On Error Formulas for Multivariate Interpolation.

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Abstract

In this paper we prove that the existence of an error formula of a form suggested in [2] leads to some very specific restrictions on an ideal basis that can be used in such formulas. As an application, we provide a negative answer to one version of the question posed by Carl de Boor (cf. [2]) regarding the existence of certain minimal error formulas for multivariate interpolation.

1 Introduction

The various forms of “error formulas” for multivariate interpolation is a popular subject of discussion in the literature (cf. [2]–[5] and [8]–[14]). In particular, a possible algebraic nature of such formulas was suggested in [2], [10], [11] and [12].

In this paper we prove that the existence of error formula of the form suggested in [2] leads to some very specific restrictions on an ideal basis that can be used in such formulas. As an application, we supply a (very) negative answer to one version of the question posed by Carl de Boor (cf. [2]) regarding the existence of certain minimal error formulas for multivariate interpolation. We will need some notations:

The symbol \( \mathbb{F} \) stands for the real or complex field, \( \mathbb{F}^d[x] = \mathbb{F}[x_1, x_2, ..., x_d] \) stands for polynomials of \( d \)-variables (\( \mathbb{F}[x], \mathbb{F}[x, y], \mathbb{F}[x, y, z] \) denote the polynomials of one, two and three variables respectively). An element \( f \in \mathbb{F}^d[x] \) is written as a finite sum \( \sum \hat{f}(k_1, ..., k_d) x_1^{k_1} x_2^{k_2} ... x_d^{k_d} \) or in the multiindex notations \( \sum \hat{f}(\mathbf{k}) \mathbf{x}^\mathbf{k} \) with \( \hat{f}(\mathbf{k}) \in \mathbb{F} \). For a polynomial \( f \in \mathbb{F}^d[x] \), we use

\[
f(D) := \sum \hat{f}(k_1, ..., k_d) \frac{1}{k_1! k_2! ... k_n!} \frac{\partial^{k_1 + k_2 + ... + k_n}}{\partial x_1^{k_1} \partial x_2^{k_2} ... \partial x_d^{k_d}}
\]

to denote the differential operator on \( \mathbb{F}^d[x] \). The space of polynomials of degree less than \( n \) is denoted by \( \mathbb{F}^d_{<n}[x] \), The set of polynomials of degree \( n \) is \( \mathbb{F}^d_n[x] \),
while the space of homogeneous polynomials of degree \( n \) is denoted as \( \mathbb{F}^d_{[n]}[x] \).

Finally the set of monomials of degree \( n \) is \( M^d_{[n]}[x] \) and \( M^d[x] \) is the set of all monomials in \( \mathbb{F}^d[x] \).

Every polynomial \( f \in \mathbb{F}^d[x] \) can be written (uniquely) as a finite sum \( f = \sum f^{[k]} \) with \( f^{[k]} \in \mathbb{F}^d_{[k]}[x] \) being homogeneous component of \( f \). The non-zero homogeneous component that correspond to the largest \( k \) is called the leading term of the polynomial \( f \) and is denoted by \( \text{Lt}(f) \). Hence, the leading term of a polynomial \( f \in \mathbb{F}^d[x] \) is the unique homogeneous polynomial \( \text{Lt}(f) \) such that \( \deg(f - \text{Lt}(f)) < \deg f \). Similarly, the non-zero homogeneous component that correspond to the least \( k \) is the least term of \( f \) and is denoted by \( \text{lt}(f) \).

For every ideal \( J \subset \mathbb{F}[x] \) we use \( Z(J) \) to denote the associated variety

\[ Z(J) = \{ z \in \mathbb{F}^d : f(z) = 0, \forall f \in J \}. \]

The ideal \( J \) is called zero-dimensional (cf. [7]) if

\[ \dim(\mathbb{F}^d[x]/J) < \infty, \]

which implies (and for \( \mathbb{F} = \mathbb{C} \)) the condition that the set \( Z(J) \) is finite.

Likewise, with every set \( Z \subset \mathbb{F}^d \) we associate an ideal

\[ J(Z) := \{ f \in \mathbb{F}^d[x] : f(z) = 0, \forall z \in Z \}. \]

It is easy to see (cf. [6]) that \( J \subset J(Z(J)) \). An ideal \( J \) is called a radical ideal if \( J(Z(J)) = J \). Equivalently (cf. [6]) an ideal \( J \) is a radical ideal if an only if \( f^m \in J \) for some integer \( m \) implies \( f \in J \).

For a subset \( B \subset \mathbb{F}^d[x] \) we use \( \langle B \rangle \) to denote the ideal generated by \( B \). The set \( B \) is called the basis of the ideal \( \langle B \rangle \). By the Hilbert basis theorem, for any ideal \( J \subset \mathbb{F}^d[x] \) there exist finite basis \( B \) such that \( J = \langle B \rangle \). There are several notions of minimal bases. For clarity we will call a basis \( B \) reduced if \( \langle B \rangle \neq \langle B_0 \rangle \) for any proper subset \( B_0 \subset B \). A basis \( B \) is called a minimal basis for an ideal \( J = \langle B \rangle \subset \mathbb{F}^d[x] \) if \( \#B < \#B \) implies that \( B_0 \) is not a basis for \( J \). For an ideal \( J \subset \mathbb{F}^d[x] \) we set

\[ m(J) := \#B, \text{ with } B \text{ being a minimal basis for } J. \]

A basis \( B \) for an ideal \( J \in \mathbb{F}^d[x] \) is called an \( H \)-basis if for every \( f \in J \) there exist \( \{ g_{b,f} \in \mathbb{F}^d[x], b \in B \} \) such that

\[ f = \sum_{b \in B} g_{b,f}b \quad \text{and} \quad \deg g_{b,f} + \deg b \leq \deg f \quad \text{for all } b \in B. \]

**Definition 1.1** (Birkhoff, [1]). Let \( E \) be a subspace of \( \mathbb{F}^d[x] \). A projector \( P \) from \( \mathbb{F}^d[x] \) onto \( E \) is called ideal if \( \ker P \) is an ideal in \( \mathbb{F}[x] \).

The following characterization of ideal projectors is due to de Boor (cf. [2]):
Theorem 1.2 A linear mapping $P : \mathbb{F}^d[x] \rightarrow \mathbb{F}^d[x]$ is an ideal projector if and only if the equality

$$P(fg) = P(fPg)$$

holds for all $f, g \in \mathbb{F}^d[x]$.

The standard example of an ideal projector is a Lagrange projector, i.e., a projector $P$ for which $Pf$ is the unique element in its range that agrees with $f$ at a certain finite set $Z$ in $\mathbb{F}^d$. For its kernel consists of exactly those polynomials that vanish on $Z$, i.e., it is the zero-dimensional radical ideal whose variety is $Z$.

In [2] and [4] Carl de Boor asked for the existence of the error formula of the following form:

$$f(x) - Pf(x) = \sum_{b \in B} b(x)\mu_{b,x}(H_b(D)f),$$

where $P$ is an ideal projector onto $\mathbb{F}_d^d\leq_n[x]$, $B$ is a (minimal) basis for the ideal $\ker P$, $H_b$ is a homogeneous polynomial satisfying

$$H_b(D)c = \delta_{b,c} \text{ for } b, c \in B$$

and $\mu_{b,x}$ is a functional on $\mathbb{F}^d[x]$ that depends on $b$ and $x$, but not on the function $f$.

For $d = 1$ such formulas exist (cf. [2] and [17]) and the minimal basis $B$ consists of one (unique monic) polynomial of degree $n$.

In this paper we will show that (1.1) and (1.2) implies that the sets

$$\{H_b : b \in B\} \text{ and } \{Lt(b) : b \in B\}$$

form (dual) linear bases for the linear space $\mathbb{F}_d^d\leq_n[x]$ of homogeneous polynomials of degree $n$. In particular, this implies that the cardinality of $B$,

$$\#B = N(n) := \binom{n + d - 1}{d - 1},$$

which is the number of monomials of degree $n$ in $\mathbb{F}^d[x]$. Since (as we will show in section 3) for all Lagrange projectors $P$ onto $\mathbb{F}_d^d\leq_n[x]$, there exists a basis $B$ such that $\langle B \rangle = \ker P$ and $\#B = d$, and since

$$\binom{n + d - 1}{d - 1} > d$$

for $d > 1$, hence for these projectors (1.1) and (1.2) can not be valid with minimal $B$.

In the last section we discuss a stronger possibility, that a minimal bases for the kernel of an ideal projector $P$ onto $\mathbb{F}_d^d\leq_n[x]$ admits an error formula of type (1.1), (1.2) if and only if $P$ is the Taylor projector.

We will need an analog of the theorem 1.1 for the projector $P' := I - P$.
Theorem 1.3 A linear mapping $P$ on $\mathbb{F}[x]$ is an ideal projector if and only if

$$P'(fg) = fP'g + P'(fPg), \forall f, g \in \mathbb{F}[x].$$

(1.5)

Proof. We have

$$P'(fg) = fg - P(fg)$$

and

$$fP'g + P'(fPg) = f(g - Pg) + fPg - P(fPg) = fg - P(fPg).$$

Hence (1.5) is equivalent to (1.0). ■

2 The Bases for Error Formulas

We will start with a simple observation:

Lemma 2.1 Let $P$ be an ideal projector onto $\mathbb{F}_{<n}^d[x]$ and let (1.1) holds with $(b) = \ker P$ and $H_b$ is a homogeneous polynomial satisfying (1.2). Then

1) $b(x)H_b, x(1) = b(x)$ for all $b \in B$.

2) The set $B$ is $\mathbb{F}$-linearly independent.

3) $\deg H_b \geq n$ for all $b \in B$.

Proof. Since $b \in \ker P$, hence

$$b(x) = (b - Pb)(x) = b(x)H_b, x(1)$$

by (1.1) and (1.2), which proves 1).

To prove 2), assume that

$$\sum_{b \in B} \alpha_b b = 0$$

for some $\alpha_b \in \mathbb{F}$.

Fix a $b^* \in B$. Then (by linearity of $H_b, x$, 1) and (1.2) we have

$$0 = H_b^*(\sum_{b \in B} \alpha_b b) = \alpha_{b^*}b^* \implies \alpha_{b^*} = 0.$$

Now, suppose that

$$m := \min\{\deg H_b : b \in B\} < n \}.$$ 

and $H^* \in \{H_b : b \in B\}$ be such that $\deg H^* = m$. Then

$$0 \neq H^*(D)H^* \in \mathbb{F}$$

and $\alpha_b := H_b(D)H^* \in \mathbb{F}$ for all $b \in B$.

Since $H^* \in \mathbb{F}_{<n}^d[x]$,

$$0 = H^*(x) - PH^*(x) = \sum_{b \in B} b(x)\mu_{b, x}(H_b(D)H^*) = \sum_{b \in B} \alpha_b b(x)$$

which contradicts 2) thus proves 3). ■

We now proceed with the main theorem of this section.
Theorem 2.2 Let $P$ be an ideal projector onto $\mathbb{P}^d_{\leq n}[x]$ and let (1.1) holds with $(B) = \ker P$ and homogeneous polynomials $H_b$ satisfying (1.2). Then the sets

$$\{H_b : b \in B\} \text{ and } \{Lt(b) : b \in B\}$$

form (dual) linear bases for the linear space $\mathbb{P}^d_{\leq n}[x]$ of homogeneous polynomials of degree $n$. In particular $\#B = N(n)$.

Proof. Let $M^d_n[x]$ be the set of monomials of degree $n$. For every $w \in M^d_n[x]$, let

$$u_w := w - Pw \in \ker P. \quad (2.1)$$

Since $\text{ran}P = \mathbb{P}^d_{< n}[x]$, hence polynomials $\{u_w, w \in W_n\}$ are linearly independent polynomials of degree $n$ and

$$\dim \text{span}\{u_w, w \in M^d_n[x]\} = N(n). \quad (2.2)$$

Now, let $B$ satisfies the assumptions of the theorem. From $\mathbb{P}^d_{< n}[x] \cap (B) = \{0\}$ we conclude that $\deg b \geq n$ and, from the lemma above, $\deg H_b \geq n$ for every $b \in B$. Hence

$$H_b(u_w) =: c_{b,w} \in \mathbb{F}. \quad (2.3)$$

Let

$$\mathcal{H}_n := \{H_b : b \in B, \deg H_b = n\} \text{ and } B_n := \{b \in B : H_b \in \mathcal{H}_n\} \quad (2.4)$$

Since $H \in \mathbb{P}^d_{\leq n}[x]$, $m > n$ implies $H(D)w = 0$ for all $w \in \mathbb{P}^d_{\leq n}[x]$, hence (1.1) implies

$$Pu_w = u_w = \sum_{b \in B_n} b(x)\mu_{b,x}(H_b(D)u_w) = \sum_{b \in B_n} c_{b,w}(x).$$

and from (2.2) and (2.3) we conclude

$$\text{span}\{u_w : w \in M^d_n[x]\} \subset \text{span}\{b : b \in B_n\}$$

and thus by (2.2)

$$N(n) \leq \dim \text{span}\{b : b \in B_n\} = \#B_n, \quad (2.5)$$

where the last equality is by the Lemma 2.1.

Once again from the Lemma 2.1, it follows that $\#B_n \leq \dim \mathbb{P}^d_{\leq n}[x] = N(n)$, hence $\{H_b : b \in B_n\}$ is a basis for $\mathbb{P}^d_{\leq n}[x]$. Now, suppose that $\tilde{b} \in B_n$ is such that $\deg \tilde{b} > n$, then for some

$$f = \sum_{b \in B_n} c_b H_b \in \mathbb{P}^d_{\leq n}[x], c_b \in \mathbb{F}, \quad (2.6)$$

we have

$$\deg \text{lt}(f(D)\tilde{b}) > 0. \quad (2.7)$$
On the other hand, by (1.2)
\[ \text{lt} \left( \sum_{b \in B_n} c_b H_b(D) \tilde{b} \right) = c_k \in \mathbb{F} \]
which contradicts (2.7). In other words for every \( b \in B_n \) we have \( \deg b = n \), which proves that the sets
\[ \{ H_b : b \in B_n \} \text{ and } \{ Lt(b) : b \in B_n \} \]
form linear bases for the linear space \( \mathbb{F}^d_n[x] \).

It remains to show that \( B \setminus B_n = \emptyset \). Indeed if not, then some \( \tilde{b} \in B \setminus B_n \) has \( \deg \tilde{b} > n \) and once again we have (2.7) for some \( f \) satisfying (2.6). On the other hand, from (1.2) we have
\[ \sum_{b \in B_n} c_b H_b(D) \tilde{b} = 0 \]
which gives the desired contradiction.

Let \((w_1, w_2, ..., w_{N(n)})\) be a fixed ordering of monomials in \( M^d_n[x] \). For an arbitrary mapping \( \phi : M^d_n[x] \to \mathbb{F}^d[x] \) we use \( \phi(w) \) to denote the vector \( (\phi(w_1), \phi(w_2), ..., \phi(w_{N(n)})) \in (\mathbb{F}^d[x])^{N(n)} \).

**Corollary 2.3** Let \( P \) be an ideal projector onto \( \mathbb{F}^d_{<n}[x] \) that admits the error formula (2.1),(2.2) for some bases \( B \). Then there exists an \( N(n) \times N(n) \) invertible scalar matrix \( F_P \) such that the elements of \( B \) form a vector \( F_T P (w - Pw) \) and the polynomials in \( \{ H_b : b \in B_n \} \) can be written as a vector \( F_T P w(D) \).

**Corollary 2.4** Let \( P \) be an ideal projector onto \( \mathbb{F}^d_{<n}[x] \) that admits the error formula (2.1),(2.2) for some basis \( B \). Then \( B \) is an \( H \)-basis for \( \ker P \).

**Proof.** It is known (cf. [10]) that a basis \( B \) is an \( H \)-basis if and only if \( \langle Lt(b), b \in B \rangle = \langle Lt(f), f \in \langle B \rangle \rangle \). By Theorem 2.2, if \( B \) admits an error formula, then
\[ \langle Lt(b), b \in B \rangle = \langle M^d_n[x] \rangle . \]
Suppose that for some non-zero \( f \in \langle B \rangle \), we have \( Lt(f) \notin \langle M^d_n[x] \rangle \). Then \( \deg f < n \) and hence \( f \in \langle B \rangle \cap \mathbb{F}^d_{<n}[x] = \ker P \cap \text{ran} P \) which is a contradiction.

**3 Computation of \( m(J) \)**

To fulfill the promise made in the introduction, it remains to observe that for Lagrange projectors onto \( \mathbb{F}^d_{<n}[x] \) there exists a basis \( B \) such that \( \langle B \rangle = \ker P \) and \#\( B = d \). The idea of the proof is very simple. Assume that \( Z(\ker P) = \{ x^{(j)}, j = 1, ..., \dim \mathbb{F}^d_{<n}[x] \} \) and that the first coordinates \( x_1^{(j)} \) of the \( x^{(j)} \) are all distinct. Let
\[ p_1(x) = \prod_{j=1}^{\dim \mathbb{F}^d_{<n}[x]} (x_1 - x_1^{(j)}) \]
and for $k = 2, \ldots, d$, let $p_k(x) \in \text{span}\{1, x_1, \ldots, x_{\text{dim } F^d[x]}\}$ be polynomials that interpolate $x_k$ at the points $\{x^{(j)}, j = 1, \ldots, \text{dim } F^d[x]\}$. Then the polynomials $\{p_1(x), x_k - p_k(x), k = 2, \ldots, d\}$ form a basis for the ideal $\ker P$ of cardinality $d$. The general case is reduced to the above argument by change of variables. Actually, it yields a bit more:

**Theorem 3.1** Let $J \subset F^d[x] = F^d[x_1, x_2, \ldots, x_d]$ be a zero-dimensional radical ideal. Then $m(J) = d$.

**Proof.** Let $\mathcal{H}_d$ be a hyperplane

$$\mathcal{H}_d := \{(z_1, \ldots, z_d) \in F^d : z_d = 0\}.$$ 

With every two distinct points $u, v \in Z(J) \subset F^d$ we associate a unique hyperplane $\mathcal{H}_{u,v} \subset F^d$ orthogonal to the non-zero vector $(u - v) \in F^d$. Since the set $Z(J)$ is finite, there are only finitely many such hyperplanes and thus there exists a vector $y = (y_1, \ldots, y_{d-1}, 1) \in F^d$ such that the inner product

$$<u - v, y> \neq 0 \text{ for all distinct } u, v \in Z(J) \text{ and } y_i \neq 0, \forall i = 1, \ldots, d. \quad (3.1)$$

We introduce a (linear) polynomial $g \in F^d[x_1, x_2, \ldots, x_d]$ defined by

$$g(x) = <x, y> \quad (3.2)$$

and consider the linear subspace $L \subset F^d[x]$ defined by

$$L := \text{span}\{g^k, k = 0, \ldots, \#Z(J) - 1\}. \quad (3.3)$$

Thus

$$\text{dim } L = \#Z(J) = \text{dim}(F^d[x]/J). \quad (3.4)$$

We claim that

$$L \oplus J = F^d[x]. \quad (3.6)$$

In view of (3.4), it suffices to prove that $L \cap J = \{0\}$. Indeed if $f \in J \cap L$ then the polynomial

$$f = \sum_{j=0}^{\#Z(J)-1} a_j g^j(u) = \sum_{j=0}^{\#Z(J)-1} a_j z^j$$

equals to zero for $\#Z(J)$ distinct values of

$$z = <u, y>, u \in Z(J).$$

Hence $f = 0$.

Let $Q$ be an ideal projector from $F^d[x]$ onto $L$ determined by the decomposition (3.6). Thus $\ker Q = J$ and $Q$ is a Lagrange projector. We claim that the set $B$ of $d$ polynomials

$$B = \{g^{\#Z(J)} - Q(g^{\#Z(J)}) \text{ and } x_j - Q(x_j), j = 1, \ldots, d - 1\} \quad (3.7)$$

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is a basis for the ideal \( J \), i.e. \( J = \langle B \rangle \).

Clearly, \( \langle B \rangle \subset J \). Since \( Q' = I - Q \) is a projector onto \( J \), it suffices to prove that
\[
Q' f \in \langle B \rangle \quad \text{for every } f \in F^d[x].
\] (3.8)

We will do so in several steps. Let \( A_j \) be a subalgebra of \( F^d[x] \) generated by \\( \{g, x_1, ..., x_j\} \).

**Step 1:** \( Q' f \in \langle B \rangle \) for every \( f \in A_0 \).

Since \( A_0 \) is a subalgebra generated by one polynomial \( g \), we have to prove that \( Q' g^m \in \langle B \rangle \) for all integers \( m \). It is obviously so for \( m \leq \#Z(J) \). We now proceed by induction. Assume that \( m > \#Z(J) \) and
\[
Q' g^k \in \langle B \rangle \quad \text{for all } k \leq m.
\]

Then by (1.5),
\[
Q' g^{m+1} = g \cdot Q' g^m + Q'(g \cdot Qg^m),
\]
where the first term is in \( \langle B \rangle \) by inductive assumption (\( Q' g^m \in \langle B \rangle \)) and the second terms belongs to \( \langle B \rangle \), since \( g \cdot Qg^m \) contains only scalar multiples of \( g \cdot g^k \) for \( k < \#Z(J) \).

**Step 2:** \( Q' f \in \langle B \rangle \) for every \( f \in A_j \) with \( j = 0, 1, ..., d - 1 \).

Assume that the result is proven for a fixed \( j \leq d - 2 \). We will use induction on \( k \) to prove that \( Q'(x_j^{k+1} \cdot A_j) \subset \langle B \rangle \) for all integers \( k \). Let \( f \in A_j \). Using (1.5) once more, we have
\[
Q'(x_j^{k+1} \cdot f) = x_j^k \cdot f \cdot Q'(x_j) + Q'(x_j^k \cdot f \cdot Q(x_j)).
\]

Again, the first term is in \( \langle B \rangle \), since, by (3.7), \( Q'(x_j) \) is. The second term belongs to \( \langle B \rangle \) since \( f \cdot Q(x_j) \in A_j(L) \) and by inductive assumption \( Q'(x_j^k \cdot f \cdot Q(x_j)) \in B \). Thus we proved that for the algebra \( A_{d-1} \) generated by \( g \) and \( x_2, ..., x_d \), we have
\[
Q'(A_{d-1}) \subset \langle B \rangle.
\]

**Step 3:** \( Q' f \in \langle B \rangle \) for every \( f \in A_d \).

It is left to prove that \( Q'(x_d^{k} \cdot A_{d-1}) \subset \langle B \rangle \). Observe that by the choice of the vector \( y \) and by (3.2),
\[
x_d = g - \sum_{k=1}^{d-1} y_k x_k
\]
and \( x_d^k \cdot A_{d-1} = (g - \sum_{k=1}^{d-1} y_k x_k)^k \cdot A_{d-1} \subset A_{d-1} \).

Since \( A_d = F[x] \), the last step proves (3.8) and the inequality \( m(J) \leq d \) with it.

To prove the reverse inequality, suppose that \( J = \langle b_1, ..., b_d \rangle \). Then by a well-known theorem from algebraic geometry (cf. [6], Proposition 5., p.460) we have \( \dim Z(J) = d - (d - 1) = 1 \) which contradicts the assumption that \( Z(J) \) is finite. ■
4 Final Remarks

In the previous section we computed \( m(J) \) for radical ideals \( J \). This result is not valid for arbitrary ideals. In fact there is an ideal \( J \) complemented to \( \mathbb{F}_d[x_1, x_2, \ldots, x_d] \) for which

\[
m(J) = N(n) := \binom{n + d - 1}{d - 1}
\]
as the error formula requires.

**Theorem 4.1** Let \( B := M_n^d[x] \). Then the ideal \( J := \langle B \rangle \) is complemented to \( \mathbb{F}_d[x] \) and

\[
m(\langle B \rangle) = N(n) = \binom{n + d - 1}{d - 1}.
\]

**Proof.** Observe that \( q \in \langle B \rangle \) if and only if \( q = \sum_{k \geq n} q^{[k]} \) and \( p \in \mathbb{F}_d[x] \) if and only if \( p = \sum_{k < n} p^{[k]} \). Since every \( f \in \mathbb{F}_d[x] \) can be written uniquely as

\[
f = \left( \sum_{k < n} f^{[k]} \right) + \left( \sum_{k \geq n} f^{[k]} \right),
\]
it follows that \( \langle B \rangle \) is complemented to \( \mathbb{F}_d[x] \).

By way of contradiction, assume that \( \langle B \rangle = \langle b_1, \ldots, b_{N(n) - 1} \rangle \). Then for every \( x^\lambda \in M_n^d[x] \) with \( |\lambda| = n \),

\[
x^\lambda = \sum_{k=1}^{N(n)-1} a_k \cdot b_k
\]
for some polynomials \( a_k \in \mathbb{F}_d[x] \). Since \( n \) is “the least degree” for each \( b_k \in \langle B \rangle \), it follows that

\[
x^\lambda = \sum_{k=1}^{N(n)-1} a_k^{[0]} \cdot b_k^{[n]}
\]
But \( a_k^{[0]} \in \mathbb{F} \) and thus

\[
\text{span}\{x^\lambda : |\lambda| = n\} \subset \text{span}\{b_1, \ldots, b_{N(n) - 1}\}.
\]

Since \( \{x^\lambda : |\lambda| = n\} \) is a linearly independent set of polynomials, the space on the left has dimension \( N(n) \) and the space on the right has dimension at most \( N(n) - 1 \), contradicting the embedding (4.1).

Observe that the ideal projector \( P \) onto \( \mathbb{F}_{d,n}^d[x] \) with \( \ker P = \langle B \rangle \) considered in the previous theorem, is the Taylor projector.

**Conjecture 4.2** For \( d > 1 \), the Taylor projector is the unique ideal projector \( P \) onto \( \mathbb{F}_{d,n}^d[x] \) that admits the error formula (1.1), (1.2) with minimal \( B \).

In partial support of this conjecture, let us mention that the conjecture is true in the bivariate case:
Theorem 4.3  Let $P$ be a bivariate ideal projector onto $\mathbb{F}_n[x,y]$ that admits the error formula (2.1),(2.2) for minimal bases $B$. Then $P$ is a bivariate Taylor projector.

The cumbersome derivation involves explicit formulas (cf. [16]) for bivariate ideal projectors and will be published elsewhere.

We want to use this opportunity to mention one more general conjecture supported by results in [17].

Conjecture 4.4  Let $P$ be a bivariate ideal projector onto $\mathbb{F}_n[x,y]$. Let $B$ be a basis for $\ker P$ that admits the error formula (2.1),(2.2). Then the basis $B$ is a reduced basis.

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References


