On a Conjecture of Tomas Sauer Regarding Nested Ideal Interpolation

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Abstract
Tomas Sauer conjectured in [4] that if an ideal complements polynomials of degree less than $n$ then it is contained in a larger ideal that complements polynomials of degree less than $n - 1$. We construct a counterexample to this conjecture for polynomials in three variables with $n = 3$.

1 Introduction and Preliminaries

Let $\mathbb{C}[x_1, ..., x_d]$ denote the space of polynomials in $d$ variables with complex coefficients, let $\mathbb{C}_{<n}[x_1, ..., x_d]$ denotes its subspace of polynomials of degree less than $n$ and let $(\mathbb{C}[x_1, ..., x_d])' = \text{Hom}_\mathbb{C}(\mathbb{C}[x_1, ..., x_d], \mathbb{C})$ denotes the algebraic dual of $\mathbb{C}[x_1, ..., x_d]$, i.e., the space of all linear functionals on $\mathbb{C}[x_1, ..., x_d]$.

Definition 1 Let $\Lambda$ be a subspace of $(\mathbb{C}[x_1, ..., x_d])'$ and $E$ be a subspace of $\mathbb{C}[x_1, ..., x_d]$. We say that $\Lambda$ is correct for $E$ if for every $f \in \mathbb{C}[x_1, ..., x_d]$ there exists unique $g \in E$ such that $\lambda(g) = \lambda(f)$ for every $\lambda \in \Lambda$.

With every subspace $\Lambda \subset (\mathbb{C}[x_1, ..., x_d])'$ we associate a subspace ker $\Lambda \subset \mathbb{C}[x_1, ..., x_d]$ defined by

$$\text{ker } \Lambda := \{f \in \mathbb{C}[x_1, ..., x_d] : \lambda(f) = 0 \text{ for all } \lambda \in \Lambda\}.$$ 

The purpose of this note is to construct a counterexample to the following conjecture of Tomas Sauer:

Conjecture 2 ([4], Conjecture 4.1): Let $\Lambda$ be a subspace of $(\mathbb{C}[x_1, ..., x_d])'$ such that $\Lambda$ is correct for $\mathbb{C}_{<n}[x_1, ..., x_d]$ and ker $\Lambda$ is an ideal in $\mathbb{C}[x_1, ..., x_d]$. Then there exists a subspace $\Lambda_0 \subset \Lambda$ such that $\Lambda_0$ is correct for $\mathbb{C}_{<n-1}[x_1, ..., x_d]$ and ker $\Lambda$ is an ideal in $\mathbb{C}[x_1, ..., x_d]$. 

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Some specific spaces \( \Lambda \) for which the conjecture is valid can be found in [5] and [6]. In particular if \( \ker \Lambda \) is the radical ideal then the conjecture is verified. If \( \ker \Lambda \) is radical, then the associated variety consists of the maximal possible number of points.

The counterexample, presented in the next section, is constructed in the space of polynomials of three variables with \( n = 3 \). The corresponding ideal \( \ker \Lambda \) is primary, that is the other extreme, the variety of \( \ker \Lambda \) consists of unique point.

We will use the rest of this section to recall some well-known facts regarding duality (apolarity, inverse systems) for \( \mathbb{C}[x_1, \ldots, x_d] \).

We will identify the space \( (\mathbb{C}[x_1, \ldots, x_d])^0 \) with the space \( \mathbb{C}[[x_1, \ldots, x_d]] \) of formal power series as follows:

With every element \( F(x_1, \ldots, x_d) \in \mathbb{C}[[x_1, \ldots, x_d]] \) we associate a differential operator \( \hat{F} \in (\mathbb{C}[x_1, \ldots, x_d])' \) by formally replacing variables in \( F \) with the appropriate powers of differential operators. For instance, if \( F(x_1, x_2) = x_1^4 + 3x_1^3x_2 + 1 \) then

\[
F(D) = \frac{\partial^4}{\partial x_1^4} + 3 \frac{\partial^3}{\partial x_1^3 \partial x_2} + I.
\]

Now, for every \( F \in \mathbb{C}[[x_1, \ldots, x_d]] \) we define the functional \( \hat{F} \in (\mathbb{C}[x_1, \ldots, x_d])' \) by

\[
\hat{F}(f) := (F(D)f)(0) \quad \text{for every} \quad f \in \mathbb{C}[x_1, \ldots, x_d] \tag{1}
\]

It is well-known (cf. [1], [2] and [3]) that a pairing \( F \mapsto \hat{F} \) defined by (1) is an isomorphism between \( \mathbb{C}[[x_1, \ldots, x_d]] \) and \( (\mathbb{C}[x_1, \ldots, x_d])' \). If \( M \subset \mathbb{C}[[x_1, \ldots, x_d]] \) we use \( \ker M \) to denote the space \( \ker \hat{M} \), i.e.,

\[
\ker M := \{ f \in \mathbb{C}[x_1, \ldots, x_d] : \hat{F}(f) = 0 \text{ for all } F \in M \}.
\]

**Definition 3** A subspace \( M \subset \mathbb{C}[[x_1, \ldots, x_d]] \) is called \( D \)-invariant if for every \( F \in M \)

\[
\partial_{x_j} F \in M \text{ for all } j = 1, \ldots, d.
\]

**Theorem 4** (cf. [1], [2] and [3] in its original form): Let \( M \) be a finite-dimensional subspace of \( \mathbb{C}[[x_1, \ldots, x_d]] \). Then \( \ker M \) is an ideal in \( \mathbb{C}[x_1, \ldots, x_d] \) if and only if \( M \) is \( D \)-invariant.

In this terminology, to construct a counterexample to the Conjecture 2, we need to construct a \( D \)-invariant subspace \( M \subset \mathbb{C}[[x_1, \ldots, x_d]] \) such that \( M \) is correct for \( \mathbb{C}_{<n}[x_1, \ldots, x_d] \) while no \( D \)-invariant subspace \( N \subset M \) is correct for \( \mathbb{C}_{<n-1}[x_1, \ldots, x_d] \).

As a warm-up, consider the following simple example that already gives a counterexample to the Conjecture 5. 14 of [4] in two variables:

**Example 5** Let \( M \) be a subspace of \( \mathbb{C}[[x, y]] \) spanned by four polynomials:

\[
F_1 = 1, F_2 = x, F_3 = x^3 - y, F_4 = x^2.
\]
It is easy to check that $M$ is $D$-invariant and that $M$ is correct for the space
\[ E := \text{span} \{f_1 = 1, f_2 = x, f_3 = y, f_4 = x^2 \} \subset \mathbb{C}[x, y] \]

since $F_j(f_k) = \delta_{j,k}$ for all $j, k = 1, \ldots, 4$. On the other hand no three-dimensional $D$-invariant subspace $N \subset M$ is correct for the space $E := \text{span} \{1, x, y\}$. Indeed, by virtue of being $D$-invariant and three dimensional, $N$ could not contain a polynomial of degree 3 for; if it does, the cosecutive partial derivatives of such polynomial would span a four-dimensional subspace. Hence $N = \text{span}\{1, x, x^2\}$ is the only three-dimensional $D$-invariant subspace of $M$. This space is not correct for $E$ since every functional associated with a polynomial in $N$ vanishes on $y \in E$.

2 Counterexample in three variables:

Expanding on the last example, we will now describe a $D$-invariant 10-dimensional subspace $M \subset \mathbb{C}[\{x, y, z\}]$ that is correct for $\mathbb{C}_{\leq 3}[x, y, z]$, such that no 4-dimensional, $D$-invariant subspace $N \subset M$ is correct for $\mathbb{C}_{\leq 2}[x, y, z]$.

Consider the space $M \subset \mathbb{C}[\{x, y, z\}]$ spanned by 10 polynomials $F_1, \ldots, F_{10}$ given by

\[
F_1 = 1, \ F_2 = x, \ F_3 = \frac{1}{2}x^2, \ F_4 = x^3 + y, \ F_5 = \frac{1}{4}x^4 + xy, \ F_6 = z, \ F_7 = \frac{1}{2}z^2, \ F_8 = \frac{1}{20}x^5 + \frac{1}{2}x^2y + xz, \ F_9 = \frac{1}{2}y^2 + x^3y + \frac{1}{20}x^6 + 3x^2z, \ F_{10} = \frac{1}{140}x^7 + \frac{1}{4}x^4y + zz^3 + \frac{1}{2}xy^2 + zy.
\]

The verification that $M$ is $D$-invariant is by straight-forward computations presented in the table below:

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The polynomials $f_1, \ldots, f_{10}$ defined by

\[
f_1 = 1, \ f_2 = x, \ f_3 = x^2, \ f_4 = y, \ f_5 = xy, \ f_6 = z, \ f_7 = z^2, \ f_8 = xz, \ f_9 = y^2, \ f_{10} = zy,
\]

in this order, form a basis for $\mathbb{C}_{\leq 3}[x, y, z]$ and $\hat{M}$ is correct for $\mathbb{C}_{\leq 3}[x, y, z]$ since $F_j(f_k) = \delta_{j,k}$.

It remains to prove that no $D$-invariant subspace $N \subset M$ is correct for

\[
\mathbb{C}_{\leq 2}[x, y, z] = \text{span} \{1, x, y, z\}.
\]
Proof. Observe that if \( N \) is correct for \( \mathbb{C}_{<2}[x, y, z] \) and \( D \)-invariant then \( N \) is 4-dimensional and thus cannot contain polynomials of degree 4 or larger, since the consecutive partial derivatives of a polynomial of degree \( k \) span a subspace of dimension \( k + 1 \). Thus \( N \) is in the linear span of

\[
F_1 = 1, F_2 = x, F_3 = \frac{1}{2}x^2, F_4 = x^3 + y, F_6 = z, F_7 = \frac{1}{2}z^2.
\] (2)

Now assume that \( N \) is correct for \( \mathbb{C}_{<2}[x, y, z] \). Then \( N \) contains a polynomial \( F \) such that the functional \( \hat{F} \) satisfies

\[
\hat{F}(y) = 1, \quad \hat{F}(0) = \hat{F}(x) = \hat{F}(z) = 0.
\] (3)

The polynomial \( F \in N \) that corresponds to such functional must be of the form

\[
F = F_4 + aF_3 + bF_7
\]

for some constants \( a \) and \( b \), for, if it contains a non-zero summand of any other polynomial in (2), then at least one of the last three equalities in (3) would fail. By \( D \)-invariance,

\[
\partial_x(F) = 3x^2 + ax \in N
\]

and by repeated differentiation, 1 and \( x \) belong to \( N \). Since \( N \) is four-dimensional, it follows that

\[
N = \text{span} \{1, x, x^2, F = x^3 + y + \frac{a}{2}x^2 + \frac{b}{2}z^2\}
\]

and hence every functional that corresponds to a polynomial in this space annihilate \( z \in \mathbb{C}_{<2}[x, y, z] \). This contradict the assumption that \( N \) is correct for \( \mathbb{C}_{<2}[x, y, z] \). \( \blacksquare \)

References


