COVERING BY COMPLEMENTS OF SUBSPACES, II

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Abstract. Let \( V \) be an \( n \)-dimensional vector space over an algebraically closed field \( K \). Define \( \gamma(k, n, K) \) to be the least positive integer \( t \) for which there exists a family \( E_1, E_2, \ldots, E_t \) of \( k \)-dimensional subspaces of \( V \) such that every \( (n-k) \)-dimensional subspace \( F \) of \( V \) has at least one complement among the \( E_i \)'s. Using algebraic geometry we prove that \( \gamma(k, n, K) = k(n-k) + 1 \).

1. Introduction

Take \( V = V(n, K) \) to be an \( n \)-dimensional vector space over the algebraically closed field \( K \). As usual a subspace \( F \) of \( V \) is a complement of the subspace \( E \) of \( V \) if \( V = E \oplus F \), i.e., if \( E + F = V \) and \( E \cap F = \{0\} \). We let \( c(E) \) denote the set of all complements of \( E \) in \( V \) and we write \( G(k, n) \) for the set of all \( k \)-subspaces (= \( k \)-dimensional subspaces) of \( V \). If \( E \in G(k, n) \) then \( c(E) \subseteq G(n-k, n) \). Define \( \gamma(k, n, F) \) to be the least positive integer \( t \) such that there exist \( k \)-subspaces \( E_1, E_2, \ldots, E_t \) of \( V \) satisfying

\[
c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n-k, n).
\]

If (1) holds we say that all \( (n-k) \)-subspaces of \( V \) are covered by the \( E_i \)'s.

In [1] we studied this problem for an arbitrary field \( K \). Among other things we showed that in general \( \gamma(k, n, K) \) depends on the field \( K \). In particular, we showed that \( \gamma(2, 4, K) \) is 5 if \( K \) is quadratically closed and is 4 otherwise. We conjectured that \( \gamma(k, n, K) = k(n-k) + 1 \) if \( K \) is algebraically closed. Here we prove this conjecture using results from algebraic geometry.

2. The lower bound \( k(n-k) + 1 \leq \gamma(k, n, K) \)

Let \( \Lambda^k(V) \) denote the \( k \)-vectors in the exterior algebra \( \Lambda(V) \) of \( V \). We let \( D(k, n) \) denote the set of all non-zero decomposable \( k \)-vectors \( \alpha = v_1 \wedge v_2 \wedge \cdots \wedge v_k \) where \( v_1, v_2, \ldots, v_k \) are linearly independent vectors in \( V \). Let \( \langle \alpha \rangle \) denote the 1-dimensional subspace of \( \Lambda^k(V) \) generated by \( \alpha \) and write

\[
\overline{D(k,n)} = \{ \langle \alpha \rangle | \alpha \in D(k, n) \}.
\]

If \( v_1, v_2, \ldots, v_k \) is a basis for \( E \in G(k, n) \), then the mapping \( E \mapsto \langle v_1 \wedge \cdots \wedge v_k \rangle \) is a bijection from \( G(k, n) \) to \( \overline{D(k,n)} \). It is well-known that this gives \( G(k, n) \) the
structure of an irreducible projective variety (the Grassmannian) of dimension 
\( k(n-k) \) in \( \mathbb{P}^N = \mathbb{P}(k^k(V)) \) where \( N = \binom{n}{k} - 1 \). We identify \( G(k, n) \) with \( D(k, n) \).

Now for any positive integer \( t \) let \( G(k, n)^t \) be the product variety of \( G(k, n) \) with itself \( t \) times. Let \( E = (E_1, \ldots, E_t) \in G(k, n)^t \). For each \( i \) let \( E_i = \langle \epsilon_i \rangle \) for some decomposable \( \epsilon_i \in \Lambda^k(V) \). Define the mappings:

\[ \varphi_i : \Lambda^{n-k}(V) \to \Lambda^n(V) \quad \text{by} \quad \varphi_i(\xi) = \epsilon_i \wedge \xi \]

for \( i = 1, \ldots, t \) and let

\[ \mathcal{K}(E) = \ker(\varphi_1) \cap \ker(\varphi_2) \cap \cdots \cap \ker(\varphi_t). \]

Note that \( \mathcal{K}(E) \) is a subspace of \( \Lambda^{n-k}(V) \).

**Lemma 1.** For \( E \in G(k, n)^t \) the following two conditions are equivalent:

(a) \( c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n-k, n) \),

(b) \( D(n-k, n) \cap \mathcal{K}(E) = \emptyset \).

**Proof.** This is an immediate consequence of the fact that if \( F = \langle \alpha \rangle \in G(n-k, n) \) for some \( \alpha \in D(n-k, n) \), then \( E_1 \cap F = \{0\} \) if and only if \( \epsilon_i \wedge \alpha \neq 0 \).

**Lemma 2.** If \( \gamma(k, n, K) = t \) and \( E = (E_1, \ldots, E_t) \in G(k, n)^t \) satisfies

\[ c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n-k, n), \]

then

\[ \dim(\mathcal{K}(E)) = \binom{n}{k} - t. \]

**Proof.** Since \( \varphi_i \) is a linear mapping from the \( \binom{n}{k} \)-dimensional vector space \( \Lambda^{n-k}(V) \) to the 1-dimensional vector space \( \Lambda^n(V) \), it suffices to show that the mappings

\[ \varphi_i \in \text{hom}(\Lambda^{n-k}(V), \Lambda^n(V)), \quad i \in \{1, \ldots, t\}, \]

are linearly independent. To see this we first note that the elements \( \epsilon_i \) are linearly independent in \( \Lambda^{n-k}(V) \). Suppose not; then we can assume that \( \epsilon_i = \sum_{j=1}^{t} a_j \epsilon_j \). It follows that \( \bigcap_{i=1}^{t} \ker(\varphi_i) = \bigcap_{i=1}^{t} \ker(\varphi_i) \). This implies by Lemma 1 that

\[ c(E_1) \cup c(E_2) \cup \cdots \cup c(E_{t-1}) = G(n-k, n) \]

and hence \( \gamma(k, n, K) \leq t - 1 \), a contradiction. Now assume that the mappings \( \varphi_1, \ldots, \varphi_t \) are linearly dependent. Say, \( \sum_{i=1}^{t} a_i \varphi_i = 0 \). This means that for all \( \xi \in \Lambda^{n-k}(V) \) we have

\[ 0 = \sum_{i=1}^{t} a_i (\epsilon_i \wedge \xi) = (\sum_{i=1}^{t} a_i \epsilon_i) \wedge \xi. \]

So it suffices to observe that if \( \delta \in \Lambda^k(V) \) and \( \delta \wedge \xi = 0 \) for all \( \xi \in \Lambda^{n-k}(V) \) then \( \delta = 0 \).

**Lemma 3.** If \( K \) is any algebraically closed field, then

\[ k(n-k) + 1 \leq \gamma(k, n, K). \]

**Proof.** Suppose \( \gamma(k, n, K) = t \leq k(n-k) \). Then there exists \( E = (E_1, \ldots, E_t) \in G(k, n)^t \) such that \( c(E_1) \cup \cdots \cup c(E_t) = G(n-k, n) \). So by Lemmas 1 and 2 there is a linear subspace \( \mathcal{K}(E) \) of \( \Lambda^{n-k}(V) \) such that \( D(k, n) \cap \mathcal{K}(E) = \emptyset \) and \( \mathcal{K}(E) \) has affine dimension \( \binom{n}{k} - t \) which is at least \( \binom{n}{k} - k(n-k) \). Let \( \mathcal{K}' \) denote the
corresponding projective subspace of \( \mathbb{P}(\Lambda^{n-k}(V)) \). Then \( \mathcal{K}' \cap G(n-k,n) = \emptyset \). But using projective dimensions we have [3, Proposition 11.4]

\[
dim(\mathcal{K}') + \dim(G(n-k,n)) \geq \binom{n}{k} - k(n-k) - 1 + k(n-k) \\
\geq \binom{n}{k} - 1 = \dim(\mathbb{P}(\Lambda^{n-k}(V)))
\]

and it follows that \( \mathcal{K}' \cap G(n-k,n) \neq \emptyset \) which is a contradiction. \( \square \)

3. The upper bound \( \gamma(k,n,K) \leq \nu(k(n-k)) + 1 \)

**Lemma 4.** If \( K \) is algebraically closed, then

\[
\gamma(k,n,K) \leq \nu(k(n-k)) + 1.
\]

**Proof.** Let \( \nu = k(n-k) \) denote the dimension of \( G(k,n) \) (and \( G(n-k,n) \)) as a projective variety. Let

\[
A = G(k,n)^{\nu+1}.
\]

Then \( A \) is a projective variety of dimension \( \nu(\nu+1) \). For every \( F \in G(n-k,n) \) define

\[
B(F) = \{ E \in G(k,n) | E \cap F \neq 0 \}.
\]

Now \( B(F) \) is an irreducible projective variety with

\[
\dim(B(F)) = \nu - 1.
\]

For \( F \in G(n-k,n) \) define

\[
C(F) = B(F)^{\nu+1}.
\]

Then

\[
\dim(C(F)) = (\nu+1)(\nu-1) = \nu^2 - 1.
\]

Now set

\[
C = \bigcup_{F \in G(n-k,n)} C(F).
\]

Note that if \( C \) is properly contained in \( A \), then there exists \( E = (E_1, \ldots, E_{\nu+1}) \in A - C \). Then for all \( F \in G(n-k,n) \) we have \( E \notin C(F) \) so there must exist an index \( i \in \{1, \ldots, \nu+1\} \) such that \( E_i \cap F = 0 \). Hence \( c(E_1) \cup \cdots \cup c(E_{\nu+1}) = G(n-k,n) \) and so \( \gamma(k,n,K) \leq \nu + 1 \), as desired. So it remains only to show that \( C \) is properly contained in \( A \). In fact we claim that \( C \) is a variety of dimension at most \( \dim(A) - 1 = \nu^2 + \nu - 1 \).

To complete the proof we fix \( F_0 \in G(n-k,n) \) and consider the projective variety

\[
D := C(F_0) \times \text{PGL}_n(K).
\]

We note that

\[
\dim(D) = \dim(C(F_0)) + \dim(\text{PGL}(n,K)) = \nu^2 - 1 + n^2 - 1.
\]

An element \( M \) of \( \text{PGL}(n,K) \) induces a linear automorphism of \( \mathbb{P}(\Lambda^k(V)) \) which induces in turn an automorphism of \( G(k,n) \). Abusing notation we write \( U \mapsto MU \)
to indicate the latter automorphism. Now we define \( \varphi : D \to C \) as follows: For \((E,M) \in D\) set

\[
\varphi(E, M) = (ME_1, ME_2, \ldots, ME_{\nu+1}).
\]

Clearly \( \varphi \) is a regular surjection. Hence by [3, Theorem 11.12]

\[
dim(D) = \dim(C) + \mu
\]

where

\[
\mu = \min \{ \dim(\varphi^{-1}(E')) \}, \quad E' \in C.
\]

This shows that

\[
dim(C) = \nu^2 - 1 + n^2 - 1 - \mu.
\]

So to prove that \( \dim(C) \leq \nu^2 + \nu - 1 \) it suffices to prove that \( n^2 - \nu - 1 \leq \mu \). To see this consider the subset \( G(F) \) of \( PGL_n(K) \) whose elements map the fixed \((n-k)\)-subspace \( F_0 \) to the \((n-k)\)-space \( F \). It is easy to see that \( \dim(G(F)) = n^2 - \nu - 1 \).

Now if \( E' = (E'_1, \ldots, E'_{\nu+1}) \in C(F) \subseteq C \) then for each \( M \in G(F) \) we have

\[
(M^{-1}E', M) = (M^{-1}E'_1, \ldots, M^{-1}E'_{\nu+1}, M) \in \varphi^{-1}(E').
\]

The mapping \( M \mapsto (M^{-1}E', M) \) is a regular injection from \( G(F) \) into the fiber \( \varphi^{-1}(E') \). It follows that each fiber has dimension at least that of \( G(F) \) and this completes the proof.

\[ \blacksquare \]

Remarks. 1. The above proof shows that almost all \((E_1, \ldots, E_{\nu+1}) \in G(k,n)^{\nu+1}\) satisfy

\[
c(E_1) \cup c(E_2) \cup \cdots \cup c(E_{\nu+1}) = G(n-k,n)
\]

since the complement \( C \) of the set of such \((\nu+1)\)-tuples forms a variety of dimension smaller than \( \dim(G(k,n)^{\nu+1}) \).

2. As shown in [1] \( \gamma(2,4,K) = 4 \) when \( K \) is not quadratically closed. So the lower bound \( \gamma(k,n,K) \geq k(n-k)+1 \) proved here for algebraically closed fields will not hold in general. On the other hand, we suspect that the upper bound \( \gamma(k,n,K) \leq k(n-k)+1 \) does hold for arbitrary fields. In fact we have verified this for finite fields of sufficiently large order using counting arguments [2]. However, as the referee pointed out it is slightly worrying that the conjecture fails in the “thin” case, that is, if we replace \( n \)-space by \( n \)-set, \( k \)-subspace by \( k \)-subset and vector space complement by set complement. However, the upper bound of \( \binom{\nu+1}{k} \) given in [1] holds in both cases.

References


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