

# The Integer Sequence Transform $a \mapsto b$ Where $b_n$ is the Number of Real Roots of the Polynomial

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

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## Abstract

We discuss the integer sequence transform  $a \mapsto b$  where  $b_n$  is the number of real roots of the polynomial  $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ . Many sequences  $a$  are shown to give the trivial sequence  $b = (0, 1, 0, 1, 0, 1, \cdots)$ , i.e.,  $b_n = n \bmod 2$ . We also look at some sequences  $a$  for which  $b$  is more interesting such as  $a_n = (n + 1)^k$ , for  $k \geq 3$ .

## 1. Preliminaries

In this paper  $a = (a_0, a_1, a_2, \cdots, a_n, \cdots)$  will, unless otherwise indicated, denote an integer sequence with  $a_0 \neq 0$ . Given such a sequence  $a$  of integers, define the sequence  $RR(a) = b$  where  $b_n$  is the number of real roots counting multiplicities of the polynomial  $\sum_{i=0}^n a_i x^i$ . Some obvious properties of  $b$  are

- (1)  $b_0 = 0$ .
- (2)  $b_1 = 1$  if  $a_1 \neq 0$ .
- (3)  $0 \leq b_n \leq n$  if  $n$  is even.
- (4)  $1 \leq b_n \leq n$  if  $n$  is odd.
- (5) if  $a_n = 0$  then  $b_n = b_{n-1}$ .
- (6)  $b_n \equiv n \pmod{2}$ .

For many integer sequences  $a$  the sequence  $b$  is the minimal sequence  $b_n = 0$  for  $n$  even and  $b_n = 1$  for  $n$  odd, i.e.,  $b_n = n \bmod 2$  (see Section 5). On the other hand, it is impossible to have  $b_n = n$  for all  $n$  for any integer sequence  $a$ . As pointed out to me by Mourad Ismail, it turns out that some related questions have been considered in analysis, e.g., [1],[3],[5], and [7]. For example, let

$$\sum_{k=0}^{\infty} a_k z^k$$

be a formal power series. The polynomial

$$S_n(z) := \sum_{k=0}^n a_k z^k$$

is called its  $n$ -th *section*. According to [7] the following theorem goes back to Polya in 1913.

**Theorem 1.** (See [3] and [7]) *If for the formal power series*

$$\sum_{k=0}^{\infty} a_k z^k, \quad a_k > 0 \tag{*}$$

*all of the roots of the sections  $S_n(z)$  are real for all sufficiently large  $n$  then the radius of convergence of the power series (\*) is infinite.*

A stronger result without the assumption that the  $a_k$  in  $\sum_{k=0}^{\infty} a_k z^k$  are positive is given in the following theorem.

**Theorem 2.** (See Ganelius [1]) *Let  $f_n(z) = \sum_{k=0}^n a_k z^k$  be the partial sums of the power series  $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ,  $a_0 = 1$ . Let  $N(S, n)$  be the number of zeros of the polynomial  $f_n(z)$  in a sector  $S = \{z : \beta \leq \arg(z) \leq \alpha + \beta\}$  for some  $\beta$  and some  $\alpha > 0$  and assume  $N(S, n) = o(n)$ , then  $f(z)$  is an entire function of order zero.*

If all zeros of the  $n$ -sections are on the real line then obviously there are many sectors which have no zeros at all. Hence from Theorem 2 we have the following.

**Corollary 1.** *Under the conditions of Theorem 2,  $\lim_{n \rightarrow \infty} a_n = 0$ . Thus if the  $a_n$  are integers  $f(z)$  must be a polynomial and  $b_n = n$  for all  $n$  is not possible..*

**Remark 1.** However  $b_n = n$  for all  $n$  is possible if we allow the  $a_n$  to be rational numbers. For example, from [3] if we take  $c^2 \geq 4$  then for all  $n$

$$\sum_{k=0}^n c^{-k^2} z^k$$

has only real roots.

Lest one is left with the impression that the considerations of this paper are solely in the province of analysis, we point out that in [10] Stanley has considered the combinatorial significance of having all real roots for a polynomial

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \quad a_i \in \mathbb{N}.$$

See also Liu and Wang ([11]) for applications to combinatorics. We also note in Section 3 that the computation of  $RR(a)$  for integer sequences can be accomplished by exact rational number arithmetic.

## 2. Possible Generalizations

In this paper we define  $RR(a) = b$  where  $b_n$  is the number of real zeros of the  $n$ -th section  $\sum_{i=0}^n a_i x^i$  of the ordinary generating function  $G(x) = \sum_{i=0}^{\infty} a_i x^i$  of the sequence  $a$ . We could just as well have replaced the ordinary generating function by some other type of generating function, e.g., exponential generating function, Lambert series, Bell series, or Dirichlet series, [13]. We have not investigated these other generating functions.

## 3. Computational Considerations

Given a polynomial  $p \in \mathbb{Z}[x]$ , the number of real roots may be computed using exact rational number arithmetic. We briefly describe this here. For more details see [14] and [4]. In order to count the number of real roots considering multiplicity it is convenient to first find the square free decomposition of  $p$ :

$$p = u f_1^{e_1} f_2^{e_2} \cdots f_k^{e_k}$$

where  $f_i \in \mathbb{Z}[x]$ ,  $e_i \in \mathbb{N}$ ,  $|u|$  is the greatest common divisor of the coefficients of  $p$ , each  $f_i$  has no multiple roots, i.e.,  $\gcd(f_i, f_i') = 1$  and when  $i \neq j$ ,  $\gcd(f_i, f_j) = 1$ . Maple's procedure **sqrfree** produces this decomposition. Next we use a procedure (in Maple, **realroot**) which produces for each  $f_i$  a list of pairwise disjoint intervals each containing exactly one root of  $f_i$ . If  $N_i$  is the number of such intervals in the list for  $f_i$  then  $\sum_{i=1}^k N_i e_i$  is the number of real roots of  $p$  counting multiplicities of the roots. Our Maple procedure, **NumRealRoots**, for computing the number of real roots including multiplicities for a polynomial  $p \in \mathbb{Z}[x]$ , is the following:

```
NumRealRoots := proc(p)
local q, k, u;
  if p = 0 then error "zero polynomial not allowed"; fi;
  q := sqrfree(p);
  k := 0;
  for u in q[2] do
```

```

        k := k+nops(realroot(u[1]))*u[2]
    end do;
    k
end proc

```

We note that in the above procedure, **NumRealRoots**, we could substitute for **nops(realroot(u[1]))** the Maple procedure based on Sturm's Theorem: **sturm(sturmseq(u[1], x), x, -infinity, infinity)**, but this produces a slower procedure.

#### 4. Motivating Example

This paper was motivated by the following problem raised on Mathematics Stack-Exchange by *Russian Bot 2.0* [9]. Let  $p_n$  denote the  $n$ th prime  $n \geq 1$ . Consider the sequence

$$a_n = p_{n+1}, n \geq 0.$$

Consider the sequence  $b$  where  $b_n$  is the number of real roots (counting multiplicities) of the polynomial

$$P_n(x) = 2 + 3x + 5x^2 + 7x^3 + \dots + a_n x^n$$

. The sequence  $b$  starts out

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots \quad (1)$$

That is  $b_n = n \bmod 2$ . The question was asked whether or not this pattern continues. It was pointed out that the pattern is broken for the first time for  $n = 2436$  when  $b_{2436} = 2$  and thereafter at least to  $n = 2730$ ,  $b_n = 2$  for  $n$  even,  $b_n = 1$  for  $n$  odd. How far this new pattern continues is unknown. However jumping ahead we find that  $b_{4000} = 2$ ,  $b_{4001} = 1$ ,  $b_{5000} = 2$ ,  $b_{5001} = 1$ ,  $b_{10000} = 2$ ,  $b_{10001} = 1$ ,  $b_{20000} = 2$ , and  $b_{20001} = 1$ .

An anonymous contributor to the answer to this problem namely, *user3733558*, mentioned that that the pattern,  $b_n = n \bmod 2$ , seems to hold at least initially for several other sequences. We consider a few such sequences in the next section.

#### 5. Some sequences for which $b = RR(a)$ is, or appears to be, the periodic sequence $0, 1, 0, 1, 0, 1, 0, 1, \dots$ , that is, $b_n = n \bmod 2$

Some of these results have been proved, unless otherwise stated they are only conjectured. But as the motivating example in Section 4 shows computations can be misleading .

### 5.1. Fibonacci Numbers:

Let  $F = (0, 1, 1, 2, 3, 5, \dots)$  be the sequence of Fibonacci numbers  $F_n$ . Then we define  $a_n = F_{n+1}$  for  $n \geq 0$ . If  $b = RR(a)$  it follows from the results of Garth, Mills and Mitchell, [2] that  $b_n = 0$  for even  $n$  and  $b(n) = 1$  for odd  $n$ . We note that the polynomials  $p(x)$  of degree  $n$  they consider are the reciprocal polynomials  $x^n S_n(1/x)$  of the  $n$ -th sections  $S_n(x)$  of the generating function of  $a$ , but this has no effect on the number of real roots. Thus  $b_n = n \bmod 2$  for  $n \geq 0$ .

### 5.2. k-Fibonacci Numbers:

As in [6] by Mansour and Shattuck, let the recursive sequence  $a_n, n \geq 0$ , of order  $k, k \geq 2$  be defined by the initial values  $a_0 = a_1 = \dots = a_{k-2} = 0$  and  $a_{k-1} = 1$  and the linear recursion

$$a_n = a_{n-1} + \dots + a_{n-k}, n \geq k.$$

Mansour and Shattuck define for  $k \geq 2$  and  $n \geq 1$ , the polynomial

$$P_{n,k}(x) = a_{k-1}x^n + a_kx^{n-1} + \dots + a_{n+k-2}x + a_{n+k-1}.$$

and they prove that for each  $k \geq 2$  the polynomial  $P_{n,k}$  has 0 real roots if  $n$  is even and 1 real root if  $n$  is odd. As in the paper of Garth, Mills and Mitchell these are the reciprocal polynomials of the  $n$ -th sections of the generating function  $\sum_{i=0}^{\infty} a_{i+k-1}x^i$  of the sequences that are of interest to us. But again this has no effect on the number of real roots. So once again if we begin the sequence at  $n = k - 1$ , we find that  $b_n = n \bmod 2$ .

### 5.3. Exponential sequences $a_n = c^n, c \in \mathbb{R}$ and $c \neq 0$ :

Note that this includes the sequence  $a_n = 1$  for all  $n$ . We seek the number of real zeros of the polynomials:

$$p_n(x) := 1 + (cx)^1 + (cx)^2 + \dots + (cx)^n = \frac{(cx)^{n+1} - 1}{(cx - 1)}$$

Note that the polynomial  $x^{n+1} - 1$  has only one real root if  $n + 1$  is odd, namely, 1, and two real roots 1 and  $-1$  if  $n + 1$  is even. It follows that  $p_n(x)$  has 0 real roots if  $n$  is even and 1 real root, namely,  $x = -1/c$  if  $n$  is odd, that is,  $b_n = n \bmod 2$ .

### 5.4. Binomial Sequences: For fixed $k \in \mathbb{N}$ , $a_n = \binom{n+k}{k}$ :

Computations suggest that for all  $k, b = RR(a)$  is the sequence

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots \quad (2)$$

We seek the number of real zeros  $b_n$  of the polynomials:  $p_n = \sum_{i=0}^n \binom{i+k}{k} x^i =$

$$\frac{1}{(1-x)^{k+1}} - \binom{n+1+k}{k} x^{n+1} {}_2F_1(1, n+2+k; n+2; x).$$

The latter equation given by Maple. It was verified by computer that  $b_n = n \bmod 2$  for  $k$  from 1 to 20 and  $n$  from 0 to 500.

**5.5.  $a_n = n + 1$**

This is the case  $k = 1$  of  $\binom{n+k}{k}$  of Section 5.4 and of the case  $k = 1$  of  $(n + 1)^k$  of Section 6. Computations suggest that for all  $k$ ,  $b = RR(a)$  is the sequence

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots \quad (3)$$

We seek the number of real zeros  $b_n$  of the polynomials

$$p_n = \frac{1 + x^{n+1}((n+1)x - n - 2)}{(x-1)^2}$$

It was verified by computer that  $b_n = n \bmod 2$  for  $n$  up to 1000.

**5.6.  $a_n = (n + 1)^2$**

This is case  $k = 2$  of  $(n + 1)^k$  of Section 6. Computations suggest that  $b = RR(a)$  is the sequence

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots \quad (4)$$

We seek the number of real zeros of the polynomials:

$$p_n = \sum_{i=0}^n (i+1)^2 x^i = \frac{(n+1)^2 x^{3+n} + (n+2)^2 x^{n+1} + (-2n^2 - 6n - 3) x^{n+2} - x - 1}{(x-1)^3}.$$

Verified by computer that  $b_n = n \bmod 2$  for  $n$  up to 1000.

**5.7. Catalan Numbers:  $a_n = \frac{1}{n+1} \binom{2n}{n}$**

Computations suggest that for all  $k$ ,  $b = RR(a)$  is the sequence

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots \quad (5)$$

Verified by computer that  $b_n = n \bmod 2$  for  $n$  up to 1000.

## 6. $k$ -th Power Sequences $a_n = (n + 1)^k$ for $k \geq 1$ .

Consider a typical sequence of this type: Let  $k = 50$  then for  $a_n = (n + 1)^{50}$ , the sequence  $b = RR(a)$  begins:

0,	1,	2,	3,	4,	5,	6,	5,	6,	7,	6,	7,	8,	9,	8
9,	8,	9,	10,	9,	10,	11,	10,	11,	10,	11,	12,	11,	12,	11
12,	13,	12,	13,	12,	13,	14,	13,	14,	13,	14,	13,	14,	13,	14
15,	14,	15,	14,	15,	14,	15,	14,	15,	16,	15,	16,	15,	16,	15
16,	15,	16,	15,	16,	17,	16,	17,	16,	17,	16,	17,	16,	17,	16
17,	16,	17,	16,	17,	16,	17,	18,	17,	18,	17,	18,	17,	18,	17
18,	17,	18,	17,	18,	17,	18,	17,	18,	17,	18,	19,	18,	19,	18
19,	18,	19,	18,	19,	18,	19,	18,	19,	18,	19,	18,	19,	18,	19
18,	19,	18,	19,	18,	19,	18,	19,	18,	19,	20,	19,	20,	19,	20
19,	20,	19,	20,	19,	20,	19,	20,	19,	20,	19,	20,	19,	20,	...

Note that  $b_n = n$  for  $n = 0, 1, 2, 3, 4, 5, 6$ .

Computations suggest that for all  $n$  there is a  $k$  such that if  $a_n = (n+1)^k$  then the sequence  $b = RR(a)$  satisfies  $b_i = i$  for  $i = 0, 1, 2, \dots, n$ . If for the smallest such  $k$  we write  $c_n = k$ , we get the sequence  $c$  beginning  $(c_0, c_1, c_2, \dots, c_{23}) = (0, 0, 5, 12, 21, 33, 47, 63, 81, 102, 125, 151, 179, 209, 242, 277, 314, 354, 396, 440, 487, 536, 587, 641)$ . Alternatively the values  $n, c_n$  are given by Table 1.

For example, since  $c_{10} = 125$ , the sequence  $b$  corresponding to  $a_i = (i + 1)^{125}$  begins:

0	1	2	3	4	5	6	7	8	9	10	9	10	11	12	11	12
13	14	13	14	15	16	15	16	17	18	17	18	19	18	19	20	19
20	21	20	21	22	21	22	23	22	23	24	23	24	23	24	25	24
25	26	25	26	27	26	27	26	27	28	27	28	27	28	29	28	29
28	29	30	29	30	29	30	31	30	31	30	31	30	31	32	31	32

## 7. Completely Real Polynomials

**Definition 1.** We shall say that a polynomial  $p = \sum_{i=0}^n a_i x^i \in \mathbb{R}[x]$  is **completely real** if all sections  $p_i = \sum_{j=0}^i a_j x^j$  for  $0 \leq i \leq n$  have  $i$  real roots. As in the case of formal power series we write  $b_i$  for the number of real roots of  $p_i$ .

In this section we exhibit some completely real polynomials in  $\mathbb{Z}[x]$ . The first example shows that for all natural numbers  $n$  there exist completely real polynomials in  $\mathbb{Z}[x]$  of degree  $n$ .

**Remark 2.** While it is the case that the reciprocal polynomial  $p^*(x) = x^n p(1/x)$  of a completely real polynomial  $p(x)$  is often completely real, this is not always the case. For example the polynomial  $p = x^3 + 99x^2 + 2456x + 13827$  is completely real, but its reciprocal  $p^* = 13827x^3 + 2456x^2 + 99x + 1$  is not.

$n$	$c_n$
0	0
1	0
2	5
3	12
4	21
5	33
6	47
7	63
8	81
9	102
10	125
11	151
12	179
13	209
14	242
15	277
16	314
17	354
18	396
19	440
20	487
21	536
22	587
23	641

Table 1:

**7.1.**  $\sum_{k=0}^n c^{n^2-k^2} z^k, c^2 \geq 4$

As pointed out in [3], if  $c^2 \geq 4$  then for all  $i \geq 0$  the polynomials

$$\sum_{k=0}^i c^{-k^2} z^k$$

have only real roots. Clearly then for  $n \in \mathbb{N}$ , the polynomials

$$c^{n^2} \sum_{k=0}^i c^{-k^2} z^k = \sum_{k=0}^i c^{n^2-k^2} z^k$$

have only real roots. The latter are the  $i$ -sections of the polynomial

$$\sum_{k=0}^n c^{n^2-k^2} z^k.$$

which is therefore completely real and in  $\mathbb{Z}[x]$  if  $c \in \mathbb{Z}$ .

**7.2.**  $\sum_{i=0}^n (i+1)^k x^i$  if  $c_n = k$

From Section 6 it follow that if  $c_n = k$  then the polynomials

$$\sum_{i=0}^n (i+1)^k x^i$$

are completely real.

**7.3.**  $p = \prod_{i=1}^n (i^k x + 1), k \in \mathbb{N}$

These polynomials in  $\mathbb{Z}[x]$  are designed to have  $n$  real roots  $-i^{-k}$ ,  $i = 1, 2, \dots, n$ . It turns out that for the pairs  $(n, k)$  in Table 2 computations show that  $b_i = i$  for  $i = 0, 1, \dots, n$ . In a pair  $(n, k)$ ,  $k$  is the smallest such that works for  $n$ .

We know of no reason that the table cannot be continued indefinitely.

**7.4. Eulerian polynomials**

The Eulerian numbers ([12]) are given by

$$A(n, m) = \sum_{k=0}^{m+1} ((-1)^k \binom{n+1}{k}) (m+1-k)^n$$

and the Eulerian polynomials ([12]) are

$$A_n(x) = \sum_{m=0}^{n-1} A(n, m) x^m$$

n	k	n	k	n	k
1	0	16	17	31	34
2	0	17	18	32	35
3	2	18	19	33	36
4	4	19	20	34	37
5	5	20	21	35	39
6	6	21	22	36	40
7	7	22	24	37	41
8	8	23	25	38	42
9	9	24	26	39	43
10	10	25	27	40	44
11	11	26	28	41	45
12	12	27	29	42	47
13	14	28	30	43	48
14	15	29	32	44	49
15	16	30	33	45	50

Table 2:

Direct computer calculations for  $n$  from 0 to 9 show that  $A_n(x)$  is completely real. But for  $n > 9$  for those  $n$  tested,  $A_n(x)$  has a few  $i$ -sections which have non-real roots. However as mentioned in Stanley ([10]),  $A_n(x)$  has only real roots. Note that from [12] the Eulerian polynomials  $A_n(x)$  are connected to the generating function of the sequence  $a_i = (i + 1)^k$  by

$$\sum_{i=0}^{\infty} (i + 1)^k x^i = \frac{A_k(x)}{(1 - x)^{k+1}}.$$

which may help to explain some of the behavior in Section 6.

## 8. Conclusion

We leave the proofs of the values of  $b = RR(a)$  suggested here and the computation of  $b$  for the remainder of the 345751<sup>1</sup> sequences  $a$  in *The On-Line Encyclopedia of Integer Sequences* ([8]) to the interested reader.

## 9. Acknowledgment

I wish to thank Mourad Ismail for steering me to relevant papers in analysis and for obtaining a copy of the difficult to locate paper by Ostrovskii [7]. I also wish to

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<sup>1</sup>Number of sequences as of July 4, 20:48 EDT, 2021.

thank Larry Dunning for providing me with a copy of the paper by Kobel, Lovava and Sagraloff. ([4]).

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