On Simultaneous Block-Diagonalization of Cyclic Commuting Matrices

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Abstract
We study simultaneous block-diagonalization of cyclic $d$-tuples of commuting matrices. Some application to ideal projectors are also presented. In particular we extend Hans Stetter’s theorem characterizing Lagrange projectors.

1 Introduction

Let $V$ be a finite-dimensional space over complex field $\mathbb{C}$ and let $L := (L_1, ..., L_d)$ be a $d$-tuple of pairwise commuting operators on $V$. Every polynomial $p(x_1, ..., x_d) = \sum c_{k_1, ..., k_d} x_1^{k_1} ... x_d^{k_d} \in \mathbb{C}[x_1, ..., x_d]$ defines an operator

$$p(L) := \sum c_{k_1, ..., k_d} L_1^{k_1} ... L_d^{k_d}$$

on $V$. A $d$-tuple $L$ is called cyclic if there exists a vector $v_0 \in V$ such that

$$\{p(L)v_0, p \in \mathbb{C}[x_1, ..., x_d]\} = V. \quad (1.2)$$

A vector $v_0$ satisfying (1.2) is called a cyclic vector for $L$.

A vector $v \in V$ is called a common eigenvector for $L$ if for all $j = 1,...,d$ there exist $\lambda_j \in \mathbb{C}$ such that $L_j v = \lambda_j v$. The $d$-tuple $\lambda = (\lambda_1, ..., \lambda_d) \in \mathbb{C}^d$ is called an eigentuple of $L$. The set of all eigentuples of $L$ is denoted by $\sigma(L)$.

In case $d = 1$, an operator $L$ is cyclic if and only if $L$ is 1-regular, i.e., every eigenspace of $L$ is at most one-dimensional. For $d > 1$ this is false in both directions as the following example (already used in [4] for different purposes) demonstrates:
Example 1.1 First consider \( L = (L_1, L_2) \) on \( \mathbb{C}^3 \) given by

\[
L_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad
L_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\] (1.3)

This is a cyclic \( d \)-tuple, \( \sigma(L) = \{(0, 0)\} \) and vectors \((0, 1, 0)\) and \((0, 0, 1)\) are common eigenvectors for \( L \). On the other hand \( L^t = (L^t_1, L^t_2) \) is given by

\[
L^t_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad
L^t_2 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\] (1.4)

is not cyclic, yet the only common eigenspace is one-dimensional, spanned by the vector \( v = (1, 0, 0) \).

Observe that \( L \) is 1-regular means that the Jordan form of \( L \) does not contain two Jordan blocks with the same eigenvalue. In other words, the number of Jordan blocks in the Jordan form of \( L \) is precisely the same as the number of distinct eigenvalues of \( L \): \( \#(\sigma(L)) \).

The main result of this paper (Theorem 2.6) shows that a cyclic \( d \)-tuple \( L \) of \( d \) commuting operators is simultaneously block-diagonalizable into \( \#(\sigma(L)) \) blocks and \( \#(\sigma(L)) \) is the largest number of blocks in any simultaneous block-diagonalization of \( L \). The converse is still false. Indeed, the pair \( L = (L_1, L_2) \) in the preceding example can be decomposed into exactly as many blocks as the pair \( L^t = (L^t_1, L^t_2) \).

Definition 1.2 Let \( L := (L_1, ..., L_d) \) be a \( d \)-tuple of operators on \( V \). A direct sum decomposition

\[
V = V_1 \oplus V_2 \oplus ... \oplus V_t
\] (1.5)
is \( L \)-invariant if each subspace \( V_k \), \( k = 1, ..., t \) is an invariant subspace for each of the operators \( L_j \), \( j = 1, ..., d \).

Letting \( L_{j,k} := L_j \mid_{V_k} \) denote the restriction of \( L_j \) onto \( V_k \) we write

\[
L_k = L \mid_{V_k} := (L_{1,k}, ..., L_{d,k}).
\] (1.6)

The simultaneous block-diagonalization of \( L \) into \( t \) blocks amounts to nothing more then the \( L \)-invariant decomposition (1.5) of \( V \): Indeed, for an appropriately chosen bases, the matrix \( \tilde{L}_j \) of \( L_j \) can be written in a block-diagonal form

\[
\tilde{L}_j = \begin{bmatrix}
\tilde{L}_{j,1} & 0 & \cdots & 0 \\
0 & \tilde{L}_{j,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \tilde{L}_{j,t}
\end{bmatrix},
\]

where each \( \tilde{L}_{j,k} \) is a square matrix of size \( \text{dim} V_k \) representing operator \( L_{j,k} \) on \( V_k \). We write \( L = \text{diag}(L_k) \).
Thus our main theorem shows that for a cyclic $d$-tuple $L$ of commuting operators on $V$, the maximal number $t$ in an $L$-invariant decomposition (1.5) of $V$ is exactly the same as $\#\sigma(L)$.

In Section 3 we apply this theorem to study decompositions of ideal projectors and associated multiplication operators. In particular, we extend Stetter’s characterization of Lagrange projectors (cf. [10], [2]) to general ideal projectors.

We use the rest of this section to recall a few well-known facts from commutative algebra (cf. [5], [6]):

For every ideal $J \subset \mathbb{C}[x_1, \ldots, x_d]$ we use $Z(J)$ to define the associated variety

$$Z(J) := \{ z \in \mathbb{C}^d : p(z) = 0, \text{ for all } p \in J \}$$

in $\mathbb{C}^d$. The ideal $J$ is called zero-dimensional if $\dim \mathbb{C}[x_1, \ldots, x_d]/J < \infty$. An ideal $J$ is zero-dimensional if and only if the set $Z(J)$ is finite. In fact $\#Z(J) \leq \dim \mathbb{C}[x_1, \ldots, x_d]/J$. An ideal $J$ is radical if $f^m \in J$ for some integer $m$ implies $f \in J$. A zero-dimensional ideal $J$ is radical if and only if $\#Z(J) = \dim \mathbb{C}[x_1, \ldots, x_d]/J$. An ideal $J \subset \mathbb{C}[x_1, \ldots, x_d]$ is called primary if $fg \in J$ implies $f \in J$ or $g^m \in J$ for some integer $m$. A zero-dimensional ideal is primary if and only if the variety $Z(J)$ consists of a single point in $\mathbb{C}^d$. The Lasker-Noether theorem, applied to zero-dimensional ideals, states that every zero-dimensional ideal has a unique minimal primary decomposition, that is

$$J = \bigcap_{k=1}^{\#Z(J)} J_k$$

where each $J_k$ is a primary zero-dimensional ideal and $Z(J_k) \cap Z(J_n) = \emptyset$ for $n \neq k$.

2 Cyclic $d$-tuples.

Let $V$ be a finite-dimensional space over the complex field $\mathbb{C}$ and let $L := (L_1, \ldots, L_d)$ be a $d$-tuple of pairwise commuting operators on $V$. A $d$-tuple $L$ defines an ideal

$$J_L := \{ p \in \mathbb{C}[x_1, \ldots, x_d] : p(L) = 0 \} \subset \mathbb{C}[x_1, \ldots, x_d].$$

(2.1)

Proposition 2.1. The ideal $J_L$ is zero-dimensional, hence $Z(J_L)$ is finite.

Proof. Let $\mathcal{L}(V)$ be the space of all linear operators on $V$. Since $V$ is finite dimensional, so is $\mathcal{L}(V)$. Let $\phi_L : \mathbb{C}[x_1, \ldots, x_d] \rightarrow \mathcal{L}(V)$ be a mapping defined by

$$\phi_L(p) := p(L) \in \mathcal{L}(V).$$

Since $\ker \phi_L = J_L$, the factorization

$$\begin{array}{ccc}
\mathbb{C}[x_1, \ldots, x_d] & \xrightarrow[\alpha]{} & \mathcal{L}(V) \\
\downarrow \phi_L & & \downarrow \beta \\
\mathbb{C}[x_1, \ldots, x_d]/J_L
\end{array}$$

is an isomorphism.
induces an injection $\beta$ into a finite-dimensional space. Thus $\dim \mathbb{C}[x_1, \ldots, x_d]/J_L < \infty$ and hence $\#Z(J_L) < \infty$. ■

The following proposition collects a few simple and well-known properties of commuting $d$-tuples of operators. The proofs are given purely for convenience.

**Proposition 2.2** Let $L$ be a $d$-tuple of pairwise commuting operators on $V$. Then

(i) $L$ has a common eigenvector, i.e., $\sigma(L) \neq \emptyset$.

(ii) If $v \in V$ is a common eigenvector for $L$ that corresponds to an eigentuple $\lambda$ then $p(L)v = p(\lambda)v$

(iii) If $\lambda \in \sigma(L)$ and $p \in J_L$ then $p(\lambda) = 0$, i.e., $\sigma(L) \subset Z(J_L)$.

(iv) If $L$ is cyclic and $v_0$ is a cyclic vector for $L$ then

$$J_L = \{ p \in \mathbb{C}[x_1, \ldots, x_d] : p(L)v_0 = 0 \}.$$  \hfill (2.3)

**Proof.** (i) By induction. For $d = 1$ the statement is obvious. Let $(\lambda_1, \ldots, \lambda_{d-1}) \in \mathbb{C}^{d-1}$ be an eigentuple for $(L_1, \ldots, L_{d-1})$. Then the subspace $H := \cap_{j=1}^{d-1} \ker(L_j - \lambda_j I) \subset V$ is non-zero and invariant with respect to $L_d$. Indeed if $h \in H$ then $(L_j - \lambda_j I)L_d h = L_d(L_j - \lambda_j I)h = 0$, hence $L_d h \in H$ and any eigenvector of $L_d$ in $H$ is a common eigenvector for $(L_1, \ldots, L_d)$.

(ii) Follows from applying $p(L)$ in the form (1.1) to $v$.

(iii) Let $v \in V$ is a common eigenvector for $L$ that corresponds to an eigentuple $\lambda \in \sigma(L)$. For any $p \in J_L$ we have $0 = p(L)v = p(\lambda)v$ and, since $v \neq 0$, $p(\lambda) = 0$.

(iv) Assume that $p(L)v_0 = 0$ and $v \in V$. Then, by cyclicity, there exists a polynomial $q \in \mathbb{C}[x_1, \ldots, x_d]$ such that $v = q(L)v_0$. We have $p(L)v = p(L)q(L)v_0 = q(L)p(L)v_0 = 0$. Hence $p(L)v = 0$ for all $v \in V$ and $p \in J_L$. ■

**Remark 2.3** In the Theorem 2.6 below we will show that $\sigma(L)$ is actually equal to $Z(J_L)$.

The next lemma is the key to our analysis of $L$-invariant decomposition of $V$:

**Lemma 2.4** Let $L := (L_1, \ldots, L_d)$ be a cyclic $d$-tuple of pairwise commuting operators on $V$. Let $V = V_1 \oplus V_2$ be an $L$-invariant decomposition of $V$. Then $L_k := L | V_k$ ($k = 1, 2$) is cyclic on $V_k$ and $\sigma(L_1) \cap \sigma(L_2) = \emptyset$. In other words common eigenvectors for $L$ in $V_1$ and $V_2$ correspond to different eigentuples.

**Proof.** Let $v_0$ be a cyclic vector for $L$ and let $v_0 = v'_0 + v''_0$ with $v'_0 \in V_1$ and $v''_0 \in V_2$. Then clearly $v'_0$ is a cyclic vector for $L_1$ and $v''_0$ is a cyclic vector for $L_2$. Let $p \in \mathbb{C}[x_1, \ldots, x_d]$ be such that $p(L)v_0 = v'_0$. Then

$$v'_0 = p(L)v_0 = p(L)(v'_0 + v''_0) = p(L)v'_0 + p(L)v''_0,$$

hence

$$(1 - p)(L)v''_0 = p(L)v''_0.$$
Since \( V_1 \cap V_2 = \{0\} \) and \( V_1 \) and \( V_2 \) are invariant with respect to \( L \), it follows that 
\[(1 - p)(L)v_0 = p(L)v_0 = 0. \text{ But } (1 - p)(L)v_0 = (1 - p)(L_1)v_0 = 0 \text{ and } p(L)v_0 = p(L_2)v_0 = 0. \] Hence, by Proposition 2.2 (iv), \((1 - p) \in J_{L_1} \) and \( p \in J_{L_2} \). Now, if \( \lambda \in \sigma(L_1) \cap \sigma(L_2) \) then, by Proposition 2.2 (iii), \( p(\lambda) = 0 \) and \( (1 - p)(\lambda) = 0 \) which is clearly not possible. 

**Remark 2.5** The converse does not hold. Since the operators \( L^t = (L_1^t, L_2^t) \) from Example 1.1 have (vacuously) the following property: For any decomposition of \( C^3 = V_1 \oplus V_2 \) into \( L \)-invariant subspaces, \( V_1 \) and \( V_2 \) cannot each have an eigenvector that correspond to the same eigenvalue.

**Theorem 2.6** Let \( L := (L_1, ..., L_d) \) be a cyclic \( d \)-tuple of pairwise commuting operators on \( V \). Let 
\[ V = V_1 \oplus V_2 \oplus ... \oplus V_l \tag{2.4} \]
be an \( L \)-invariant decomposition of \( V \). Then
\begin{enumerate}
  
  \item \( l \leq \#\sigma(L) \leq \#Z(J_L) \).
  
  \item There exists an \( L \)-invariant decomposition of \( V \):
  \[ V = V_1 \oplus V_2 \oplus ... \oplus V_m \tag{2.5} \]
  with \( m = \#Z(J_L) \).
  
  \item In particular, \( \sigma(L) = Z(J_L) \) and \( m = \#\sigma(L) = \#Z(J_L) \) is a maximal number of subspaces in any \( L \)-invariant decomposition of \( V \).
  
  \item The decomposition (2.5) with \( m = \#Z(J_L) \) is unique and \( \sigma(L_k) \), where \( L_k := L | V_k \), is a singleton.
\end{enumerate}

**Proof.** The first inequality in (i) follows from Lemma 2.4 by pigeonhole principle, the second from Proposition 2.2 (iii).

To prove the second statement of the theorem, let \( Z(J_L) = \{z_1, ..., z_m\} \subset C^d \) and let 
\[ J_L = \cap_{k=1}^m J_k \tag{2.6} \]
be the primary decomposition of \( J_L \), i.e., each \( J_k \) is a primary ideal with \( Z(J_k) = \{z_k\} \). We use \( J^{(k)} \) to denote the ideal \( (\cap_{s \neq k} J_s) \). Let \( v_0 \) be a cyclic vector for \( L \) and define 
\[ V_k = \{p(L)v_0, p \in J^{(s)}\}. \tag{2.7} \]
We first claim each \( V_k \) is an invariant subspace for each \( L_j \). Indeed \( L_j p(L)v_0 = (x_j p)(L)v_0 \) and if \( p \in J^{(s)} \), so is \( x_j p \) thus showing that \( L_j p(L)v_0 \in V_k \).

Now if \( k \neq l \) and \( v \in V_k \cap V_l \) we have \( v = p(L)v_0 = q(L)v_0 \) with \( p \in (\cap_{s \neq k} J_s) \) and \( q \in (\cap_{s \neq l} J_s) \). We have \( (p(L) - q(L))v_0 = 0 \) implying (by proposition 2.2 (iv)) \( (p(L) - q(L))v = 0 \) hence \( p - q \in J_L = \cap_{k=1}^m J_k \). In particular \( p - q \in J^{(k)} \) and so \( p \in J^{(k)} \), so \( v \in V_k \cap V_l \). Thus \( q \in J^{(k)} \cap J^{(l)} = J_L \) and, by definition of \( J_L \), \( q(L) = 0 \) implying \( v = q(L)v_0 = 0 \). This shows that \( V_k \cap V_l = \{0\} \). It remains to prove that 
\[ V_1 + V_2 + ... + V_m = V \tag{2.8} \]
Let $h_k \in \mathbb{C}[x_1, \ldots, x_d]$ be such that
\[ h_k(z_s) = \delta_{k,s}. \] (2.9)

Since $h_k$ is equal to zero for each point of $Z(J^{(k)})$, by Hilbert’s Nullstellensatz, there exists an integer $n$ such that $h_k^n \in J^{(k)}$ for all $k$. From (2.9)
\[ (1 - \sum_{k=1}^{m} h_k^n)(z_s) = 0 \text{ for all } s = 1, \ldots, m \]
and again, by the Nullstellensatz, there exists an integer $l$ so that $(1 - \sum_{k=1}^{m} h_k^n)^l \in J_L$, thus $(1 - \sum_{k=1}^{m} h_k^n)^l(L) = 0$. Expanding $(1 - \sum_{k=1}^{m} h_k^n)^l$ we obtain
\[ (1 - \sum_{k=1}^{m} h_k^n)^l = 1 - \sum_{k=1}^{m} h_k^n p_k \in J_L \]
for some polynomials $p_k \in \mathbb{C}[x_1, \ldots, x_d]$ ($p_k$ are polynomials in $h_k^n$). Hence $I = \sum_{k=1}^{m} h_k^n p_k(L)$ and every $v \in V$ has a decomposition $v = \sum_{k=1}^{m} h_k^n p_k(L)v$. Since $v = f(L)v_0$ for some $f \in \mathbb{C}[x_1, \ldots, x_d]$, we obtain
\[ v = \sum_{k=1}^{m} h_k^n p_k(L)f(L)v_0 = \sum_{k=1}^{m} (h_k^n p_k f)(L)v_0. \]

Since $h_k^n \in J^{(k)}$, it follows that $(h_k^n p_k f) \in J^{(k)}$ and $(h_k^n p_k f)(L)v_0 \in V_k$ thus proving (2.8) and (ii).

(iii) follows immediately from parts (i) and (ii) of the theorem.

To prove (iv), suppose that
\[ V = U_1 \oplus U_2 \oplus \ldots \oplus U_m \] (2.10)
is an $L$-invariant decomposition with $m = \#Z(J_L)$ and let $L_k$ be the restriction of $L$ to $U_k$. The ideal $J_{L_k}$ is primary, for otherwise the primary decomposition of this ideal would lead to an $L_k$-invariant (hence $L$-invariant) decomposition of $U_k$ and thus decomposition of $V$ into more then $m$ subspaces. This would contradict part (i) of the theorem. It is now easy to check that
\[ J_L = \cap_{k=1}^{m} J_{L_k} \] (2.11)
is the primary decomposition of $J_L$, hence coincides with decomposition (2.6).
Without loss of generality, let $J_{L_k} = J_k$ for all $k = 1, \ldots, m$. Let
\[ v_0 = u_0^{(1)} + \ldots + u_0^{(m)} \] (2.12)
be the decomposition of the cyclic vector \( v_0 \) according to (2.9). For every \( v \in V \) there exists a polynomial \( p \in \mathbb{C}[x_1, \ldots, x_d] \) such that

\[
v = p(L)v_0 = p(L)u_0^{(1)} + \ldots + p(L)u_0^{(m)} = p(L_1)u_0^{(1)} + \ldots + p(L_m)u_0^{(m)}.
\] (2.13)

It follows from (2.10) that \( v \in U_k \) if and only if

\[
p(L_s)u_0^{(s)} = 0 \quad \text{for all } s \neq k.
\] (2.14)

But, clearly, \( u_0^{(k)} \) is a cyclic vector for \( L_k \), hence (2.14) is equivalent to \( p \in \cap_{p \neq k}(J_{L_k}) = J^{(k)} \). Thus \( p(L)v_0 \in U_k \) if and only if \( p \in J^{(k)} \) and \( U_k = V_k \). ■

Let us finish this section with another observation on cyclic commuting d-tuples:

For a \( d \)-tuple \( L := (L_1, \ldots, L_d) \) of pairwise commuting operators on \( V \) define \( C(L) \) to be the set of all operators that commute with every operator \( L_1, \ldots, L_d \). In case \( d = 1 \), an operator \( L \) is cyclic if and only if every operator in \( C(L) \) is a polynomial in \( L \):

\[
C(L) = \{ p(L), \ p \in \mathbb{C}[x_1] \}.
\]

**Theorem 2.7** Let \( L := (L_1, \ldots, L_d) \) be a cyclic \( d \)-tuple of pairwise commuting operators on \( V \). If \( T \in C(L) \) then \( T = q(A) \) for some \( q \in \mathbb{C}[x_1, \ldots, x_d] \). The converse does not hold. The \( d \)-tuple \( L^t := (L_1^t, L_2^t) \) defined in (1.4) is not cyclic, yet

\[
C(L^t) = \{ p(L^t), \ p \in \mathbb{C}[x_1, x_2] \}.
\]

**Proof.** Assume that \( L \) is cyclic and let \( v_0 \) be a cyclic vector for \( L \). If \( T \in C(L) \), let \( q \in \mathbb{C}[x_1, \ldots, x_d] \) be a polynomial such that \( T v_0 = q(L)v_0 \). We claim that \( T = q(L) \). Indeed, let

\[
\{v_j = f_j(L)v_0, \ j = 1, \ldots, N\}
\]

be a basis for \( V \). Then

\[
q(L)v_j = q(L)f_j(L)v_0 = f_j(L)q(L)v_0 = f_j(L)T v_0 = T f_j(L)v_0 = T v_j
\]

for every \( j = 1, \ldots, N \), which shows that \( q(L) = T \).

As to the converse, let \( T \) commutes with \( L_1^t \) and \( L_2^t \) from example ???. Then \( T^t \) commutes with \( L_1 \) and \( L_2 \). By the first part of the theorem, there exists a polynomial \( q \in \mathbb{C}[x_1, x_2] \) such that \( T^t = q(L_1, L_2) \). Hence \( T = q(L_1^t, L_2^t) \). ■

### 3 Decomposition of Ideal projectors

In this section we use Theorem 2.6 to extend Stetter’s characterization of Lagrange projectors(cf. [10], [2]) to general ideal projectors acting in the space \( \mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_d] \) of polynomials in \( d \) variables.

**Definition 3.1** (cf. [1]) A linear idempotent map \( P \) on \( \mathbb{C}[x] \) is called an **ideal projector** if \( \ker P \) is an ideal in \( \mathbb{C}[x] \).
Theorem 3.2 (de Boor [2]) A linear operator $P$ on $\mathbb{C}[x]$ is an ideal projector if and only if
\begin{equation}
P(fg) = P(fPg)
\end{equation}
for all $f, g \in \mathbb{C}[x]$.

A standard example of an ideal projector onto an $N$-dimensional subspace $V \subset \mathbb{C}[x]$ is a Lagrange projector, i.e., a linear projector $P$ for which $Pf$ is the unique element in $V$ such that $f(z_k) = Pf(z_k)$, $j = 1, \ldots, N$ for some set $\{z_1, \ldots, z_N\}$ of $N$ distinct points in $\mathbb{C}^d$. In this case the ideal $\ker P$ is a radical ideal, its associated variety
\[ Z(\ker P) := \{z \in \mathbb{C}^d : f(z) = 0, \ \forall f \in \ker P\} = \{z_1, \ldots, z_N\}. \]
The minimal primary decomposition for the ideal $\ker P$ is
\[ \ker P = J_1 \cap J_2 \cap \ldots \cap J_N \]
where each $J_j$ is a maximal ideal $J_j = \{f \in \mathbb{C}[x] : f(z_j) = 0\}$.

Every ideal projector $P$ onto $V$ generates a $d$-tuple $M_P = (M_1, \ldots, M_d)$ of $d$ multiplication operators on $V$ defined by
\[ M_j(v) := P(x_jv) \]
for every $v \in V$. The $d$-tuple $M_P$ is a cyclic $d$-tuple of pairwise commuting operators on $V$ (cf. [2]) and
\[ \{p(M_1, \ldots, M_d)v_0, \ p \in \mathbb{C}[x]\} = V \]
with $v_0 := P1 \in V$. Some insight into the relation between $P$ and $M_P$ is shed by a beautiful observation of Stetter [10] (cf. also [2], [4], [6]):

Theorem 3.3 The ideal projector $P$ is a Lagrange projector if and only if $M_1, \ldots, M_d$ are simultaneously diagonalizable, i.e., there exists a basis $\{v_1, \ldots, v_N\}$ in $V$ consisting of common eigenvectors of $M_j$ such that:
\[ M_jv_k = z_{j,k}v_k, \ j = 1, \ldots, d, \ k = 1, \ldots, N \]
for some $z_{j,k} \in \mathbb{C}$. In this case the projector $P$ interpolates at sites $z_k := (z_{j,k}, \ j = 1, \ldots, d) \in \mathbb{C}^d$ and the eigenvectors $v_k$ are the fundamental polynomials of Lagrange interpolation, i.e., $v_k(z_s) = 0$ if $k \neq s$.

Normalizing $v_k$ in the above theorem so that $v_k(z_s) = \delta_{k,s}$ we can write the projector $P$ as
\[ P = \sum_{k=1}^N P_k \]
where each $P_k$ is a one-dimensional Lagrange projector defined by $P_kf = f(z_s)v_k$ satisfying the orthogonality relations:
\[ P_kP_s = 0 \text{ if } k \neq s. \]

As an immediate application of the Theorem 2.6, we obtain the following generalization of the Stetter’s theorem to arbitrary ideal projectors:
Theorem 3.4 Let $P$ be an ideal projector onto the $N$-dimensional subspace $V$. Let
\[ \ker P = J_1 \cap J_2 \cap \ldots \cap J_m, \quad m \leq N \]
be the minimal primary decomposition of $\ker P$. Then
(i) $M_P$ has a unique (up to order of blocks) block diagonalization $M_P = \text{diag}(M_k)$ consisting of $m$ blocs and $m$ is a maximal number of blocks in any block-diagonalization of $M_P$.
(ii) Each block $M_k$ defines a distinct prime ideal
\[ I_k = \{ p \in \mathbb{C}[x] : p(M_k) = 0 \} \]
and
\[ \ker P = I_1 \cap I_2 \cap \ldots \cap I_m \]
is the minimal primary decomposition of the $\ker P$.

Remark 3.5 If $P$ is an ideal projector, $M_P = (M_1, \ldots, M_d)$ and the operators $M_j$ are simultaneously diagonalizable, then the number of blocks $m = N$ is clearly maximal, hence we obtain the Stetter’s theorem.

Let us illustrate this theorem on a simple example:

Example 3.6 Let $P$ be an ideal projector from $\mathbb{C}[x,y]$ onto its subspace $V := \text{span}\{1,x,y\}$ such that $(Pf)(0,0) = f(0,0), \quad \frac{\partial}{\partial x}(Pf)(0,0)) = \frac{\partial}{\partial x}(f)(0,0)), \quad (Pf)(0,1) = f(0,1)$. It is easy to check that $Px^2 = 0, \quad Pxy = 0$ and $Py^2 = y$. Hence the two multiplication operators are
\[
M_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]
Let $S = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}$, hence $S^{-1} = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & -1 & 1
\end{bmatrix}$. Then
\[
SM_1 S^{-1} = \begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]
and
\[
SM_2 S^{-1} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]
is a simultaneous block-diagonalization of $M_1$ and $M_2$ consisting of two blocks corresponding to an $M_P$-invariant decomposition
\[ V = \text{span}\{1,x\} \oplus \text{span}\{y\}. \]
This is the maximal $M_P$-invariant decomposition and thus a maximal block-diagonalization.
We conclude this paper by discussing the relationship between ideal decomposition of an ideal projector $P$ onto the space $V$ and the $M_P$-invariant decomposition of $V$.

**Definition 3.7** Let $P$ be an operator from $\mathbb{C}[x]$ onto its subspace $V$. We say that

$$P = \sum_{k=1}^{t} P_k$$

(3.2)
is an ideal decomposition of $P$ if each $P_k$, $(k = 1, ..., t)$ is an ideal projector and

$$P_k P_s = 0 \text{ if } k \neq s.$$ 

(3.3)

**Theorem 3.8** If (3.2) is an ideal decomposition of $P$ then $P$ is an ideal projector and

$$V = \oplus_{k=1}^{t} (\text{ran } P_k)$$

(3.4)
is an $M_P$-invariant decomposition of $V$.

Conversely, if $P$ is an ideal projector onto $V$ and

$$V = V_1 \oplus V_2 \oplus ... \oplus V_t$$

(3.5)
is an $M_P$-invariant decomposition of $V$ then it generates an ideal decomposition (3.2) of $P$ with $\text{ran } P_k = V_k$.

**Proof.** We have

$$P(fg) - P(fPg) = \sum_{k=1}^{t} P_k(fg) - \sum_{k=1}^{t} P_k(f \sum_{s=1}^{t} P_s g)$$

by (3.1)

$$= \sum_{k=1}^{t} P_k(fg) - \sum_{k=1}^{t} P_k(f \sum_{s=1}^{t} P_k P_s g)$$

by (3.3)

$$= \sum_{k=1}^{t} P_k(fg) - \sum_{k=1}^{t} P_k(f) P_k g$$

by (3.1)

$$= \sum_{k=1}^{t} P_k(fg) - \sum_{k=1}^{t} P_k(fg) = 0$$

and by Theorem 3.2, $P$ is an ideal projector. Decomposition (3.4) easily follows from (3.2) and (3.3). It remains to show that decomposition (3.4) is $M_P$-invariant. Let $f \in (\text{ran } P_k)$. Then

$$M_j f := P(x_j f)$$

by (3.1)

$$= P(x_j P f)$$

by (3.3)

$$= P(x_j P_k f)$$

$$= \sum_{s=1}^{t} P_s(x_j P_k f)$$

by (3.1)

$$= \sum_{s=1}^{t} P_s(x_j P_s P_k f)$$

by (3.3)

$$= P_k(x_j P_k f)$$

and $M_j f \in (\text{ran } P_k)$.

Conversely, let $P$ be an ideal projector onto $V$ and suppose that (3.5) is an $M_P$-invariant decomposition of $V$. Then

$$M_j (g) = P(x_j g) \in V_k$$

for every $g \in V_k$.
and thus
\[ P(fg) \in V_k \text{ for every } g \in V_k. \]

Let \( Q_k \) be the projector from \( V \) onto \( V_k \) parallel to \( \oplus_{s \neq k} V_s \) and define \( P_k := Q_k P \). We have
\[ I_V = \sum Q_k \text{ and } Q_k Q_s = 0 \text{ for } k \neq s \quad (3.6) \]
from which (3.2) and (3.3) follows. Clearly \( P_k \) is a projector onto \( V_k \) and we only have left to check that \( \ker P_k \) is an ideal. This follows from the following sequence of implications:

\[
\begin{align*}
& f \in \ker P_k \Rightarrow P f \in \ker Q_k \Rightarrow P f \in \oplus_{s \neq k} V_s \quad \text{by (3.1) and (3.6)} \\
& P(gf) = P(gf) f \in \oplus_{s \neq k} V_s \text{ for every } g \in \mathbb{C}[x] \Rightarrow \\
& Q_k P(gf) = 0 \Rightarrow gf \in \ker P_k.
\end{align*}
\]

This proves the theorem. \( \blacksquare \)

Combining theorems 2.6 and 3.8 we immediately obtain

**Theorem 3.9** Let \( P \) be an ideal projector onto the \( N \)-dimensional subspace \( V \). Let
\[ \ker P = J_1 \cap J_2 \cap \ldots \cap J_m \quad (3.7) \]
be the minimal primary decomposition of \( \ker P \). Then the projector \( P \) has a unique ideal decomposition
\[ P = \sum_{k=1}^m P_k \]
and this decomposition is maximal in the sense that if
\[ P = \sum_{k=1}^t \tilde{P}_k \]
is an ideal decomposition of \( P \) then \( t \leq m \).

**Example 3.10** Let \( P \) be the ideal projector defined in the example 3.6. Define ideal projectors \( P_1 \) onto \( \text{span}\{1,x\} \) and \( P_2 \) onto \( \text{span}\{y\} \) by requiring \((P_1 f)(0,0) = f(0,0), \quad \frac{\partial}{\partial x}(P_1 f)(0,0)) = \frac{\partial}{\partial x}(f)(0,0)), \quad (P_2 f)(0,1) = f(0,1)\). Then \( P_1 P_2 = 0 \) and \( P = P_1 + P_2 \) is the maximal ideal decomposition of \( P \).

**Remark 3.11** The existence (but not uniqueness or maximality) of ideal decomposition (3.2) also follows from the description of ideal projectors in [8], cf. also [3]. Thus the size of the blocks in the maximal block-diagonalization of \( M_P \) is the multiplicity of zeroes of the corresponding primary ideals in (3.7).
References


