2$.

P5.1 $s_1 = 0 \Rightarrow s_2 = 5$.

P5.2 $s_i = 0 \Rightarrow s_{i-1} + s_{i+1} \geq 5$ for $2 \leq i \leq n-1$.

P6.1 $s_i \geq 1$ for i=1, 2 $\Rightarrow \Sigma_1^2 \geq 3$.

P6.2 $s_i \geq 1$ for $k \leq i \leq k+4 \Rightarrow \Sigma_k^{k+4} \geq 6$.

P6.3 $s_i \geq 1$ for $1 \leq i \leq 6 \Rightarrow \Sigma_1^6 \geq 8$.

P7 If $s_j = 0$ for some j, $2 \leq j \leq n-1$ and $s_i \geq 1$ for all $i < j$, then

$$\Sigma_1^{j+1} \geq \begin{cases} \gamma_{5,j-1} + 4 & \text{if j=4 or 8} \\ \gamma_{5,j-1} + 3 & \text{otherwise} \end{cases}$$

P8.1 If $n = 5$ and $s_1 \geq 3$ then $\Sigma_1^5 > 7$.

P8.2 If $n = 9$ and $s_1 \geq 3$ then $\Sigma_1^9 > 12$.

P9 If $(s_1, s_2, \ldots, s_n)$ is a dominating sequence for $P_k \times P_n$, then so is the reversed sequence $(s_n, s_{n-1}, \ldots, s_1)$.

We now prove the above statements P1-P9. P1 is an immediiate consequence of the following lemma which was proved in section 3.2.

Lemma 4.1 *For all $n \geq 1$, there is a dominating set S for $P_5 \times P_n$ containing at most*

$$\phi(n) = n + 1 + \lfloor \frac{n+3}{5} \rfloor$$

elements.

Proof of P2.1. The values for $\gamma_{5,n}$ for $1 \leq n \leq 9$ are well-known. They were determined by E. O. Hare[5] by computer. We omit the proofs. However, these values are easy to establish using the method of this paper. ∎

Proof of P2.2. Immediate from P2.1. ∎

The following lemma is crucial.

Lemma 4.2 *Let S be a dominating set for $P_5 \times P_n$ and assume that each of columns i, i+1, i+2, i+3 contains precisely one element of S. Then $2 \leq i$, $i+3 \leq n-1$ and there are only the two possible configurations for S in columns i, i+1, i+2, i+3 shown in figure 4.1. It follows that the vertices indicated by white circles in the columns i-1 and i+4 must lie in S.*

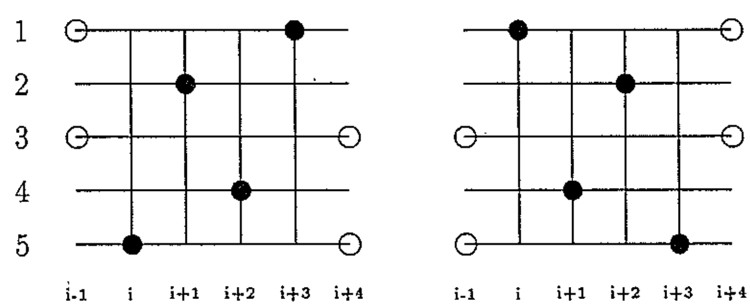


Figure 4.1: The configurations of Lemma 4.2

Proof. We call vertices in S *stones* and we say that a stone v *covers* vertex w if $w = v$ or w is adjacent to v. If v is a stone in the i-th column, then v covers exactly one vertex in column $i-1$, exactly one vertex in column $i+1$ and two or three vertices in column i depending on its location. Let v_j denote the single stone in the j-th column, $j = i, i+1, i+2, i+3$.

We note that the stone v_{i+1} in column $i+1$ cannot lie in row 1 or row 5. For if so it would cover only 2 vertices in column $i+1$, leaving 3 vertices to be covered by only two stones in the adjacent columns. Similarly, v_{i+2} cannot lie in row 1 or 5.

Now if v_{i+1} were in row 3 then it would leave uncovered $(1, i+1)$ and $(5, i+1)$ to be covered by the stones in columns i and $i+2$. But as just noted the stone in column $i+2$ cannot lie in either row 1 or row 5, so this cannot happen.

This leaves only row 2 and 4 for v_{i+1}. Suppose $v_{i+1} = (2, i+1)$. This leaves $(4, i+1)$ and $(5, i+1)$ to be covered by v_i and v_{i+2}. Since v_{i+2} cannot be in row 5 we must have $v_i = (5, i)$ and $v_{i+2} = (4, i+2)$. This forces $v_{i+3} = (1, i+3)$ as desired. This gives the configuration on the left. By symmetry, if $v_{i+1} = (4, i+1)$ we get the configuration on the right. ∎

Proof of P3. Obvious. ∎

Proof of P4.1. If the elements of S in columns $1, \ldots, j-1$ do not cover column $j-1$ we can move the vertices of S that lie in column j to column $j-1$. This will give a cover with Σ_1^j elements of $P_5 \times P_{j-1}$. ∎

Proof of P4.2. Similar to P4.1. ∎

Proof of P4.3. As in the proof of P4.1 if $(s_1, \ldots, s_i, \ldots, s_j, \ldots, s_n)$ is a dominating sequence for $P_5 \times P_n$, then

$$(s_i + s_{i+1}, s_{i+2}, \ldots, s_{j-2}, s_{j-1} + s_j)$$

is a dominating sequence for $P_5 \times P_{j-i-1}$ and P4.3 follows. ∎

Proof of P5.1 and P5.2. Obvious. ∎

Proof of P6.1. If $s_1 = s_2 = 1$, then at most 4 elements of the first column can be covered. Hence $s_1 \geq 2$ or $s_2 \geq 2$ and so $\Sigma_1^2 \geq 3$. ∎

Proof of P6.2. Immediate from Lemma 4.2. ∎

Proof of P6.3. If $\Sigma_1^6 \leq 7$, then $s_1 + s_2 = 3$ by P6.1 and hence $s_3 = s_4 = s_5 = s_6 = 1$. Then by Lemma 4.2 we have $s_2 = 2$ and we can assume (1,2) and (3,2) are the only elements of S in column 2. This leaves $(2,1)$, $(4,1)$ and $(5,1)$ uncovered in column 1. There is no way to cover these three with a single vertex in column 1. ∎

Proof of P7. We first handle separately the cases $j = 2, 3, 4, 5, 6, 7, 8$ and then we complete the proof for $j \geq 9$ with a single argument.

For the cases $j = 2, 3, \ldots, 7$ we use the exact value of $\gamma_{5,j}$ from P2.1 as well as P5.2 and P6.1.

j=2:

$$\Sigma_1^3 = s_1 + 0 + s_3 \geq 5 = \gamma_{5,1} + 3$$

j=3:

$$\begin{aligned}\Sigma_1^4 &= s_1 + \Sigma_2^4 \\ &\geq 1 + 5 = 6 = \gamma_{5,2} + 3\end{aligned}$$

j=4:

$$\begin{aligned}\Sigma_1^5 &= \Sigma_1^2 + \Sigma_3^5 \\ &\geq 3 + 5 = 8 = \gamma_{5,3} + 4\end{aligned}$$

j=5:

$$\begin{aligned}\Sigma_1^6 &= \Sigma_1^2 + s_3 + \Sigma_4^6 \\ &\geq 3 + 1 + 5 = 9 = \gamma_{5,4} + 3\end{aligned}$$

j=6:

$$\begin{aligned}\Sigma_1^7 &= \Sigma_1^2 + s_3 + s_4 + \Sigma_5^7 \\ &\geq 3 + 1 + 1 + 5 = 10 = \gamma_{5,5} + 3\end{aligned}$$

j=7:

$$\begin{aligned}\Sigma_1^8 &= \Sigma_1^2 + s_3 + s_4 + s_5 + \Sigma_6^8 \\ &\geq 3 + 1 + 1 + 1 + 5 = 11 = \gamma_{5,6} + 3\end{aligned}$$

j=8: Here we need P6.3 to obtain

$$\Sigma_1^9 = \Sigma_1^6 + \Sigma_7^9 \geq 8 + 5 = 13 = \gamma_{5,7} + 4.$$

Now let $j \geq 9$. Consider first the sequence

$$s_7, s_8, \ldots, s_{j-2}$$

of length $n = j - 8$. Let $n = 5q + r$, $0 \leq r \leq 4$. Then grouping the elements by fives and using P6.2 we get

$$\Sigma_7^{j-2} \geq 6q + r = 5q + r + q = n + \lfloor \frac{n}{5} \rfloor$$

That is,

$$\Sigma_7^{j-2} \geq j - 8 + \lfloor \frac{j-8}{5} \rfloor.$$

Using this, P6.2 and P5.2 we get

$$\begin{aligned}
\Sigma_1^{j+1} &= \Sigma_1^6 + \Sigma_7^{j-2} + \Sigma_{j-1}^{j+1} \\
&\geq 8 + j - 8 + \lfloor \frac{j-8}{5} \rfloor + 5 \\
&= j + \lfloor \frac{j+2}{5} \rfloor + 3 \\
&= j - 1 + 1 + \lfloor \frac{j-1+3}{5} \rfloor + 3.
\end{aligned}$$

Then using P1 we get

$$\Sigma_1^{j+1} \geq \gamma_{5,j-1} + 3. \blacksquare$$

Proof of P8.1. If $s_4 \geq 1$ and $s_5 \geq 1$ we can use P9 (which is obvious) and apply P6.1 to the reverse of $(s_1, s_2, \ldots, s_5)$ to get $\Sigma_4^5 \geq 3$. Now if $s_i \geq 1$, $2 \leq i \leq 5$, we have

$$\Sigma_1^5 \geq s_1 + s_2 + s_3 + \Sigma_4^5 \geq 3 + 2 + 3 > 7.$$

Otherwise, let j be the largest integer i such that $s_i = 0$. Then $s_{j-1} + s_{j+1} \geq 5$ ($s_{j-1} = 5$ if $j = 5$) and hence if $j = 3$, 4 or 5 we have

$$\Sigma_1^5 \geq s_1 + s_{j-1} + s_{j+1} \geq 3 + 5 > 7.$$

If $j = 2$, then $s_4 \geq 1$, $s_5 \geq 1$ so as above

$$\Sigma_1^5 \geq s_1 + s_3 + \Sigma_4^5 \geq 5 + 3 > 7. \blacksquare$$

Proof of P8.2. Suppose first that $s_i \geq 1$, $1 \leq i \leq 9$. Then since the reverse of a dominating sequence is a dominating sequence we have $\Sigma_4^9 \geq 8$ by P6.3, hence

$$\begin{aligned} \Sigma_1^9 &= s_1 + s_2 + s_3 + \Sigma_4^9 \\ &\geq 3 + 1 + 1 + 8 = 13 > 12. \end{aligned}$$

Otherwise we let j be the largest integer i such that $s_i = 0$. We consider in turn the eight cases $j = 9, 8, \ldots, 2$.

j=9: By P4.3 we have $\Sigma_2^7 \geq \gamma_{5,4} = 6$ and by P5.2 we have $s_8 = 5$. Hence

$$\Sigma_1^9 = s_1 + \Sigma_2^7 + s_8 \geq 3 + 6 + 5 = 14 > 12.$$

j=8: $s_8 = 0$ implies $s_7 + s_9 \geq 5$. If we take any vertex not covered by the elements of S in columns 1-5 we can cover them by adding at most s_6 vertices to S. Now it follows from P8.1 that $\Sigma_1^6 > 7$. Hence

$$\Sigma_1^9 = \Sigma_1^6 + s_7 + s_9 > 7 + 5 = 12.$$

j=7: We must divide this case into several subcases depending on the values of s_2, s_3, s_4, s_5.

subcase 1: $s_i \geq 1$ for i=2, 3, 4, 5: In this case we have

$$\begin{aligned} \Sigma_1^9 &= s_1 + \Sigma_2^5 + \Sigma_6^8 + s_9 \\ &\geq 3 + 4 + 5 + 1 = 13 > 12. \end{aligned}$$

subcase 2: $s_3 = 0$: In this case we have

$$\begin{aligned} \Sigma_1^9 &= s_1 + \Sigma_2^4 + s_5 + \Sigma_6^8 + s_9 \\ &\geq 3 + 5 + 0 + 5 + 1 = 14 > 12. \end{aligned}$$

subcase 3: $s_4 = 0$: In this case we have

$$\begin{aligned} \Sigma_1^9 &= s_1 + s_2 + \Sigma_3^5 + \Sigma_6^8 + s_9 \\ &\geq 3 + 0 + 5 + 5 + 1 = 14 > 12. \end{aligned}$$

subcase 4: $s_5 = 0$, $s_2 = 0$, $s_3 \geq 1$, $s_4 \geq 1$: Hence

$$\begin{aligned} \Sigma_1^9 &= \Sigma_1^3 + \Sigma_4^6 + \Sigma_8^9 \\ &\geq 5 + 5 + 3 = 13 > 12. \end{aligned}$$

Here we use P9 and P6.1 to get $\Sigma_8^9 \geq 3$. The other two inequalities come from P5.2.

subcase 5 : $s_5 = 0$, $s_2 \geq 1$, $s_3 \geq 1$, $s_4 \geq 1$: Hence

$$\begin{aligned} \Sigma_1^9 &= s_1 + \Sigma_2^3 + \Sigma_4^6 + \Sigma_8^9 \\ &\geq 3 + 2 + 5 + 3 = 13 > 12. \end{aligned}$$

subcase 6 : $s_2 = 0$, $s_5 \geq 1$, $s_4 \geq 1$, $s_3 \geq 1$: Hence

$$\begin{aligned} \Sigma_1^9 &= \Sigma_1^3 + s_4 + s_5 + \Sigma_6^8 + s_9 \\ &\geq 5 + 1 + 1 + 5 + 1 = 13 > 12. \end{aligned}$$

j=6:

$$\begin{aligned} \Sigma_1^9 &= s_1 + \Sigma_2^4 + \Sigma_5^7 + \Sigma_8^9 \\ &\geq 3 + \gamma_{5,1} + 5 + 3 = 3 + 2 + 5 + 3 = 13 > 12. \end{aligned}$$

j=5: In this case we consider the reverse sequence:

$$(s_9, s_8, \ldots, s_6, 0, s_4, s_3, s_2, s_1).$$

Since $s_5 = 0$ and $s_i \geq 1$ for i=9, 8, 7, 6, we can apply P7 to get $\Sigma_4^9 \geq \gamma_{5,4} + 3$. Hence

$$\begin{aligned} \Sigma_1^9 &= s_1 + s_2 + s_3 + \Sigma_4^9 \\ &\geq s_1 + s_2 + s_3 + \gamma_{5,4} + 3 \\ &\geq s_1 + s_2 + s_3 + 9. \end{aligned}$$

If $s_2 = 0$, then $s_1 + s_3 \geq 5$ so $\Sigma_4^9 = 14 > 12$.

If $s_2 \geq 1$, then $s_1 + s_2 + s_3 \geq 4$ so $\Sigma_4^9 = 13 > 12$.

j=4: As in the previous case we apply P7 to

$$(s_9, s_8, \ldots, s_5, 0, s_3, s_2, s_1).$$

to get $\Sigma_3^9 \geq \gamma_{5,5} + 3 = 10$. Hence

$$\Sigma_1^9 = s_1 + s_2 + \Sigma_3^9 \geq 3 + 10 = 13 > 12.$$

j=3: As in the previous case $\Sigma_2^9 \geq \gamma_{5,6} + 3 = 11$. So

$$\Sigma_1^9 = s_1 + \Sigma_2^9 \geq 3 + 11 = 14 > 12.$$

j=2: As in the previous case

$$\Sigma_1^9 \geq \gamma_{5,7} + 4 = 13 > 12. \blacksquare$$

Lemma 4.3 *If $(s_1, s_2, \ldots, s_n)$ is a dominating sequence for $P_5 \times P_n$ such that $s_i \geq 1$ for all i, then for $n \geq 8$,*

$$\Sigma_1^n \geq n + 1 + \lfloor \tfrac{n+3}{5} \rfloor.$$

Proof. We use repeatedly in this proof the fact that if $s_i \geq 1$, for $k \leq i \leq l$ and $l - k + 1$, the number of terms in Σ_k^l, is divisible by 5, then by P6.2 we have

$$\Sigma_k^l \geq 6\frac{l-k+1}{5}. \tag{4.3}$$

Let $n = 5q + r$, $0 \leq r \leq 4$. We consider the five cases $r = 0, 1, 2, 3, 4$.

r=0: By (4.3), and P6.1 twice (once applying P9),

$$\begin{aligned} \Sigma_1^n &= \Sigma_1^2 + \Sigma_3^{n-3} + s_{n-2} + \Sigma_{n-1}^n \\ &\geq 3 + 6\frac{n-5}{5} + 1 + 3 = n + 1 + \frac{n}{5} \\ &= n + 1 + \lfloor \tfrac{n+3}{5} \rfloor \end{aligned}$$

r=1: Similarly,

$$\begin{aligned} \Sigma_1^n &= \Sigma_1^2 + \Sigma_3^{n-4} + s_{n-3} + s_{n-2} + \Sigma_{n-1}^n \\ &\geq 3 + 6\frac{n-6}{5} + 1 + 1 + 3 = n + 1 + \frac{n-1}{5} \\ &= n + 1 + \lfloor \tfrac{n+3}{5} \rfloor. \end{aligned}$$

r=2: Using (4.3) and P6.3 twice (once applying P9),

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^6 + \Sigma_7^{n-6} + \Sigma_{n-5}^n \\ &\geq 8 + 6\frac{n-12}{5} + 8 = n + 1 + \frac{n+3}{5} \\ &= n + 1 + \lfloor \tfrac{n+3}{5} \rfloor.\end{aligned}$$

r=3: Similarly,

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^6 + \Sigma_7^{n-2} + \Sigma_{n-1}^n \\ &\geq 8 + 6\frac{n-8}{5} + 3 = n + 1 + \frac{n+2}{5} \\ &= n + 1 + \lfloor \tfrac{n+3}{5} \rfloor.\end{aligned}$$

r=4: Finally,

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^2 + \Sigma_3^{n-2} + \Sigma_{n-1}^n \\ &\geq 3 + 6\frac{n-4}{5} + 3 = n + 1 + \frac{n+1}{5} \\ &= n + 1 + \lfloor \tfrac{n+3}{5} \rfloor. \blacksquare\end{aligned}$$

Theorem 4.1

$$\gamma_{5,n} = \begin{cases} n + 1 + \lfloor \frac{n+1}{5} \rfloor & \textit{for } n = 2,\ 3,\ \textit{or } 7 \\ n + 1 + \lfloor \frac{n+3}{5} \rfloor & \textit{otherwise} \end{cases} \tag{4.4}$$

Proof. For $n \leq 8$ the theorem follows immediately from P2.2. So we may assume $n \geq 9$. By P1 it suffices to show that if $(s_1, s_2, \ldots, s_m)$, $m \geq 9$, is any dominating sequence for $P_5 \times P_m$ then

$$\Sigma_1^m \geq m + 1 + \lfloor \tfrac{m+3}{5} \rfloor. \tag{4.5}$$

We prove (4.5) by induction starting with $m = 8$. Let $n \geq 9$ and assume (4.5) holds for $8 \leq m < n$.

If $s_i \geq 1$ for $1 \leq i \leq n$, then by Lemma 4.3, (4.5) holds. So we can assume $s_i = 0$ for at least one i. Now let $s'_i = s_{n-i+1}$, so $(s'_1, s'_2, \ldots, s'_n)$ is the reverse of $(s_1, s_2, \ldots, s_n)$. We set

$$j := \min\{i | s_i = 0 \; or \; s'_i = 0\},$$

that is, j gives the location of a zero closest to one of the two ends of $(s_1, s_2, \ldots, s_n)$.

For the arguments below we use repeatedy the simple result

$$\lfloor \frac{a}{b} \rfloor + c = \lfloor \frac{a + bc}{b} \rfloor \tag{4.6}$$

which holds for all positive integers a, b, c.

Now we show that (4.5) holds for $m = n$ for all values of j. We do the cases $j = 1, 2, \ldots, 10$ successively and finally handle $j \geq 11$ in a single argument. Of course j is always at most $\lceil \frac{n}{2} \rceil$.

j=1: By symmetry assume $s_1 = 0$. This implies $s_2 = 5$ by P5.1. Hence by P4.2 and the induction hypothesis or P2.2 (whichever applies) we have

$$\begin{aligned}
\Sigma_1^n &= s_2 + \Sigma_3^n \geq 5 + \gamma_{5,n-3} \\
&\geq 5 + (n-3) + 1 + \lfloor \frac{(n-3)+1}{5} \rfloor \\
&\geq n + 1 + (2 + \lfloor \frac{n-2}{5} \rfloor) \\
&\geq n + 1 + \lfloor \frac{n+8}{5} \rfloor \\
&\geq n + 1 + \lfloor \tfrac{n+3}{5} \rfloor.
\end{aligned}$$

j=2: Again we can assume $s_2 = 0$, by symmetry. Using P5.2 and P4.2 we have

$$\Sigma_1^n = s_1 + s_3 + \Sigma_4^n \geq 5 + \gamma_{5,n-4}.$$

Since $n \geq 9$, we have $n - 4 \geq 5$ so by P2.2 or the induction hypothesis

$$\gamma_{5,n-4} \geq (n-4) + 1 + \lfloor \frac{(n-4)+2}{5} \rfloor.$$

Hence

$$\begin{aligned} \Sigma_1^n &\geq n + 1 + 1 + \lfloor \frac{n-2}{5} \rfloor \\ &\geq n + 1 + \lfloor \tfrac{n+3}{5} \rfloor. \end{aligned}$$

Now for $j \geq 3$ we use the following pattern. First we note that by P7

$$\Sigma_1^{j+1} \geq \gamma_{5,j-1} + 3 + \delta \tag{4.7}$$

where $\delta = 1$ if $j = 4$ or 8 and $\delta = 0$ otherwise. Next we take

$$\Sigma_1^n = \Sigma_1^{j+1} + \Sigma_{j+2}^n$$

and use P4.2 and (4.7) to obtain

$$\Sigma_1^n \geq \gamma_{5,j-1} + 3 + \delta + \gamma_{5,n-j-2}. \tag{4.8}$$

Now we apply the induction hypothesis to obtain lower bounds on $\gamma_{5,j-1}$ and $\gamma_{5,n-j-2}$ if $j-1 \geq 8$ or $n-j-2 \geq 8$, otherwise we use P2.2 to get a lower bound. Unfortunately we must consider separately the cases $j = 3, 4, \ldots, 10$.

We note for further use that once we have established $j = 1, 2, \ldots, k$, then when establishing $j = k+1$ we can assume $s_i \geq 1$, $s_i' \geq 1$ for $1 \leq i \leq k$, and $s_k = 0$, so there are at least $2k+1$ entries in $(s_1, s_2, \ldots, s_m)$, i.e. $n \geq 2k+1$.

j=3: In this case since $n \geq 9$ we have $n - j - 2 = n - 5 \geq 4$. We consider two subcases: $n \neq 12$ and $n = 12$:

n $\neq$ 12: In this case $n-5 \neq 7$ and $n - 5 \geq 4$ so by P2.2 or the induction hypothesis (if $n - 5 \geq 8$) we have from (4.7) and (4.8)

$$\begin{aligned}
\Sigma_1^n &\geq \gamma_{5,2} + 3 + \gamma_{5,n-5} \\
&\geq 3 + 3 + n - 5 + 1 + \lfloor \tfrac{n-5+3}{5} \rfloor \\
&= n + 1 + 1 + \lfloor \frac{n-2}{5} \rfloor \\
&= n + 1 + \lfloor \tfrac{n+3}{5} \rfloor.
\end{aligned}$$

n = 12: We must also divide this into two subcases: $\Sigma_1^2 = 3$ and $\Sigma_1^2 \geq 4$. (note by P6.1, $\Sigma_1^2 \geq 3$).

$\Sigma_1^2 = 3$: Now since $s_1 \geq 1$ and $s_2 \geq 1$ we have $s_2 \leq 2$. Since $s_3 = 0$ by P5.2 we have $s_4 \geq 3$. Now the sequence $(s_4, s_5, \ldots, s_{12})$ is a dominating sequence for $P_5 \times P_9$ since $s_3 = 0$. Also $s_4 \geq 3$ so by P8.2, $\Sigma_3^{12} \geq 13$. Hence

$$\begin{aligned}
\Sigma_1^{12} &= \Sigma_1^2 + \Sigma_3^{12} \geq 3 + 13 = 16 \\
&= 12 + 1 + \lfloor \frac{12+3}{5} \rfloor.
\end{aligned}$$

$\Sigma_1^2 \geq 4$: Then

$$\Sigma_1^{12} = \Sigma_1^2 + \Sigma_3^{12} \geq 4 + \gamma_{4,9}$$

and by the induction hypothesis since $9 < 12$ we have $\gamma_{4,9} = 12$ so

$$\Sigma_1^{12} \geq 4 + 12 = 16, \text{ as desired.}$$

j=4: Appplying (4.7) and (4.8) we have

$$\begin{aligned}
\Sigma_1^n &\geq \gamma_{5,3} + 4 + \gamma_{5,n-6} \\
&\geq 4 + 4 + (n-6) + 1 + \lfloor \frac{(n-6)+1}{5} \rfloor \\
&= n + 1 + 2 + \lfloor \frac{n-5}{5} \rfloor \\
&= n + 1 + \lfloor \frac{n+5}{5} \rfloor \\
&\geq n + 1 + \lfloor \frac{n+3}{5} \rfloor.
\end{aligned}$$

j=5: As above,

$$\begin{aligned}
\Sigma_1^n &\geq \gamma_{5,4} + 3 + \gamma_{5,n-7} \\
&\geq 6 + 3 + (n-7) + 1 + \lfloor \frac{(n-7)+1}{5} \rfloor \\
&= n + 1 + 2 + \lfloor \frac{n-6}{5} \rfloor \\
&= n + 1 + \lfloor \frac{n+4}{5} \rfloor \\
&\geq n + 1 + \lfloor \frac{n+3}{5} \rfloor.
\end{aligned}$$

j=6: As above,

$$\begin{aligned}
\Sigma_1^n &\geq \gamma_{5,5} + 3 + \gamma_{5,n-8} \\
&\geq 7 + 3 + (n-8) + 1 + \lfloor \frac{(n-8)+1}{5} \rfloor \\
&= n + 1 + 2 + \lfloor \frac{n-7}{5} \rfloor \\
&= n + 1 + \lfloor \frac{n+3}{5} \rfloor.
\end{aligned}$$

j=7: As above,

$$\begin{aligned}\Sigma_1^n &\geq \gamma_{5,6}+3+\gamma_{5,n-9}\\ &\geq 8+3+\gamma_{5,n-9}\end{aligned}$$

and by the remark preceding the case j=3, $n \geq 2\cdot 6+1$ so $n-9 \geq 4$ so

$$\begin{aligned}\gamma_{5,n-9} &\geq (n-9)+1+\lfloor\frac{(n-9)+2}{5}\rfloor\\ &= n-8+\lfloor\frac{n-7}{5}\rfloor.\end{aligned}$$

Hence

$$\begin{aligned}\Sigma_1^n &\geq n+1+2+\lfloor\frac{n-7}{5}\rfloor\\ &= n+1+\lfloor\tfrac{n+3}{5}\rfloor.\end{aligned}$$

j=8: In this case $n \geq 2\cdot 7+1$ so $n-10 \geq 5$. Hence

$$\gamma_{5,n-10} \geq (n-10)+1+\lfloor\frac{(n-10)+2}{5}\rfloor$$

and hence

$$\begin{aligned}\Sigma_1^n &\geq \gamma_{5,7}+4+\gamma_{5,n-10}\\ &\geq 9+4+n-9+\lfloor\frac{(n-8}{5}\rfloor\\ &= n+1+3+\lfloor\frac{n-8}{5}\rfloor\\ &= n+1+\lfloor\frac{n+7}{5}\rfloor\\ &\geq n+1+\lfloor\tfrac{n+3}{5}\rfloor.\end{aligned}$$

j=9: In this case $n \geq 2\cdot 8+1$ so $n-11 \geq 6$ so

$$\gamma_{5,n-11} \geq (n-11)+1+\lfloor\frac{(n-11)+2}{5}\rfloor.$$

So

$$\begin{aligned}
\Sigma_1^n &\geq \gamma_{5,8} + 3 + \gamma_{5,n-11} \\
&\geq 11 + 3 + n - 11 + 1 + \lfloor \frac{(n-9)}{5} \rfloor \\
&= n + 1 + 3 + \lfloor \frac{n-9}{5} \rfloor \\
&= n + 1 + \lfloor \frac{n+6}{5} \rfloor \\
&\geq n + 1 + \lfloor \tfrac{n+3}{5} \rfloor.
\end{aligned}$$

j=10: In this case $n \geq 2 \cdot 9 + 1$ so $n - 12 \geq 7$. Since $n > 9$ by induction $\gamma_{5,9} = 12$ has already been established. Hence

$$\begin{aligned}
\Sigma_1^n &\geq \gamma_{5,9} + 3 + \gamma_{5,n-12} \\
&\geq 12 + 3 + (n-12) + 1 + \lfloor \frac{(n-12)+2}{5} \rfloor \\
&= n + 1 + 3 + \lfloor \frac{n-10}{5} \rfloor \\
&= n + 1 + \lfloor \frac{n+5}{5} \rfloor \\
&\geq n + 1 + \lfloor \tfrac{n+3}{5} \rfloor.
\end{aligned}$$

j $\geq$ 11: In this case $n \geq 2(j-1)+1$ so $n - j - 2 \geq j - 3 \geq 8$. So both $j - 1 \geq 8$ and $n - j - 2 \geq 8$. So we can apply induction to $\gamma_{5,j-1}$ and $\gamma_{5,n-j-2}$ in (4.7) to obtain

$$\begin{aligned}
\Sigma_1^n &\geq (j-1) + 1 + \lfloor \frac{(j-1)+3}{5} \rfloor + 3 \\
&\quad + (n-j-2) + 1 + \lfloor \frac{(n-j-2)+3}{5} \rfloor \\
&= n + 1 + (1 + \lfloor \frac{j+2}{5} \rfloor + \lfloor \frac{(n-j+1}{5} \rfloor).
\end{aligned}$$

Now we use the easily verify fact that for real numbers x and y we have

$$1 + \lfloor x \rfloor + \lfloor y \rfloor \geq \lfloor x + y \rfloor. \tag{4.9}$$

to show that $\Sigma_1^n \geq n + 1 + \lfloor \frac{n+3}{5} \rfloor$. This finishes the proof of Theorem 4.1. ∎

4.2 The domination number $\gamma_{6,n}$

First we list the following statements Q1-Q7. The proofs are given below.

Q1 $\gamma_{6,n} \leq \psi(n)$.

Q2 $\gamma_{6,1} = 2, \ \ \gamma_{6,2} = 4, \ \ \gamma_{6,3} = 5,$

$\gamma_{6,4} = 7, \ \ \gamma_{6,5} = 8, \ \ \gamma_{6,6} = 10.$

Q3. $0 \leq s_i \leq 6$.

Q4.1 $\Sigma_1^j \geq \gamma_{6,j-1}$ and $\Sigma_j^n \geq \gamma_{6,n-j}$, $2 \leq j \leq n-1$.

Q4.2 $\Sigma_i^j \geq \gamma_{6,j-i-1}$, $1 \leq i < j \leq n$, $j - i \geq 2$.

Q5.1 $s_1 = 0 \Rightarrow s_2 = 6$.

Q5.2 $s_i = 0 \Rightarrow s_{i-1} + s_{i+1} \geq 6$.

Q6 If $s_i \geq 1$ for all i in each of the summation ranges below, then the following hold.

Q6.1 $s_1 + s_2 \geq 3$ and

if $s_1 = 1$, then $s_2 \geq 3$.

Q6.2 $\Sigma_i^{i+2} \geq 4$.

Q6.3 $\Sigma_1^4 \geq 6$ and if $\Sigma_1^3 \leq 4$, then $\Sigma_1^4 \geq 7$.

Q6.4 $\Sigma_1^6 \geq 9$.

Q6.5 $\Sigma_1^7 \geq 10$.

Q6.6 $\Sigma_1^8 \geq 12$.

Q6.7 $\Sigma_1^{10} \geq 15$.

Q6.8 $\Sigma_i^{i+13} \geq 20$.

Q7 If $(s_1, s_2, \ldots, s_n)$ is a dominating sequence for $P_6 \times P_n$ then so is the reverse sequence $(s_1', s_2', \ldots, s_n')$ where $s_i' = s_{n-i+1}$.

Q1 is immediate from the following lemma which was proved in section 3.3.

Lemma 4.4 *There is a dominating set S for $P_6 \times P_n$ with $\psi(n)$ elements.*

Proof of Q2. The values of $\gamma_{6,k} = \gamma_{k,6}$ for $k = 1, 2, 3, 4$, follow from the results of Jacobson and Kinch[3]. The cases $k = 5$ and 6 were established by computer by Hare[5]. They may be easily established by the methods of this paper, however we omit the proofs. ∎

Proof of Q3-Q6.2. Obvious. ∎

To establish Q6.3-Q6.8 we need some lemmas.

Lemma 4.5 *Let S be a dominating set for $P_6 \times P_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$ Suppose for some i*

$$(s_i, s_{i+1}, s_{i+2}, s_{i+3}) = (2, 1, 1, 2).$$

Then $2 \leq i$, $i + 3 \leq n - 1$ and up to isomorphism there are only the two possible configurations for S indicated in Figure 4.2.

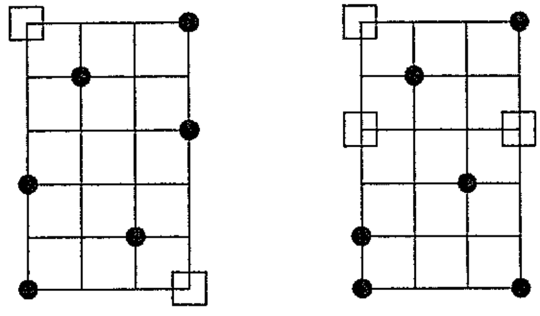


Figure 4.2: The configuration of Lemma 4.5

Proof. For a vertex v, let $N(v)$ demotes the set of vertices w such that $w = v$ or w is adjacent to v. By hypothesis we have

$$|S \cap C_i| = |S \cap C_{i+3}| = 2$$

and

$$S \cap C_{i+1} = \{v_{i+1}\},\ S \cap C_{i+2} = \{v_{i+2}\}.$$

The vertices in $S \cap C_i$ and $S \cap C_{i+3}$ cover exactly 4 of the 12 vertices in $C_{i+1} \cup C_{i+2}$. This leaves 8 vertices in these two columns to be covered by v_{i+1} and v_{i+2}. But each of these can cover at most 4 vertices in $C_{i+1} \cup C_{i+2}$. Hence each must cover exactly 4 with no overlap. Hence $N(v_{i+1}) \cap N(v_{i+2}) = \phi$ and neither v_{i+1} nor v_{i+2} can lie in the 1-st row or the 6-th row. This leaves up to isomorphism only the two possiblities shown in Figure 4.2 for v_{i+1} and v_{i+2}. In each case the location of stones of S in C_i and C_{i+3} are forced. This leaves the boxed vertices uncovered, so they must be covered by stones of S adjacent columns. ∎

Lemma 4.6 *Let S be a dominating set for $P_6 \times P_n$ with dominating sequence $(s_1, s_2, \ldots, s_n)$ Suppose for some i*

$$(s_i, s_{i+1}, s_{i+2}, s_{i+3}, s_{i+4}, s_{i+5}) = (1, 2, 1, 1, 2, 1).$$

Then $2 \leq i$, $i + 5 \leq n - 1$ and up to isomorphism there is only the one possible configuration for S shown in Figure 4.3.

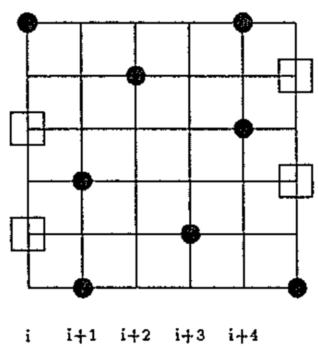


Figure 4.3: The configuration of Lemma 4.6

Proof. Immediate from lemma 4.4.

Corollary 4.1 *Let $(s_1, s_2, \ldots, s_n)$ be a dominating sequence for $P_6 \times P_n$ with $s_i \geq 1$, for all i. Then*

1. $(s_1, s_2, \ldots, s_9) = (2, 1, 2, 1, 2, 1, 1, 2, 1)$ *is not possible.*

2. $(s_{n-8}, \ldots, s_{n-1}, s_n) = (1, 2, 1, 1, 2, 1, 2, 1, 2)$ *is not possible.*

3. $(s_i, \ldots, s_{i+8}) = (2, 1, 2, 1, 2, 1, 1, 2, 1)$ *where the two stones in column i are in either row 2 and row 4 or row 3 and row 5 is not possible.*

4. $(s_i, \ldots, s_{i+8}) = (1,2,1,1,2,1,2,1,2)$ *where the two stones in column* $i+8$ *are in either row 2 and row 4 or row 3 and row 5 is not possible.*

Proof. To prove 1 we note that by Lemma 4.6 the two stones in column 3 are forced (up to isomorphism) to lie in row 3 and row 5. Now it is easy to check that no combination of two stones in column 1 and one stone in column 2 will cover the first two columns.

For 3 we note again that by Lemma 4.6 up to isomorphism the two stones in column $i+2$ are fixed. It is easy to see that neither of the two possiblities stated for the two stones in column i are possible.

Statements 2 and 4 follow respectively from 1 and 3 by symmetry. ∎

Lemma 4.7 *Let* S *be a dominating set for* $P_6 \times P_n$ *with the dominating sequence* $(s_1, s_2, \ldots, s_n)$. *Suppose for some* i*:*

$$(s_i, s_{i+1}, \ldots, s_{i+6}) = (1,1,2,1,2,1,1).$$

Then $2 \leq i,\ i+6 \leq n-1$ *and up to isomorphism there are only the two possible configurations shown in Figure 4.4 below for column* i *through* $i+6$.

Proof. Let

$$\{y\} = S \cap C_i \ \{x\} = S \cap C_{i+1} \ \{z,w\} = S \cap C_{i+2} \ \{r\} = S \cap C_{i+3}$$

$$\{s,t\} = S \cap C_{i+4} \ \{u\} = S \cap C_{i+5} \text{ and } \{v\} = S \cap C_{i+6}$$

By symmetry we can assume x lies in row 1, row 2 or row 3. Since there are only three stones of S in the columns adjacent to C_{i+1}, x must cover three vertices in C_{i+1}

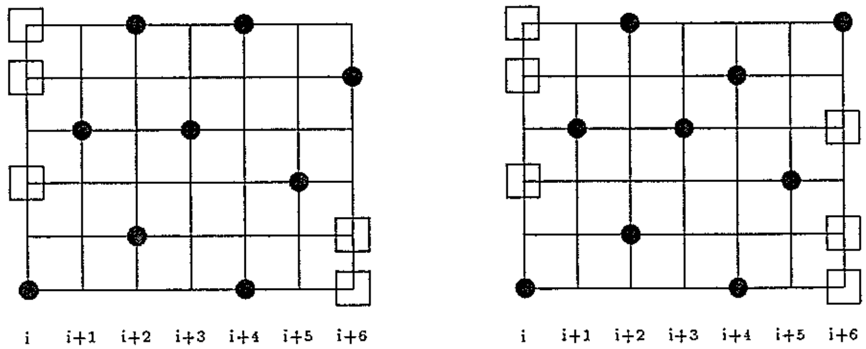


Figure 4.4: The configuration of Lemma 4.7

which is not possible if x lies in row 1. Suppose then that x lies in row 2. Then there remains the three vertices $(4, i+1)$, $(5, i+1)$ and $(6, i+1)$ that must be covered by stones of S in adjacent columns. Once y is selected, z and w are forced. So we have the three possibilities given in Figure 4.5.

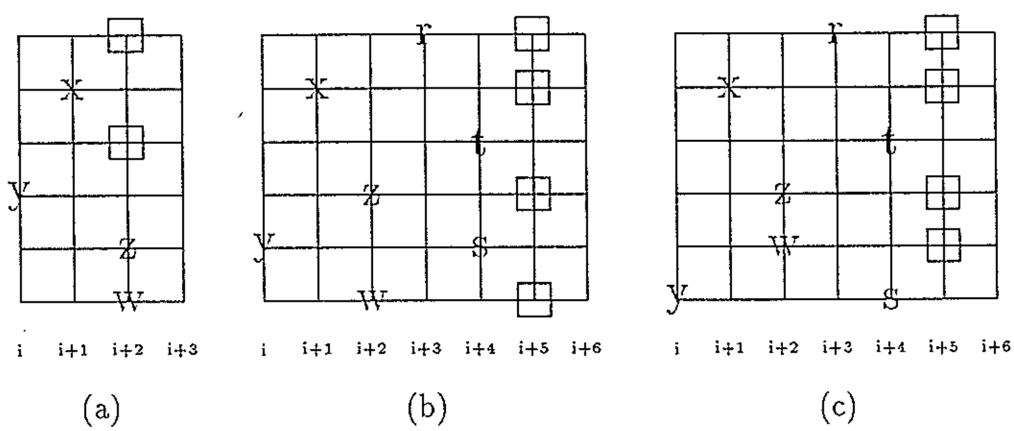


Figure 4.5: The cases of Lemma 4.7

In Figure 4.5(a) once x, y, z and w are placed there are the two boxed vertices remaining in C_{i+2} that must be covered by a single stone r in C_{i+3}, which is impossible. In cases 4.5(b) and 4.5(c) once x, y, z and w are selected, r, s and t are forced, leaving the four boxed vertices uncovered in C_{i+5} which cannot be covered by a single stone in C_{i+5} and a single stone in C_{i+6}. Hence x must lie in row 3. But again there are three uncovered vertices in C_{i+1}, namely $(1, i+1)$, $(5, i+1)$ and $(6, i+1)$. If we choose $y = (1, i)$ so as to cover $(1, i+1)$, then z and w are fixed as in Figure 4.6(a). This leaves the two boxed vertices uncovered. Since they cannot be covered by a single stone in C_{i+3} this case is not possible. This leaves us with the the two cases in Figure 4.6(b) and 4.6(c).

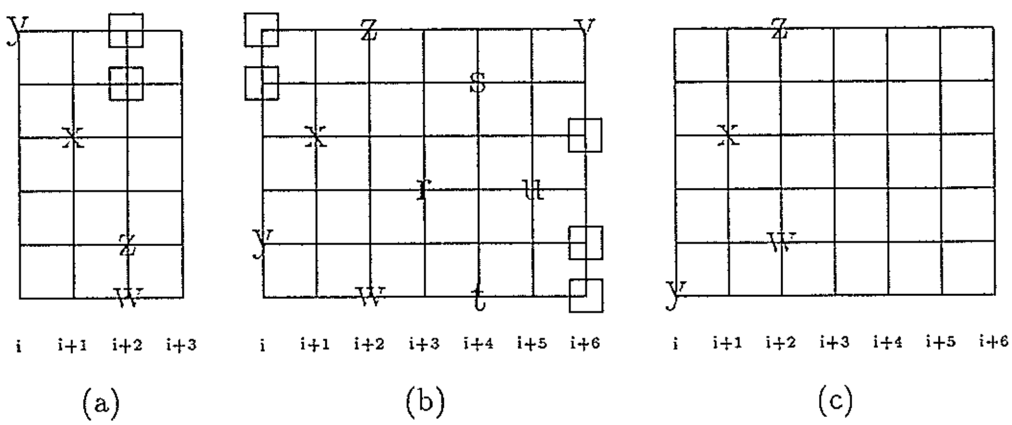


Figure 4.6: The cases of Lemma 4.7

In Figure 4.6(b), one of the stones in column $i + 4$, say s, has to be fixed in row 2. By symmetry to column $i + 1$, u must be either in row 3 or in row 4. If u is in row 3, then there are three vertices $(1, i+5)$, $(5, i+5)$ and $(6, i+5)$ left uncovered. It is not possible to cover these vertices by stone t and stone v. If u is in row 4, then t and v are forced. This is isomorphic to the left block in Figure 4.4.

In Figure 4.6(c), there are the only two possible configurations for the three right most columns shown in Figure 4.7(a) and 4.7(b). Indeed, up to symmetry all other cases have been discussed.

In Figure 4.7(a), the four boxed vertices in column $i + 3$ can not be covered by a single stone r.

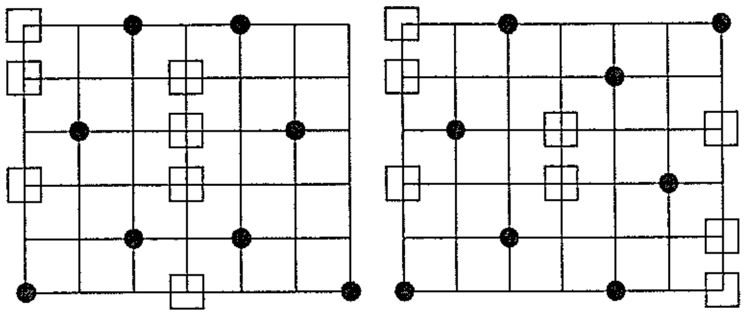


Figure 4.7: The cases of Lemma 4.7

In Figure 4.7(b), either one of the boxed vertices in column $i+3$ can be the stone r and this configuration is equal to or a symmetric image of the right of Figure 4.4. ∎

Corollary 4.2 *A dominating sequence* $(s_1, s_2, \ldots, s_n)$ *for* $P_6 \times P_n$ *with*

$$(s_i, s_{i+1}, \ldots, s_{i+13}) = (1,1,2,1,2,1,1,2,2,1,1,2,1,1)$$

is not possible.

Proof. If such a dominating sequence is possible, by Lemma 4.7 we may assume that column C_{i+5} and C_{i+6} are configured as the right most two columns of the left block in Figure 4.4. These two columns together with the specified dominating sequence forces the placement of the elements of S up to column $(i+12)$ as in Figure 4.8. However, this leaves uncovered the two boxed vertices in C_{i+12}. Both of these cannot be covered by a single vertex in C_{i+13}. ∎

Corollary 4.3 *A dominating sequence* $(s_1, s_2, \ldots, s_n)$ *for* $P_6 \times P_n$ *with*

$$(s_i, s_{i+1}, \ldots, s_{i+8}) = (1,2,1,1,2,1,2,1,1)$$

is not possible.

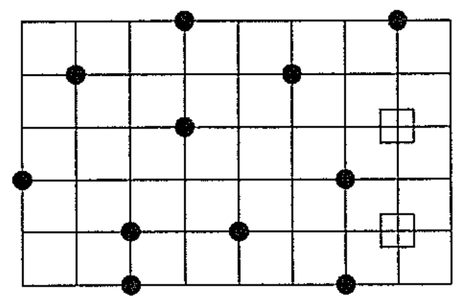


Figure 4.8: Configuration of Corollary 4.2

Proof. By Lemma 4.7 we may assume that since $s_{i+1} = 2$ that the configuration for columns $i+2$ to $i+8$ are fixed as in the left-right mirror image of Figure 4.4. This forces the configuration given in Figure 4.9. This leaves 2 uncovered vertices in C_{i+1} to be covered by a single vertex in C_i, which is not possible. ∎

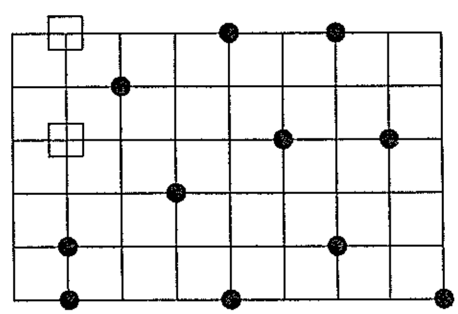


Figure 4.9: Configuration of Corollary 4.3

Corollary 4.4 *Let $(s_1, s_2, \ldots, s_n)$ be a dominating sequence for $P_6 \times P_n$ satisfying $s_j \geq 1$ for all j. Suppose*

$$(s_i, s_{i+1}, \ldots, s_{i+6}) = (1, 1, 2, 1, 2, 1, 1)$$

Then if $i \geq 3$, then $s_{i-2} + s_{i-1} \geq 4$ and if $i + 8 \leq n$, then $s_{i+7} + s_{i+8} \geq 4$.

Proof. By Lemma 4.7 we have $s_{i-1} \geq 3$ and so $s_{i-2} + s_{i-1} \geq 1 + 3 = 4$, or else $s_{i-1} = 2$. But by Corollary 4.3 $s_{i-2} > 1$ so $s_{i-2} + s_{i-1} \geq 4$. The rest follows by symmetry. ∎

Lemma 4.8 *Let $(s_1, s_2, \ldots, s_n)$ be a dominating sequence for $P_6 \times P_n$ and suppose $s_t \geq 1$ for $i \leq t \leq i+6$. Then*

1. $\Sigma_i^{i+6} \geq 9$ and

2. If $\Sigma_i^{i+6} = 9$ then

$$(s_i, s_{i+1}, \ldots, s_{i+6}) = (1, 1, 2, 1, 2, 1, 1).$$

Proof. Write $\Sigma_i^{i+6} = \Sigma_i^{i+2} + \Sigma_{i+3}^{i+5} + s_{i+6}$. By Q6.1 we cannot have $s_i = s_{i+1} = s_{i+2} = 1$ so we must have $\Sigma_i^{i+2} \geq 4$. Similarly $\Sigma_{i+3}^{i+5} \geq 4$ and (1) follows. If $\Sigma_i^{i+6} = 9$ then clearly $\Sigma_i^{i+2} = \Sigma_{i+3}^{i+5} = 4$ so $s_{i+6} = 1$. By symmetry $s_i = 1$ so $\{s_{i+1}, s_{i+2}\} = \{1, 2\}$ and again by symmetry $\{s_{i+4}, s_{i+5}\} = \{1, 2\}$. This leaves four possibilities for $(s_i, s_{i+1}, \ldots, s_{i+6})$:

a) (1,2,1,1,1,2,1)

b) (1,2,1,1,2,1,1)

c) (1,1,2,1,1,2,1)

d) (1,1,2,1,2,1,1)

a) is impossible by Q6.2 and b) and c)are ruled out by Lemma 4.6. This leaves only d) as desired. ∎

Proof of Q6.3. If $\Sigma_1^4 \leq 5$, then $\Sigma_1^3 \leq 4$. Then by Q6.2, $\Sigma_1^3 = 4$. But if $\Sigma_1^3 = 4$ by Q6.1, $s_1 = 2$, $s_2 = s_3 = 1$. By Q6.2 also $s_4 \geq 2$. But $s_4 = 2$ is not possible by Lemma 4.5. So $s_4 \geq 3$ and therefore $\Sigma_1^4 \geq 7$. ∎

Proof of Q6.4 and Q6.5.

$$\Sigma_1^6 = \Sigma_1^3 + \Sigma_4^6.$$

Hence if $\Sigma_1^3 \geq 5$ we clearly have $\Sigma_1^6 \geq 9$ by Q6.2. If $\Sigma_1^3 \leq 4$, then by Q6.2, $\Sigma_1^3 = 4$ and by Q6.3 $\Sigma_1^4 \geq 7$. Hence

$$\Sigma_1^6 \geq 7 + s_5 + s_6 \geq 9.$$

Now

$$\Sigma_1^7 = \Sigma_1^6 + s_6 \geq 9 + 1 = 10. \blacksquare$$

Proof of Q6.6. If $s_1 = 1$ then $s_2 \geq 3$. So

$$\Sigma_1^8 = s_1 + s_2 + \Sigma_3^5 + \Sigma_6^8 \geq 4 + 4 + 4 \geq 12$$

by Q6.2. Similarly, the results holds if $s_1 \geq 3$. If $s_1 = 2$ and $s_2 = 1$, then $\Sigma_1^8 < 12$ implies $\Sigma_3^5 = \Sigma_6^8 = 4$. By Lemma 4.8 this leaves only the possibility

$$(s_1, s_2, \ldots, s_8) = (2, 1, 1, 2, 1, 2, 1, 1)$$

which is impossible by Lemma 4.7. ∎

Proof of Q6.7.

$$\Sigma_1^{10} = \Sigma_1^8 + s_9 + s_{10} \geq 12 + 1 + 1 \geq 14.$$

If $\Sigma_1^4 \geq 7$, then

$$\Sigma_1^{10} = \Sigma_1^4 + \Sigma_5^7 + \Sigma_8^{10} \geq 7 + 4 + 4 = 15.$$

Suppose $\Sigma_1^4 \leq 6$, then by Q6.3, $\Sigma_1^4 = 6$ and $\Sigma_1^3 = 5$. This gives $\Sigma_1^3 = 5$, $s_4 = 1$, $\Sigma_5^7 = \Sigma_8^{10} = 4$. By Lemma 4.8

$$(s_4, s_5, \ldots, s_{10}) = (1, 1, 2, 1, 2, 1, 1).$$

But then by Q6.1 and Q6.3 we have $s_2 = 1$ and $s_3 = 2$ so

$$(s_2, s_3, \ldots, s_{10}) = (1, 2, 1, 1, 2, 1, 2, 1, 1)$$

which is not possible by corollary 4.3. ∎

Proof of Q6.8.

$$\Sigma_i^{i+13} = \Sigma_i^{i+6} + \Sigma_{i+7}^{i+13}.$$

If $\Sigma_i^{i+6} \geq 10$ and $\Sigma_{i+7}^{i+13} \geq 10$ we are done. From Q6.2 each of Σ_i^{i+6} and Σ_{i+7}^{i+13} is at least 9. We show that if one is 9 then the other is at least 11. Suppose $\Sigma_i^{i+6} = 9$. By Lemma 4.8,

$$(s_i, s_{i+1}, \ldots, s_{i+6}) = (1, 1, 2, 1, 2, 1, 1)$$

By Lemma 4.7 we know $s_{i+7} = 2$ or $s_{i+7} = 3$. If $s_{i+7} \geq 3$ then

$$\Sigma_{i+7}^{i+13} = s_{i+7} + \Sigma_{i+8}^{i+10} + \Sigma_{i+11}^{i+13} \geq 3 + 4 + 4 = 11.$$

So we can assume $s_{i+7} = 2$ and $\Sigma_{i+8}^{i+10} = \Sigma_{i+11}^{i+13} = 4$. Now $s_{i+8} = 1$ leads to a subsequence

$$(s_i, \ldots, s_{i+8}) = (1, 1, 2, 1, 2, 1, 1, 2, 1)$$

which is ruled out by Corollary 4.3 (via symmetry). Hence $s_{i+8} = 2$ so since $\Sigma_{i+8}^{i+10} = 4$ we have $s_{i+9} = s_{i+10} = 1$ which gives

$$(s_i, \ldots, s_{i+10}) = (1,1,2,1,2,1,1,2,2,1,1)$$

Now Q6.2 rules out $s_{i+11} = 1$ so $s_{i+11} = 2$, $s_{i+12} = s_{i+13} = 1$. This gives

$$(s_i, \ldots, s_{i+13}) = (1,1,2,1,2,1,1,2,2,1,1,2,1,1)$$

which is not possible by corollary 4.2. Hence in any case $\Sigma_{i+7}^{i+13} \geq 11$. By symmetry if $\Sigma_{i+7}^{i+13} = 9$, then $\Sigma_i^{i+6} \geq 11$. ∎

Lemma 4.9 *If* $(s_1, s_2, \ldots, s_n)$ *is a dominating sequence for* $P_6 \times P_n$ *with* $s_i \geq 1$ *for all* i*, then*

$$\Sigma_1^n \geq \psi(n). \tag{4.10}$$

Proof. From Q2 the lemma is immediate for $n \leq 6$. So we may assume $n \geq 7$. Write $n = 7q + r$, $1 \leq r \leq 7$, then $\psi(n) = 10q + b_r$. We prove (4.10) by considering the 7 cases $r = 1, 2, \ldots, 7$, each divided into two subcases: q even and q odd.

In examining these cases we use $Q6.1 - Q6.8$ repeatedly without comment. In particular, by Q6.8, if the number $t - s + 1$ of terms in Σ_s^t is divisible by 14 we have

$$\Sigma_s^t \geq 20\left(\frac{t-s+1}{14}\right).$$

Observe that we leave the case $r = 3$ to last since it requires a different approach. Note that since $n \geq 7$, $q \geq 1$ except when $n = 7$.

n=7q+1, q odd:

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^8 + \Sigma_9^n \\ &\geq 12 + 20(\frac{n-8}{14}) = 10q + 2.\end{aligned}$$

n=7q+1, q even:

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^8 + \Sigma_9^{n-7} + \Sigma_{n-6}^n \\ &\geq 12 + 20(\frac{n-15}{14}) + 10 = 10q + 2.\end{aligned}$$

n=7q+2, q odd: If $\Sigma_1^3 \geq 5$, then

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^3 + \Sigma_4^{n-6} + \Sigma_{n-5}^n \\ &\geq 5 + 20(\frac{n-9}{14}) + 9 = 10q + 4.\end{aligned}$$

By symmetry the result also holds if $\Sigma_{n-2}^n \geq 5$. So we can assume $\Sigma_1^3 \leq 4$ and so $\Sigma_1^4 \geq 7$ by Q6.3. Similarly $\Sigma_{n-3}^n \geq 7$. Hence

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^4 + \Sigma_5^{n-5} + s_{n-4} + \Sigma_{n-3}^n \\ &\geq 7 + 20(\frac{n-9}{14}) + 1 + 7 \\ &\geq 14 + 20(\frac{n-9}{14}) = 10q + 4.\end{aligned}$$

n=7q+2, q even:

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^6 + \Sigma_7^{n-10} + \Sigma_{n-9}^n \\ &\geq 9 + 20(\frac{n-16}{14}) + 15 = 10q + 4.\end{aligned}$$

n=7q+4, q odd: If $\Sigma_1^3 \geq 5$, then

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^3 + \Sigma_4^{n-8} + \Sigma_{n-7}^n \\ &\geq 5 + 20(\frac{n-11}{14}) + 12 = 10q + 7.\end{aligned}$$

If $\Sigma_1^3 \leq 4$, then $\Sigma_1^4 \geq 7$ and

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^4 + \Sigma_5^{n-7} + \Sigma_{n-6}^n \\ &\geq 7 + 20(\frac{n-11}{14}) + 10 = 10q + 7.\end{aligned}$$

n=7q+4, q even:

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^{10} + \Sigma_{11}^{n-8} + \Sigma_{n-7}^n \\ &\geq 15 + 20(\frac{n-18}{14}) + 12 = 10q + 7.\end{aligned}$$

n=7q+5, q odd:

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^6 + \Sigma_7^{n-6} + \Sigma_{n-5}^n \\ &\geq 9 + 20(\frac{n-12}{14}) + 9 = 10q + 8.\end{aligned}$$

n=7q+5, q even:

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^{10} + \Sigma_{11}^{n-9} + s_{n-8} + \Sigma_{n-7}^n \\ &\geq 15 + 20(\frac{n-19}{14}) + 1 + 12 \\ &= 10q + 8.\end{aligned}$$

n=7q+6, q odd: If $\Sigma_1^3 \geq 5$ then

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^3 + \Sigma_4^{n-10} + \Sigma_{n-9}^n \\ &\geq 5 + 20(\frac{n-13}{14}) + 15 = 10q + 10.\end{aligned}$$

Suppose $\Sigma_1^3 \leq 4$, then $\Sigma_1^4 \geq 7$ so

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^4 + \Sigma_5^7 + \Sigma_8^{n-6} + \Sigma_{n-5}^n \\ &\geq 7 + 4 + 20(\frac{n-13}{14}) + 9 = 10q + 10.\end{aligned}$$

n=7q+6, q even:

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^{10} + \Sigma_{11}^{n-10} + \Sigma_{n-9}^n \\ &\geq 15 + 20(\frac{n-20}{14}) + 15 = 10q + 10.\end{aligned}$$

n=7q+7, q odd:

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^4 + \Sigma_5^{n-10} + \Sigma_{n-9}^n \\ &\geq 6 + 20(\frac{n-14}{14}) + 15 = 10q + 11.\end{aligned}$$

n=7q+7, q even: If $\Sigma_1^3 \geq 5$, then

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^3 + \Sigma_4^{n-4} + \Sigma_{n-3}^n \\ &\geq 5 + 20(\frac{n-7}{14}) + 6 = 10q + 11.\end{aligned}$$

If $\Sigma_1^3 \leq 4$, then $\Sigma_1^4 \geq 7$ so

$$\begin{aligned}\Sigma_1^n &= \Sigma_1^4 + \Sigma_5^{n-3} + \Sigma_{n-2}^n \\ &\geq 7 + 20(\frac{n-7}{14}) + 4 = 10q + 11.\end{aligned}$$

Now we consider the remaining case $n = 7q + 3$, $n \geq 10$. Since the proof of this case itself is long and complicated we will combine several lemmas and introduce some notations.

We define for sequences $A = (a_1, a_2, \ldots, a_m)$ and $B = (b_1, b_2, \ldots, b_n)$, the product

$$AB = (a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n).$$

This product is clearly an associative operation and we allow the empty sequence of length zero to act as an identity. We will write for $S = (s_1, s_2, \ldots, s_n)$:

$$\Sigma S = \sum_{i=1}^{n} s_i \text{ and } |S| = n.$$

Lemma 4.10 *Let $S = (s_1, s_2, \ldots, s_n)$, $n = 7q+3 \geq 10$, be a dominating sequence of $P_6 \times P_n$ with $s_i \geq 1$ for all i. And suppose*

$$\Sigma S < \psi(|S|) = 10q + 6.$$

Then $S = (2,1,2,1)T(1,2,1,2)$ for some sequence T.

Proof First we will show $\Sigma_1^4 \leq 6$ and $\Sigma_{n-3}^n \leq 6$. Suppose $\Sigma_1^4 \geq 7$. First let q be odd. By symmetry from Q6.4, $\Sigma_{n-5}^n \geq 9$. Then using Q6.8 as above we have

$$\begin{aligned} \Sigma_1^n &= \Sigma_1^4 + \Sigma_5^{n-6} + \Sigma_{n-5}^n \\ &\geq 7 + 20\left(\frac{n-10}{14}\right) + 9 \\ &= 7 + 10q - 10 + 9 = 10q + 6. \end{aligned}$$

Next suppose q is even. Then by Q6.2, Q6.7 and Q6.8 we have $\Sigma_{n-12}^{n-10} \geq 4$, and $\Sigma_{n-9}^n \geq 15$. Hence

$$\begin{aligned} \Sigma_1^n &= \Sigma_1^4 + \Sigma_5^{n-13} + \Sigma_{n-12}^{n-10} + \Sigma_{n-9}^n \\ &\geq 7 + 20\left(\frac{n-17}{14}\right) + 4 + 15 = 10q + 6. \end{aligned}$$

Therefore $\Sigma_1^4 \leq 6$. By Q6.3, $\Sigma_1^4 = 6$ and $\Sigma_1^3 \geq 5$. Hence $s_4 = 1$ and $\Sigma_1^3 = 5$. By symmetry $s_{n-3} = 1$ and $\Sigma_{n-2}^n = 5$.

Now we will show $\Sigma_1^2 \leq 3$ and $\Sigma_{n-1}^n \leq 3$. Suppose $\Sigma_1^2 \geq 4$. Suppose q is odd. By Q6.6, $\Sigma_{n-7}^n \geq 12$. Hence

$$\begin{aligned} \Sigma_1^n &= \Sigma_1^2 + \Sigma_3^{n-8} + \Sigma_{n-7}^n \\ &\geq 4 + 20\left(\frac{7q+3-10}{14}\right) + 12 = 10q + 6. \end{aligned}$$

Now we let q be even. We sum the s_i's as

$$\Sigma_1^n = \Sigma_1^2 + \Sigma_3^{n-15} + \Sigma_{n-14}^{n-8} + \Sigma_{n-7}^n.$$

By Lemma 4.8, $\Sigma_{n-14}^{n-8} \geq 9$. In the case of $\Sigma_{n-14}^{n-8} \geq 10$, we have

$$\Sigma_1^n \geq 4 + 20\left(\frac{n-17}{14}\right) + 10 + 12 = 10q + 6.$$

And in the case of $\Sigma_{n-14}^{n-8} = 9$, by Lemma 4.8 and Corollary 4.4, $\Sigma_{n-7}^{n-6} \geq 4$. Hence

$$\begin{aligned} \Sigma_1^n &= \Sigma_1^2 + \Sigma_3^{n-15} + \Sigma_{n-14}^{n-8} + \Sigma_{n-7}^{n-6} + \Sigma_{n-5}^n \\ &\geq 4 + 20\left(\frac{n-17}{14}\right) + 9 + 4 + 9 = 10q + 6. \end{aligned}$$

Therefore $\Sigma_1^2 \leq 3$. By Q6.1, $s_1 = 2$ and $s_2 = 1$. Symmetrically $s_{n-1} = 1$ and $s_n = 2$. Thus we have

$$S = (2,1,2,1)T(1,2,1,2). \blacksquare$$

Definition 4.2 *For given dominating sequence S as in Lemma 4.10, we let $\mathcal{B}$ be the set of all sequences B such that $S = ABC$ for some (possibly empty) sequences A and C satisfying the following:*

1. $|A| \equiv 0 \pmod 7$ *and* $|C| \equiv 0 \pmod 7$.

2. $|B| \geq 10$.

3. $B = (2,1,2,1)T(1,2,1,2)$ *for some* T.

4. $|A| = 0$ *or* $A = A'(2,1,2,1,1,2,1)$ *for some* A'.

5. $|C| = 0$ *or* $C = (1,2,1,1,2,1,2)C'$ *for some* C'.

6. $\Sigma B < \psi(|B|) = 10m + 6$ *where* $|B| = 7m + 3$.

By Lemma 4.10, the sequence S itself is in $\mathcal{B}$ so $\mathcal{B}$ is not empty. Now in what follows we let

$$B = (s_j, s_{j+1}, \ldots, s_l)$$

be a sequence in $\mathcal{B}$ chosen so that $|B|$ is minimal. We will show that this assumption implies there is a sequence in $\mathcal{B}$ shorter than B. This contradiction will establish Lemma 4.9 for the remaining case $n \equiv 3 \pmod 7$, $n \geq 10$. We break up the rest of this proof into three lemmas.

Lemma 4.11 *For any* $B = (s_j, s_{j+1}, \ldots, s_l) \in \mathcal{B}$, *we have*

R1: $\Sigma_j^{j+5} \geq 9$ *and* $\Sigma_{l-5}^{l} \geq 9$.

R2: $\Sigma_j^{j+7} \geq 12$ *and* $\Sigma_{l-7}^{l} \geq 12$.

R3: $\Sigma_j^{j+12} \geq 19$ *and* $\Sigma_{l-12}^{l} \geq 19$.

Proof By symmetry we need only to prove the first part of each property.

By Q6.2, $\Sigma_{j+3}^{j+5} \geq 4$ and by assumption $\Sigma_j^{j+2} = 5$ so $\Sigma_j^{j+5} \geq 9$. R1 follows.

If $\Sigma_j^{j+5} \geq 10$ then $\Sigma_j^{j+7} \geq 10 + s_{i+6} + s_{i+7} = 12$. If $\Sigma_j^{j+5} \leq 9$ then

$$(s_j, \ldots, s_{j+5}) = (2,1,2,1)(1,2) \text{ or } (2,1,2,1)(2,1).$$

In the second case, $s_{j+5} = 1$ forces $\Sigma_{j+6}^{j+3} \geq 3$ by Q6.2. Hence $\Sigma_j^{j+7} \geq 9 + 3 = 12$. In the first case, if $s_{j+6} = 1$, then

$$(s_{j+1}, s_{j+2}, \ldots, s_{j+6}) = (1,2,1,1,2,1).$$

This forces $s_{j+7} = 2$ by Lemma 4.6. We still have $\Sigma_j^{j+7} \geq 9+3 = 12$. R2 is completed.

If $\Sigma_{j+6}^{j+12} \geq 10$, then

$$\Sigma_j^{j+12} = \Sigma_j^{j+5} + \Sigma_{j+6}^{j+12} \geq 9 + 10 = 19.$$

If $\Sigma_{j+6}^{j+12} = 9$, then by Lemma 4.8 and Corollary 4.4, $\Sigma_{j+4}^{j+5} \geq 4$, hence

$$\Sigma_j^{j+12} = \Sigma_j^{j+3} + \Sigma_{j+4}^{j+5} + \Sigma_{j+6}^{j+12} \geq 6 + 4 + 9 = 19.$$

This proves R3. ∎

Lemma 4.12 *There does not exist a $B \in \mathcal{B}$ with $|B| = 10$.*

Proof If $|B| = 10$ and $\Sigma B < 16$ then either

$$B = (2,1,2,1)(1,2)(1,2,1,2) \text{ or } (2,1,2,1)(2,1)(1,2,1,2).$$

Note that these two possibilities for B are left-right symmetric so it suffices to show that $B = (2,1,2,1,2,1,1,2,1,2)$ is not possible. Suppose first that $S = ABC$ as in Definition 2. If $A = \phi$ then $S = BC$ which is impossible by Corollary 4.1 part 1. If $A \neq \phi$, then $A = A'(2,1,2,1,1,2,1)$ and the last 6 entries of A are 1, 2, 1, 1, 2, 1. By Lemma 4.6 the two stones in the first column of B must lie either in row 2 and row 4 or in row 3 and row 5. But this is not possible by Corollary 4.1 part 3. ∎

Lemma 4.13 *If $B \in \mathcal{B}$ then either*

$$B = (2,1,2,1,1,2,1)B' \textit{ for some } B' \in \mathcal{B}$$

or

$$B = B''(1,2,1,1,2,1,2) \textit{ for some } B'' \in \mathcal{B}.$$

Proof As above we let $|B| = 7m + 3$. By Lemma 4.12, $m \geq 2$. Now

$$B = (2,1,2,1)(s_{j+4}, \ldots, s_{l-4})(1,2,1,2).$$

We claim first that $\Sigma_j^{j+6} < 11$; suppose not. Then $\Sigma_j^{j+6} \geq 11$. We show this is impossible: If m is odd then we have

$$\begin{aligned} \Sigma_j^l &= \Sigma_j^{j+6} + \Sigma_{j+7}^{l-3} + \Sigma_{l-2}^l \\ &\geq 11 + 20\left(\frac{7m+3-10}{14}\right) + 5 = 10m + 6, \end{aligned}$$

which contradicts $B \in \mathcal{B}$. If m is even, by Lemma 4.8 and Corollary 4.4, $s_{l-2}+s_{l-1} = 3$ forces $\Sigma_{l-9}^{l-3} \geq 10$. Hence

$$\begin{aligned} \Sigma_j^l &= \Sigma_j^{j+6} + \Sigma_{j+7}^{l-10} + \Sigma_{l-9}^{l-3} + \Sigma_{l-2}^l \\ &\geq 11 + 20\left(\frac{7m+3-17}{14}\right) + 10 + 5 = 10m + 6, \end{aligned}$$

which is again a contradiction. Hence $\Sigma_j^{j+6} \leq 10$. By Q6.2, $\Sigma_{j+4}^{j+6} \geq 4$ and hence $\Sigma_{j+4}^{j+6} = 4$ and $\Sigma_j^{j+6} = 10$. By symmetry $\Sigma_{l-6}^{l-4} = 4$ and $\Sigma_{l-6}^{l} = 10$. It is also clear that $s_{j+4} \leq 2$ and $s_{l-4} \leq 2$.

Now we can not have both $s_{j+4} = 2$ and $s_{l-4} = 2$. Suppose that $s_{j+4} = s_{l-4} = 2$. Let m be odd. Then

$$\begin{aligned} \Sigma_j^l &= \Sigma_j^{j+4} + \Sigma_{j+5}^{l-5} + \Sigma_{l-4}^l \\ &\geq 6 + 2 + 20\left(\frac{7m+3-10}{14}\right) + 2 + 6 = 10m + 6, \end{aligned}$$

a contradiction. If m is even, then $(s_{j+3}, s_{j+4}) = (1, 2)$ forces $\Sigma_{j+5}^{j+11} \geq 10$ by Lemma 4.8 and Corollary 4.4. Hence

$$\begin{aligned} \Sigma_j^l &= \Sigma_j^{j+4} + \Sigma_{j+5}^{j+11} + \Sigma_{j+12}^{l-5} + \Sigma_{l-4}^l \\ &\geq 8 + 10 + 20\left(\frac{7m+3-17}{14}\right) + 8 = 10m + 6, \end{aligned}$$

a contradiction. This shows $s_{j+4} = 1$ or $s_{l-4} = 1$.

Suppose $s_{j+4} = 1$. Then $s_{j+5} = 2$ and $s_{j+6} = 1$. Therefore

$$B = (2, 1, 2, 1, 1, 2, 1)B,'$$

where $B' = (s_{j+7}, \ldots, s_{l-4})(1, 2, 1, 2)$.

To prove $B' \in \mathcal{B}$, we need only to show $\Sigma B < \psi(|B|)$ and B' has the form $(2, 1, 2, 1)T'(1, 2, 1, 2)$. Clearly

$$\Sigma B = 10 + \Sigma B'$$

and $|B'| = |B| - 7 = 7(m-1) + 3 \geq 10$. Therefore

$$\Sigma B' = \Sigma B - 10 < 10m + 6 - 10 = 10(m-1) + 6 = \psi(|B'|).$$

We claim that $\Sigma_{j+7}^{j+10} \leq 6$ and $\Sigma_{j+7}^{j+8} \leq 3$. Suppose $\Sigma_{j+7}^{j+10} \geq 7$. Suppose $(m-1)$ is odd. By R1, $\Sigma_{l-5}^{l} \geq 9$. Hence

$$\begin{aligned}
\Sigma_{j+7}^{l} &= \Sigma_{j+7}^{j+10} + \Sigma_{j+11}^{l-6} + \Sigma_{l-5}^{l} \\
&\geq 7 + 20\left(\frac{7m - 7 + 3 - 10}{14}\right) + 9 \\
&= 10(m-1) + 6 = \psi(|B'|),
\end{aligned}$$

a contradiction. Suppose $(m-1)$ is even. By R3 $\Sigma_{l-12}^{l} \geq 19$. Hence

$$\begin{aligned}
\Sigma_{j+7}^{l} &= \Sigma_{j+7}^{j+10} + \Sigma_{j+11}^{l-13} + \Sigma_{l-12}^{l} \\
&\geq 7 + 20\left(\frac{7m - 7 + 3 - 17}{14}\right) + 19 \\
&= 10(m-1) + 6 = \psi(|B'|),
\end{aligned}$$

a contradiction. Hence $\Sigma_{j+7}^{j+10} \leq 6$.

Suppose $\Sigma_{j+7}^{j+8} \geq 4$. Suppose $(m-1)$ is odd. By R2 $\Sigma_{l-7}^{l} \geq 12$ Hence

$$\begin{aligned}
\Sigma_{j+7}^{l} &= \Sigma_{j+7}^{j+8} + \Sigma_{j+9}^{l-8} + \Sigma_{l-7}^{l} \\
&\geq 4 + 20\left(\frac{7m - 7 + 3 - 10}{14}\right) + 12 = \psi(|B'|),
\end{aligned}$$

a contradiction. Suppose $(m-1)$ is even. We sum the entries of B' as

$$\Sigma_{j+7}^{l} = \Sigma_{j+7}^{j+8} + \Sigma_{j+9}^{l-15} + \Sigma_{l-14}^{l-8} + s_{l-7} + s_{l-6} + \Sigma_{l-5}^{l}.$$

In the case of $\Sigma_{l-14}^{l-8} \geq 10$, then by R2, $\Sigma_{l-7}^{l} \geq 12$. Hence

$$\Sigma_{j+7}^{l} \geq 4 + 20\left(\frac{7m - 7 + 3 - 17}{14}\right) + 10 + 12 = \psi(|B'|).$$

In the case of $\Sigma_{l-14}^{l-8} = 9$, then by Lemma 4.8 and Corollary 4.4, $s_{l-7} + s_{l-6} \geq 4$ and by R1, $\Sigma_{l-5}^{l} \geq 9$. Hence

$$\Sigma_{j+7}^{l} \geq 4 + 20\left(\frac{7m - 7 + 3 - 17}{14}\right) + 9 + 4 + 9 = \psi(|B'|).$$

We now have shown $\Sigma_{j+7}^{j+10} \leq 6$ and $\Sigma_{j+7}^{j+8} \leq 3$.

Now $(s_{j+1}, \ldots, s_{j+6}) = (1, 2, 1, 1, 2, 1)$ forces $s_{j+7} \geq 2$ by Lemma 4.6. Therefore $(s_{j+7}, s_{j+8}) = (2, 1)$. There are only two choices for (s_{j+9}, s_{j+10}), either $(1, 2)$ or $(2, 1)$. If $(s_{j+9}, s_{j+10}) = (1, 2)$ then

$$B = (2)(1, 2)(1, 1, 2, 1, 2, 1, 1, 2)(s_{j+11}, \ldots, s_l).$$

This is impossible by Lemma 4.8, since

$$(s_{j+3}, \ldots, s_{j+9}) = (1, 1, 2, 1, 2, 1, 1)$$

must force $s_{j+1} + s_{j+2} \geq 4$. Thus

$$(s_{j+7}, s_{j+8}, s_{j+9}, s_{j+10}) = (2, 1, 2, 1).$$

The proof of $B' \in \mathcal{B}$ is completed.

By symmetry if $s_{l-4} = 1$, we can prove that

$$B = B''(1, 2, 1, 1, 2, 1, 2)$$

and $B'' \in \mathcal{B}$. ∎

Now combining Lemmas 4.10 through 4.13, we can complete the proof of Lemma 4.9 for the case of $n = 7q + 3 \geq 10$. If such a dominating sequence S in Lemma 4.10 exists, then there must be a sequence $B \in \mathcal{B}$ with shortest length. By Lemma 4.12, $|B| > 10$. But by Lemma 4.13, from B, we can always get B' or B'' in $\mathcal{B}$ with the length shorter than B. This contradiction implies the number of stones in S cannot be less than $10q + 6$. Thus the proof of Lemma 4.9 is completed. ∎

Theorem 4.2

$$\gamma_{6,n} = \begin{cases} n + 1 + \lfloor \frac{3n+3}{7} \rfloor & \textit{if } n \equiv 1 \bmod 7 \textit{ or } n = 3 \\ n + 1 + \lfloor \frac{3n+5}{7} \rfloor & \textit{otherwise} \end{cases} \tag{4.11}$$

Proof By Q1 it suffices to prove that for any dominating sequence $(s_1, s_2, \ldots, s_m)$ for $P_6 \times P_m$ that

$$\Sigma_1^m \geq \psi(m). \tag{4.12}$$

By Q2, it holds for $m \leq 6$. So we will assume that it holds for $1 \leq m \leq n-1$, $n \geq 7$ and prove it still holds for $m = n$. Now if $s_i \geq 1$ for all i, (4.12) holds by Lemma 4.9. So we can assume $s_j = 0$ for some j. Let j be the least positive integer i such that $s_i = 0$.

If $j = 1$, then by Q5.1 and Q4.1

$$\Sigma_1^n = s_2 + \Sigma_3^n \geq 6 + \gamma_{6,n-3}.$$

By induction $\gamma_{6,n-3} \geq \psi(n-3)$ so

$$\Sigma_1^n \;\geq\; 6 + \psi(n-3) \;.$$

$$
\begin{aligned}
&\geq 6+n-3+1+\lfloor\frac{3(n-3)+3}{7}\rfloor \\
&= n+1+\lfloor\frac{3n+15}{7}\rfloor \\
&\geq n+1+\lfloor\frac{3n+5}{7}\rfloor \geq \psi(n).
\end{aligned}
$$

For $j=2$, then as above by induction and Q5.2

$$
\begin{aligned}
\Sigma_1^n &= s_1+s_3+\Sigma_4^n \geq 6+\psi(n-4) \\
&\geq 6+n-4+1+\lfloor\frac{3(n-4)+3}{7}\rfloor \\
&> n+1+\lfloor\frac{3n+5}{7}\rfloor \geq \psi(n).
\end{aligned}
$$

By symmetry, the theorem is true if $j=n-10$ or $j=n$. So we can assume $3 \leq j \leq n-2$. Now we consider the following two cases: $s_{j+1} \geq 4$ and $s_{j+1} \leq 3$.

Case I $s_{j+1} \geq 4$.

First assume $j=n-2$. Now $s_{j+1}=s_{n-1} \geq 4$. Since $s_n=0$ is not possible, we have $s_{n-1}+s_n \geq 5$. Applying the induction hypothesis we get

$$
\begin{aligned}
\Sigma_1^n &= \Sigma_1^{n-2}+s_{n-1}+s_n \\
&\geq \gamma_{6,n-3}+5 \\
&\geq n-3+1+\lfloor\frac{3(n-3)+3}{7}\rfloor+5 \\
&\geq n+1+\lfloor\frac{3n+8}{7}\rfloor \\
&\geq \psi(n).
\end{aligned}
$$

So we can assume $3 \leq j \leq n-3$. Now by the induction hypothesis

$$\gamma_{6,j-1}=j+\lfloor\frac{3j-3+a}{7}\rfloor,$$

and

$$\gamma_{6,n-j-2} = n - j - 1 + \lfloor \frac{3n - 3j - 6 + b}{7} \rfloor,$$

where a and $b = 3$ or 5 depending on whether $j-1$ and $n-j-2$ are $3, \equiv 1 \pmod 7$, or not.

Since both $j-1$ and $n-j-2$ are between 2 and $n-1$ by Q4.1 we have

$$\Sigma_1^j \geq \gamma_{6,j-1} \text{ and } \Sigma_{j+2}^n \geq \gamma_{6,n-j-2}.$$

Hence

$$\begin{aligned} \Sigma_1^n &\geq \gamma_{6,j-1} + 4 + \gamma_{6,n-j-2} \\ &= n + 3 + \lfloor \frac{3j - 3 + a}{7} \rfloor + \lfloor \frac{3n - 3j - 6 + b}{7} \rfloor \\ &\geq n + 2 + \lfloor \frac{3n + a + b - 9}{7} \rfloor, \end{aligned}$$

by (4.9). Hence

$$\Sigma_1^n \geq n + 1 + \lfloor \frac{3n + a + b - 2}{7} \rfloor. \tag{4.13}$$

Now if $a = 5$ or $b = 5$ then by (4.13) we have

$$\begin{aligned} \Sigma_1^n &\geq n + 1 + \lfloor \frac{3n + 6}{7} \rfloor \\ &\geq n + 1 + \lfloor \frac{3n + 5}{7} \rfloor \geq \psi(n). \end{aligned}$$

So we only need to be concerned about the cases where $a = 3$ and $b = 3$. These are the following three cases:

i. $j - 1 = n - j - 2 = 3$.

ii. One of $j-1$ and $n-j-2$ is 3 and the other is congruent to 1 modulo 7.

iii. $j-1 \equiv n-j-2 \equiv 1 \pmod 7$.

In case i $n = 9$. In case ii $n \equiv 0 \pmod 7$. And in case iii $n \equiv 5 \pmod 7$. In each case one can verify that

$$\lfloor \frac{3n+4}{7} \rfloor = \lfloor \frac{3n+5}{7} \rfloor,$$

and so it follows from (4.13) that

$$\begin{aligned} \Sigma_1^n &\geq n+1+\lfloor \frac{3n+4}{7} \rfloor \\ &= n+1+\lfloor \frac{3n+5}{7} \rfloor = \psi(n). \end{aligned}$$

This completes the proof of Case I.

Case II. $s_{j+1} \leq 3$.

In this case we show that we can always find another dominating set with the same number of stones but with $s_i \geq 1$ for $1 \leq i \leq j$. It is clear that once this is established we are done.

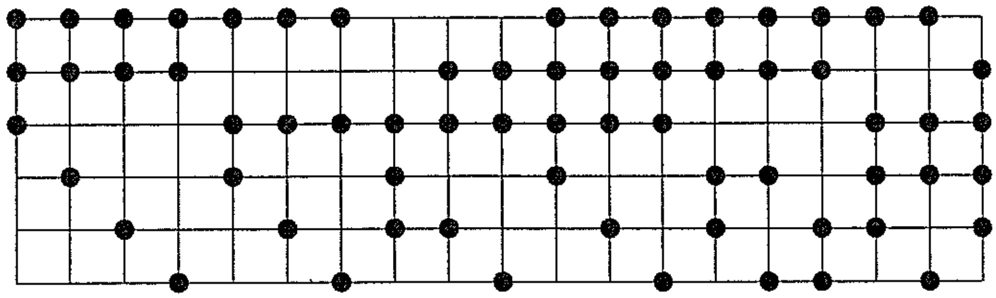
Now since $s_j = 0$ we have $s_{j-1} + s_{j+1} \geq 6$. Hence in this case we must have $s_{j-1} \geq 3$. So we need to consider the possibilities

$$s_{j-1} \in \{3,4,5,6\},\ s_j = 0,\ s_{j+1} \in \{0,1,2,3\}.$$

In all cases we show that by readjusting the stones in columns $j-1$, j, and $j+1$, we can obtain a new dominating set with at least one stone in each of these three columns.

When $s_{j-1} \geq 5$, we can assume, by symmetry, $(1, j-1)$, $(2, j-1)$ and $(3, j-1)$ are all in S. Therefore we can move one stone from $(1, j-1)$ to $(1, j-2)$ and one stone from $(2, j-1)$ to $(2, j)$. One easily checks that the new cofiguration is still a dominating set. This leaves only the cases $s_{j-1} = 3$ and $s_{j-1} = 4$.

We list in Figure 4.10, up to symmetry, all possible distributions of the stones in column $j-1$ for $s_{j-1} = 3$ and $s_{j-1} = 4$.



(1) (2) (3) (4) (5) (6) (7) (8) (9) (10)(11)(12)(13)(14)(15)(16)(17)(18)(19)

Figure 4.10: The distributions of stones on column $j-1$.

Consider the pairs of 6×5 blocks in Figure 4.11. In each block the 5 columns are numbered $j-2$, $j-1$, j, $j+1$ and $j+2$ from the left. In the left hand side of each pair, $s_j = 0$, the stones in column $j-1$ are distributed as one of the cases listed in Figure 4.10 and the stones in columns $j-2$ and $j-1$ are those additional stones that must be present to cover all vertices in columns $j-1$ and j. By moving some stones in the left hand side, we obtain a new dominating set on the right hand side as desired. For (1) and (2), we simply move the top two stones in column $j-1$, one to left the other to right, to get a new dominating set. (11), (12) and (13) are the same as (1). (14) and (15) are the same as (2). For (3) and (4), we move four stones to get a new dominating set. (5) is dual to (3) in the sense that in case (5) the pattern of three stones in column $j-1$ forces stones in rows 2, 5 and 6 of column $j+1$ and this is isomorphic to the situation in case (3). Similarly (8) is dual to (4). The adjustment of stones of (18) is shown in Figure 4.11(18). (17) and (19) are similar to (18).

We now consider the remaining cases (6), (7), (9), (10) and (16). For these cases we need to use the fact that by the definition of j we have at least one stone in column $j-2$. The existence of such a stone allows us to move one stone from column $j-1$ to column j in the same row or sometimes an adjacent row. However since we do not know in which of the six rows the stone in column $j-2$ lies we must consider all six possibilities. To facilitate this verification we use the following facts: Suppose $(i, j-2) \in S$. Then we have:

1. If $(i, j-1) \in S$ and vertex $(i+1, j-1)$ is covered by some stone other than $(i, j-1)$, (when $i > 6$ or $i \leq 0$, we regard the vertex is covered), then we can move $(i, j-1)$ to $(i-1, j)$.

2. If $(i, j-1) \in S$ and vertex $(i-1, j-1)$ is covered by some stone other than $(i, j-1)$, then we can move $(i, j-1)$ to $(i+1, j)$.

3. If $(i+1, j-1) \in S$ and vertex $(i+2, j-1)$ is covered by some stone other than $(i+1, j-1)$, then we can move $(i+1, j-1)$ to $(i+1, j)$.

4. If $(i-1, j-1) \in S$ and vertex $(i-2, j-1)$ is covered by some stone other than $(i-1, j-1)$, then we can move $(i-1, j-1)$ to $(i-1, j)$.

Now by the facts mentioned above, any placement of a stone in column $j-2$ can "push" one stone from column $j-1$ to column s_j. Figure 4.11(6), gives an example of the placement of a stone $(5, j-2)$ which pushes the stone $(5, j-1)$ to $(6, j)$. In Figure 4.11(7), stone $(5, j-2)$ pushes stone $(6, j-1)$ to $(6, j)$. And for (9), (10) and

CHAPTER 5

ALGORITHMS FOR DETERMINING $\gamma_{k,n}$

A dynamic program for finding $\gamma_{k,n}$ was created by Hare [6]. Based on Hare's idea, in the first section, we give an algorithm similar to Hare's and give a proof that the algorithm works. Recently, [10] Fisher showed that for fixed k, we only need to find $\gamma_{k,n}$ for $1 \leq n \leq N$ for some appropriate integer N and the rest of $\gamma_{k,n}$, for $n > N$, will follow recursively. We present Fisher's method in section 5.2.

5.1 Hare's method

For a given subset of the vertex set of the grid $P_k \times P_n$ we define for $u \in P_k \times P_n$ and $S \subseteq P_k \times P_n$

$$d(u, S) = \min_{v \in S} d(u, v).$$

It is clearly that if S is a dominating set then every stone u (vertex in S) satisfies $d(u, S) = 0$ and every non-stone vertex u satisfies $d(u, S) = 1$.

Let $\mathcal{A}_j$ denote the collection of all subsets A_j of

$$\{1, \ldots, k\} \times \{1, \ldots, j\} = \bigcup_{i=1}^{j} C_i$$

such that

$$d(u, A_j) \leq 1 \; for \; all \; u \in \bigcup_{i=1}^{j-1} C_i,$$

i.e., A_j is a set of vertices of $P_k \times P_n$ that lies in the first j columns and covers all vertices in the first $j-1$ columns.

Remark

> A_j is not a fixed set. We use the subscripts j to remind us that A_j lies in the first j columns of $P_k \times P_j$

If $A_j \in \mathcal{A}_j$, clearly

$$d(u, A_j) \leq 2 \; for \; u \in \bigcup_{i=1}^{j} C_i.$$

For such a subset A_j, the *terminal vector* associated with A_j is defined as the vector $x = (x_1, x_2, \ldots, x_k)$ where $x_i = d((i,j), A_j)$, $i = 1, \ldots, k$. Here we still call the elements of A_j stones. Every entry x_i of the terminal vector x is either 0 or 1 or 2 depending on the position of stones in the last two columns. It is clear that for $1 \leq i \leq k$, we have

> $x_i = 0$ if only if the vertex (i,j) is a stone in A_j;
>
> $x_i = 1$ if only if the vertex is covered but not a stone;
>
> $x_i \geq 2$ if only if vertex is uncovered.

For convenience we set $x_0 = x_{k+1} = 1$. Note that if $x_i = 0$ then $x_{i-1}, x_{i+1} \in \{0, 1\}$.

Let $T(k)$ be the set consisting of all possible terminal vectors, that is,

$$T(k) = \{(x_1, \ldots, x_k) \in \{0,1,2\}^k : (x_i, x_{i+1}) \neq (0,2) \; or \; (2,0) \text{ for } i = 1, \ldots, k-1\}.$$

The vectors in $T(k)$ can be listed as $t_1, t_2, \ldots, t_l$, where $l = |T(k)|$, the cardinality of $T(k)$ is given by

$$|T(1)| = 3, \; |T(2)| = 7,$$

$$|T(i)| = 2|T(i-1)| + |T(i-2)| \quad \text{for } i \geq 3.$$

By solving this recursion we have

$$|T(k)| = \frac{1}{2}[(1+\sqrt{2})^k + (1-\sqrt{2})^k].$$

as pointed out by Hare [6].

For a given $x \in T(k)$, let $\mathcal{A}_j(x)$ be the class consisting of elements of all sets $A_j \in \mathcal{A}_j$ having the terminal vector x. It is possible when $j = 1, 2$ that $\mathcal{A}_j(x)$ might be empty. Later we will discuss this in detail.

Now we define for given $x \in T(k)$,

$$m_j(x) = \min\{|A_j| : A_j \in \mathcal{A}_j(x)\}. \tag{5.1}$$

If $\mathcal{A}_j(x)$ is empty we let $m_j(x) = \infty$. We will show that the entries of the $l \times n$ matrix $[m_j(t_i)]$ can be built up recursively column by column.

Note if $A_j \in \mathcal{A}_j(x)$, then deleting column j from A_j results in a set $A_{j-1} \in \mathcal{A}_{j-1}(y)$ for some terminal vector y; if $A_{j-1} \in \mathcal{A}_{j-1}(y)$, then concatenating a $k \times 1$ block, which covers all uncovered vertices in column $j-1$, to A_{j-1} results in a set $A_j \in \mathcal{A}_j(x)$ for some x.

First we represent a $k \times 1$ block by a vector $z = (z_1, z_2, \ldots, z_k)$ where

$$z_i = \begin{cases} 0 & \text{if } (i,1) \text{ is a stone in the block} \\ 2 & \text{if } (i,1) \text{ is uncovered by the block} \\ 1 & \text{otherwise.} \end{cases}$$

Let R be the set consisting of the elements of all such vectors. Now when we concatenate a $k \times 1$ block with vector $z \in R$ to a set A_{j-1} with the terminal vector y

to get a set A_j with the terminal vector x, the relationship between x, y and z is a partial operation $f : T \times R \longrightarrow T$. Suppose $f(y, z) = x$, then for each i, x_i, y_i and z_i must satisfy the following table.

	$z_i = 0$	$z_i = 1$	$z_i = 2$
$y_i = 0$	0	1	1
$y_i = 1$	0	1	2
$y_i = 2$	0	undefined	undefined

Table 5.1: Partial operation $f(y, z) = x$

Now let $x \in T(k)$ be fixed, in order to have $f(y, z) = x$, y and z are restricted as following:

If $x_i = 2$ then $z_i = 2$ and $y_i = 1$.

If $x_i = 0$ then $z_i = 0$, $y_i \in \{0, 1, 2\}$.

If $x_i = 1$, $x_{i+1} > 0$ and $x_{i-1} > 0$ then $z_i = 2$ and $y_i = 0$.

If $x_i = 1$ and at least one of x_{i+1} and x_{i-1} is 0 then $z_i = 1$, $y_i \in \{0, 1\}$.

Actually z, if it exists, is uniquely determined by x. And there are very few pairs (y, z) in the set

$$f^{-1}(x) = \{(y, z) : f(y, z) = x, \text{ for some z}\}.$$

Sometimes the set $f^{-1}(x)$ might be empty since y might not be a terminal vector. When $j = 0$, we say that there is no stone in column 0 and every vertex in column 0 is covered. Therefore there is one and only one terminal vector $(1, 1, \ldots, 1)$. When $j = 1$, the set of all possible terminal vectors are still a proper subset of T. For example, when $k = 3$ and $j = 1$ there is no set A_1 terminating at $(0, 1, 1)$ since the vertex $(3, 1)$ can not be covered by any stone in column 0 or column 1. But if $j \geq 3$ then $f^{-1}(x) \neq \phi$. We prove this in the following lemma.

Lemma 5.1 *Any $x \in T(k)$ can be a terminal vector of some A_j if $j \geq 3$.*

Proof. First, by the operation table, we can concatenate the $k \times 1$ block in which all vetices are stones to any A_j resulting a set A_{j+1} with the terminal vector $(0, 0, \ldots, 0)$. That is, the vector with all entries 0 can be a terminal vector at any column. For any $x \in T(k)$ with no entry equal to 2, there always exists a z such that $f((0, \ldots, 0), z) = x$ by using a proper $k \times 1$ block. So all vectors with no entry equal to 2 can be a terminal vector at any column j, for $j \geq 2$. Therefore when $j \geq 3$, for any vector $x \in T(k)$ we can take

$$(y_i, z_i) = \begin{cases} (0,0) & \text{if } x_i = 0 \\ (0,2) \text{ or } (0,1) & \text{if } x_i = 1 \\ (1,2) & \text{if } x_i = 2. \end{cases}$$

In the second row above, $z_i = 1$ or 2 is dependent on given x. The vector y chosen above has no entry equal to 2. So any x can be a terminal vector when $j \geq 3$. ∎

Now define

$$m'_0(x) = \begin{cases} 0 & \text{if } x = (1, 1, \ldots, 1) \\ \infty & \text{otherwise,} \end{cases}$$

$$m'_j(x) = \min \{m_{j-1}(y) + |x| : (y, z) \in f^{-1}(x)\} \text{ for } j \geq 1, \tag{5.2}$$

where $|x|$ is the number of 0's in the vector x.

will lead to a set $A_j \in \mathcal{A}_j(x)$. Thus

$$\begin{aligned} m'_j(x) &= m_{j-1}(y'') + |x| \\ &= |A_{j-1}| + |x| \\ &= |A_j| \geq m_j(x). \blacksquare \end{aligned}$$

Theorem 5.1 shows the way to build the $l \times n$ matrix column by column. We restate this recursion:

$$m_0(x) = \begin{cases} 0 & \text{if } x = (1,1,\ldots,1) \\ \infty & \text{otherwise.} \end{cases}$$

$$m_j(x) = \begin{cases} \infty & \text{if } f^{-1}(x) = \phi \\ \min\{m_{j-1}(y) + |x| : (y,z) \in f^{-1}(x)\} & \text{othewise.} \end{cases}$$

Once $m_n(t_1), \ldots, m_n(t_l)$ are calculated the domination number $\gamma_{k,n}$ follows. Let

$$M_n = \min_{x \in T}\{m_n(x) : \text{x has no entry equal to } 2\}, \tag{5.3}$$

we have the following theorem.

Theorem 5.2 *For all k, n we have $\gamma_{k,n} = M_n$.*

Proof The set of all x satisfying (5.3) contains the vector $(0,0,\ldots,0)$, hence it is non-empty. Therefore M_n is a finite number. Let x_0 be a vector with no entry 2 and satisfying $m_n(x_0) = M_n$. Since $m_n(x_0)$ is a finite number, there must be a set A_n terminating at x_0 and having $m_n(x_0)$ stones. But such a set A_n is a dominating set of $P_k \times P_n$. So we have $\gamma_{k,n} \leq M_n$.

On the other hand, let S be a minimal dominating set of $P_k \times P_n$. Then S is of type A_n having some terminal vector x_0 with no entry equal to 2. Therefore

$$\gamma_{k,n} = |S| \geq m_n(x_0) \geq M_n. \blacksquare$$

Now we write an algorithm for finding $\gamma_{k,n}$. The proof of the algorithm has been done in this section. For fixed k, the computing time of this algorithm is linear in n.

Algorithm

1. List the elements of T as $t_1, \ldots, t_l$.

2. For each t_i list the elements in $f^{-1}(t_i)$ as List(t_i).

3. For $i = 1$ to l do

   if $t_i = (1, 1, \ldots, 1)$ then $m_0(t_i) = 0$

   else $m_0(t_i) = \infty$

4. For $j = 1$ to n do

   for $i = 1$ to l do

   find the minimum of $m_{j-1}(y)$, $(y, z) \in List(t_i)$.

   This is $m_j(t_i)$.

5. Find the minimum of $m_n(t_i)$ for all t_i with no entry equal to 2.

5.2 Fisher's method

Theorem 5.3 *For given k, suppose there exist two integers q and N such that $N > q$ and*

$$m_N(t_i) - m_{N-q}(t_i) = C, \text{ for all } 1 \leq i \leq l, \tag{5.4}$$

where C is a finite positive constant. Then whenever $n \geq N$, we have

$$m_n(t_i) - m_{n-q}(t_i) = C, \text{ for all } 1 \leq i \leq l. \tag{5.5}$$

Hence

$$\gamma_{k,n} = \gamma_{k,n-q} + C, \text{ for } n \geq N. \tag{5.6}$$

Proof. By the definition in Formula (5.2) and by taking the minimum of both sides of (5.4) we have, for all t_i,

$$m'_{N+1}(t_i) - m'_{N+1-q}(t_i) = C.$$

So by Theorem 1, for all t_i

$$m_{N+1}(t_i) - m_{N-q+1}(t_i) = C.$$

And by Theorem 5.2 we have

$$\gamma_{k,N+1} = \gamma_{k,N+1-q} + C.$$

Now (5.5) and (5.6) follow by induction. ∎

Theorem 5.3 comes from Fisher's idea [10]. Fisher worked on a VAX 8821 machine and he has found the period q, starting point N and the constant C in (5.6), for fixed k, $1 \leq k \leq 16$ [10]. But in general for $k \geq 17$ what should be N, q and C are still unknown.

We have tried to find N, q and C for small k on a IBM-PC 286 computer. It takes only 18 seconds to get results for $k = 5$ and 1 minute for $k = 6$. The domination numbers $\gamma_{5,n}$ and $\gamma_{6,n}$ obtained in this way agree with that found in Chapter 4.

Table 5.2 shows the results for $k = 5$. In Table 5.2, $N = 17$ is the first place such that

$$m_{17}(t_i) - m_{12}(t_i) = 6, \text{ for all } 1 \leq i \leq 99.$$

Therefore

$$\gamma_{5,n} = \gamma_{5,n-5} + 6 \qquad \text{for } n \geq 17.$$

After finding $\gamma_{5,n}$ for $1 \leq n \leq 16$, the above recursion can start at 13. Thus the domination numbers of $P_5 \times P_n$ are:

$$\gamma_{5,1} = 2,\ \gamma_{5,2} = 3,\ \gamma_{5,3} = 4,\ \gamma_{5,4} = 6,\ \gamma_{5,5} = 7,\ \gamma_{5,6} = 8,$$

$$\gamma_{5,7} = 9,\ \gamma_{5,8} = 11,\ \gamma_{5,9} = 12,\ \gamma_{5,10} = 13,\ \gamma_{5,11} = 14,\ \gamma_{5,12} = 16,$$

$$\gamma_{5,n} = \gamma_{5,n-5} + 6 \qquad \text{for } n \geq 13.$$

T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0(0,0,0,0,0)	5	5	7	8	9	11	12	13	14	16	17	18	19	21	22	23	24
1(1,0,0,0,0)	4	5	6	7	9	10	11	12	14	15	16	17	18	20	21	22	23
3(0,1,0,0,0)	4	5	6	8	9	10	11	12	13	15	16	17	19	20	21	22	23
4(1,1,0,0,0)	∞	4	6	7	8	9	11	12	13	14	15	17	18	19	20	21	23
5(2,1,0,0,0)	3	4	6	7	8	10	11	12	13	14	16	17	18	19	20	22	23
9(0,0,1,0,0)	4	5	7	7	8	10	11	12	13	15	16	17	18	20	21	22	23
10(1,0,1,0,0)	3	4	6	7	8	9	10	11	13	14	15	16	17	19	20	21	22
12(0,1,1,0,0)	3	4	6	7	8	9	11	12	13	14	16	17	18	19	20	22	23
13(1,1,1,0,0)	∞	4	5	7	8	9	10	12	13	14	15	17	18	19	20	21	23
14(2,1,1,0,0)	∞	3	5	6	7	9	10	11	12	13	15	16	17	18	19	21	22
16(1,2,1,0,0)	∞	4	5	6	7	8	10	11	12	14	15	16	17	18	20	21	22
17(2,2,1,0,0)	2	∞	5	6	7	9	10	11	13	14	15	16	17	19	20	21	22
27(0,0,0,1,0)	4	5	6	8	9	10	11	12	13	15	16	17	19	20	21	22	23
28(1,0,0,1,0)	3	5	5	7	8	9	10	12	13	14	15	16	18	19	20	21	22
30(0,1,0,1,0)	3	4	5	7	8	9	10	12	13	14	15	17	18	19	20	21	23
31(1,1,0,1,0)	∞	4	5	6	7	8	10	11	12	13	14	16	17	18	19	20	22
32(2,1,0,1,0)	2	4	5	6	7	9	10	11	12	13	15	16	17	18	19	21	22
36(0,0,1,1,0)	3	4	6	7	8	9	11	12	13	14	16	17	18	19	20	22	23
37(1,0,1,1,0)	2	4	5	6	7	9	10	11	12	13	15	16	17	18	19	21	22
39(0,1,1,1,0)	∞	3	5	6	7	9	10	11	12	14	15	16	17	19	20	21	22

Table 5.2: 1. Output of program for finding $\gamma_{5,n}$.

T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
40(1,1,1,1,0)	∞	4	5	7	8	9	10	12	13	14	15	16	18	19	20	21	22
41(2,1,1,1,0)	∞	3	5	6	7	9	10	11	12	13	15	16	17	18	19	21	22
43(1,2,1,1,0)	∞	3	5	6	7	8	9	11	12	13	14	15	17	18	19	20	21
44(2,2,1,1,0)	∞	∞	4	5	6	8	9	10	12	13	14	15	16	18	19	20	21
48(0,1,2,1,0)	2	4	5	6	7	8	10	11	12	13	15	16	17	18	19	21	22
49(1,1,2,1,0)	∞	4	4	6	7	8	10	11	12	13	14	16	17	18	19	20	22
50(2,1,2,1,0)	∞	3	4	5	6	8	9	10	11	12	14	15	16	17	18	20	21
52(1,2,2,1,0)	∞	3	4	6	7	8	10	11	12	13	14	16	17	18	19	20	22
53(2,2,2,1,0)	1	∞	5	6	7	8	9	10	12	13	14	16	17	18	19	20	22
81(0,0,0,0,1)	4	5	6	7	9	10	11	12	14	15	16	17	18	20	21	22	23
82(1,0,0,0,1)	3	5	6	6	8	9	10	12	13	14	15	17	18	19	20	22	23
84(0,1,0,0,1)	3	5	5	7	8	9	10	12	13	14	15	16	18	19	20	21	22
85(1,1,0,0,1)	∞	4	5	6	8	9	10	11	13	14	15	16	17	19	20	21	22
86(2,1,0,0,1)	2	4	5	6	8	9	10	11	12	14	15	16	17	18	20	21	22
90(0,0,1,0,1)	3	4	6	7	8	9	10	11	13	14	15	16	17	19	20	21	22
91(1,0,1,0,1)	2	4	5	6	7	9	10	11	12	14	15	16	17	18	20	21	22
93(0,1,1,0,1)	2	4	5	6	7	9	10	11	12	13	15	16	17	18	19	21	22
94(1,1,1,0,1)	∞	4	5	6	7	8	9	11	12	13	15	16	17	18	19	21	22
95(2,1,1,0,1)	∞	3	4	5	7	8	9	10	11	13	14	15	16	17	19	20	21
97(1,2,1,0,1)	∞	3	4	6	7	8	9	10	12	13	14	15	16	18	19	20	21

Table 5.2: 2. Output of program for finding $\gamma_{5,n}$.

T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
98(2,2,1,0,1)	1	∞	4	5	7	8	9	10	12	13	14	15	17	18	19	20	21
108(0,0,0,1,1)	∞	4	6	7	8	9	11	12	13	14	15	17	18	19	20	21	23
109(1,0,0,1,1)	∞	4	5	6	8	9	10	11	13	14	15	16	17	19	20	21	22
111(0,1,0,1,1)	∞	4	5	6	7	8	10	11	12	13	14	16	17	18	19	20	22
112(1,1,0,1,1)	∞	3	4	6	7	8	9	11	12	13	14	16	17	18	19	20	22
113(2,1,0,1,1)	∞	3	5	5	7	8	9	11	12	13	14	15	17	18	19	20	21
117(0,0,1,1,1)	∞	4	5	7	8	9	10	12	13	14	15	17	18	19	20	21	23
118(1,0,1,1,1)	∞	4	5	6	7	8	9	11	12	13	15	16	17	18	19	21	22
120(0,1,1,1,1)	∞	4	5	7	8	9	10	12	13	14	15	16	18	19	20	21	22
121(1,1,1,1,1)	∞	5	5	7	8	9	11	12	13	14	16	17	18	19	21	22	23
122(2,1,1,1,1)	∞	4	5	6	7	9	10	11	12	14	15	16	17	18	20	21	22
124(1,2,1,1,1)	∞	4	5	6	8	9	10	11	12	13	15	16	17	19	20	21	22
125(2,2,1,1,1)	∞	∞	4	6	7	8	9	11	12	13	14	15	17	18	19	20	21
129(0,1,2,1,1)	∞	4	4	6	7	8	10	11	12	13	14	16	17	18	19	20	22
130(1,1,2,1,1)	∞	4	5	7	7	8	10	11	12	13	15	16	17	18	20	21	22
131(2,1,2,1,1)	∞	3	4	6	7	8	9	10	11	13	14	15	16	17	19	20	21
133(1,2,2,1,1)	∞	3	4	6	7	8	9	11	12	13	14	16	17	18	19	20	22
134(2,2,2,1,1)	∞	∞	4	5	7	8	9	10	12	13	14	15	17	18	19	20	21
144(0,0,1,2,1)	∞	4	5	6	7	8	10	11	12	14	15	16	17	18	20	21	22
145(1,0,1,2,1)	∞	3	4	6	7	8	9	10	12	13	14	15	16	18	19	20	21

Table 5.2: 3. Output of program for finding $\gamma_{5,n}$.

T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
147(0,1,1,2,1)	∞	3	5	6	7	8	9	11	12	13	14	15	17	18	19	20	21
148(1,1,1,2,1)	∞	4	5	6	8	9	10	11	12	13	15	16	17	19	20	21	22
149(2,1,1,2,1)	∞	3	5	5	7	8	9	10	12	13	14	15	16	18	19	20	21
151(1,2,1,2,1)	∞	3	4	5	7	8	9	10	12	13	14	15	17	18	19	20	21
152(2,2,1,2,1)	∞	∞	4	5	6	7	8	10	11	12	13	14	16	17	18	19	20
156(0,1,2,2,1)	∞	3	4	6	7	8	10	11	12	13	14	16	17	18	19	20	22
157(1,1,2,2,1)	∞	3	4	6	7	8	9	11	12	13	14	16	17	18	19	20	22
158(2,1,2,2,1)	∞	2	4	5	6	7	9	10	11	12	13	15	16	17	18	19	21
160(1,2,2,2,1)	∞	∞	3	5	6	7	9	10	11	12	14	15	16	17	19	20	21
161(2,2,2,2,1)	∞	∞	4	5	7	8	9	10	12	13	14	15	16	18	19	20	21
189(0,0,0,1,2)	3	4	6	7	8	10	11	12	13	14	16	17	18	19	20	22	23
190(1,0,0,1,2)	2	4	5	6	8	9	10	11	12	14	15	16	17	18	20	21	22
192(0,1,0,1,2)	2	4	5	6	7	9	10	11	12	13	15	16	17	18	19	21	22
193(1,1,0,1,2)	∞	3	5	5	7	8	9	11	12	13	14	15	17	18	19	20	21
194(2,1,0,1,2)	1	3	4	5	7	8	9	10	12	13	14	15	17	18	19	20	21
198(0,0,1,1,2)	∞	3	5	6	7	9	10	11	12	13	15	16	17	18	19	21	22
199(1,0,1,1,2)	∞	3	4	5	7	8	9	10	11	13	14	15	16	17	19	20	21
201(0,1,1,1,2)	∞	3	5	6	7	9	10	11	12	13	15	16	17	18	19	21	22
202(1,1,1,1,2)	∞	4	5	6	7	9	10	11	12	14	15	16	17	18	20	21	22
203(2,1,1,1,2)	∞	3	5	6	6	8	9	10	12	13	14	15	17	18	19	20	22

Table 5.2: 4. Output of program for finding $\gamma_{5,n}$.

T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
205(1,2,1,1,2)	∞	3	5	5	7	8	9	10	12	13	14	15	16	18	19	20	21
206(2,2,1,1,2)	∞	∞	4	5	6	8	9	10	11	13	14	15	16	17	19	20	21
210(0,1,2,1,2)	∞	3	4	5	6	8	9	10	11	12	14	15	16	17	18	20	21
211(1,1,2,1,2)	∞	3	4	6	7	8	9	10	11	13	14	15	16	17	19	20	21
212(2,1,2,1,2)	∞	2	4	5	6	7	9	10	11	12	14	15	16	17	18	20	21
214(1,2,2,1,2)	∞	2	4	5	6	7	9	10	11	12	13	15	16	17	18	19	21
215(2,2,2,1,2)	∞	∞	4	5	6	7	8	9	11	12	13	15	16	17	18	19	21
225(0,0,1,2,2)	2	∞	5	6	7	9	10	11	13	14	15	16	17	19	20	21	22
226(1,0,1,2,2)	1	∞	4	5	7	8	9	10	12	13	14	15	17	18	19	20	21
228(0,1,1,2,2)	∞	∞	4	5	6	8	9	10	12	13	14	15	16	18	19	20	21
229(1,1,1,2,2)	∞	∞	4	6	7	8	9	11	12	13	14	15	17	18	19	20	21
230(2,1,1,2,2)	∞	∞	4	5	6	8	9	10	11	13	14	15	16	17	19	20	21
232(1,2,1,2,2)	∞	∞	4	5	6	7	8	10	11	12	13	14	16	17	18	19	20
233(2,2,1,2,2)	∞	∞	3	4	6	7	8	9	11	12	13	14	16	17	18	19	20
237(0,1,2,2,2)	1	∞	5	6	7	8	9	10	12	13	14	16	17	18	19	20	22
238(1,1,2,2,2)	∞	∞	4	5	7	8	9	10	12	13	14	15	17	18	19	20	21
239(2,1,2,2,2)	∞	∞	4	5	6	7	8	9	11	12	13	15	16	17	18	19	21
241(1,2,2,2,2)	∞	∞	4	5	7	8	9	10	12	13	14	15	16	18	19	20	21
242(2,2,2,2,2)	0	∞	5	5	7	8	9	11	12	13	14	16	17	18	19	21	22

Table 5.2: 1. Output of program for finding $\gamma_{5,n}$.

CHAPTER 6

CONCLUSIONS

6.1 Conjecture

The outstanding problem left unsolved is the conjecture: For k and n sufficient large

$$\gamma_{k,n} = \lfloor \frac{(k+2)(n+2)}{5} \rfloor - 4 \tag{6.1}$$

The results in chapter 2 and Fisher's results [10] show that we must have $n \geq k \geq 16$. In fact Fisher's work shows that (6.1) holds for $k = 16$ and $n \geq 14$.

6.2 Further directions

For finding the domination number of $P_k \times P_n$, we are going to work on the problems:

1. **Prove some lower bounds**

   It is possible to find some lower bounds strictly less than the standard upper bound. In section 2.3 we have mentioned two low bounds

$$\gamma_{k,n} \geq \lfloor \frac{kn+k+n}{5} \rfloor, \tag{6.2}$$

and

$$\gamma_{k,n} \geq \lfloor \frac{kn + k + 2n}{5} \rfloor. \tag{6.3}$$

2. **Attempts to prove the conjecture**

   The only way to prove the conjecture is to prove the standard upper bound is itself a lower bound for n and k sufficiently large. One way to approach the proof is discussing the behavior of the dominating sequence like P1-P9 and Q1-Q7 in Chapter 4. The complete list of the propositions of the dominating sequence for general k must be very long and very complicated. To avoid this, we have tried several different approaches, but so far none of them works.

3. **Prove $\gamma_{k,n}$ for small k**

   Even if the conjecture is true, additional work for finding $\gamma_{k,n}$ has to be done for small k. Some of this was done in Chapter 3 and Chapter 4. Fisher found $\gamma_{k,n}$ for $k \leq 16$ using the output of his program. However, more work of proving $\gamma_{k,n}$ for small k structurely seems necessary to proving the conjecture.

The product space $X = \prod_{i=1}^{n} P_{q_i}$ can be regarded as an n-dimensional grid graph. So the (complete) grid graph is a 2-dimensional grid graph. The problem treated in this dissertation is of course a special case of the following more difficult problems:

1. Determine the domination numbers of the higher dimensional grid graphs $\prod_{i=1}^{n} P_{q_i}$ for $n \geq 3$.

2. Determine the p-covering number, $p \geq 2$ of the product space X for $n \geq 2$.

And of course there is no need to restrict oneself to grid graphs. These questions may be asked for any graph or, in deed, for any finite metric space.

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