Another Note on Polynomial vs Rational Approximation

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Communicated by Peter B. Borwein

Received October 17, 1994; accepted June 6, 1995

Let \( E \) be a subspace of \( C(X) \) and let \( R(E) = \{ gh : g, h \in E; h > 0 \} \). We make a simple, yet intriguing observation: if zero is a best approximation to \( f \) from \( E \), then zero is a best approximation to \( f \) from \( R(E) \).

We also prove that if \( \{ E_n \} \) is dense in \( C(X) \) then for almost all \( f \) (in the sense of category)

\[
\limsup d(f, R(E_n)) = d(f, E_n) = 1.
\]

That extends the results of P. Borwein and S. Zhou who proved it for the case when \( E_n \) is the space of algebraic or trigonometric polynomials of degree \( n \).

1. Introduction

Consider an arbitrary function \( f \in C([-1,1]) \). Let \( P_n \) stand for the space of polynomials of degree \( n \) and let \( R_{n,n} \) stand for rational functions

\[
\left\{ \frac{g}{h} : g, h \in P_n; h > 0 \right\}.
\]

Let \( p_n^* \) be the best approximation to \( f \) from \( P_n \). Then zero is the unique best approximation from \( R_{n,n} \) to \( f - p_n^* \).

Here is a short proof:

The function \( f - p_n^* \) equioscillates i.e. there are points \( \xi_1, \ldots, \xi_{n+2} \in [-1,1] \) such that

\[
(f - p_n^*)(\xi_j) = \lambda (-1)^j \| f - p_n^* \| \text{ where } \lambda = \pm 1 \text{ (say } \lambda = -1). \]

Now if \( \| f - p_n^* - g/h \| \leq \| f - p_n^* \| \) then \( g(h(\xi_j)) \geq 0 \) for \( j \) even and \( g(h(\xi_j)) \leq 0 \) for \( j \) odd. Since \( h \) is strictly positive, the function \( g \in P_n \) should satisfy the same condition

\[
g(\xi_j) \geq 0 \text{ for } j \text{ even} \quad \text{and} \quad g(\xi_j) \leq 0 \text{ for } j \text{ odd}.
\]
That forces \( g \) to have \( n+1 \) zeros and hence \( g = 0 \). Examining this proof it is easy to conclude that it has nothing to do with the nature of \( h \), as long as it is strictly positive. The only property of \( g \) that we used is that \( g \) and \( p^* \) belong to the same Chebyshev subspace of \( C[0,1] \).

It turns out that this statement (aside from uniqueness) holds true for rational functions where the numerator and denominator come from arbitrary subspaces of \( C(X) \). This is the content of Theorem 2.1.

We then use this theorem to prove that for most of the functions (in the sense of category) in \( C(X) \) the rate of best approximation and the rate of best rational approximation is the same. This is known for specific subspaces (cf [1], [3]). We prove it for arbitrary subspaces of \( C(X) \).

2. The Best Rational Approximation

Let \( X \) be a compact Hausdorff space, let \( C(X) \) be the space of real-valued continuous functions on \( X \). If \( G \) and \( H \) are subspaces of \( C(X) \) we use

\[
R(G, H) := \{ g/h : g \in G, h \in H, h(x) > 0 \text{ for all } x \in X \}.
\]

To avoid trivialities we will always assume that \( H \) contains a strictly positive function.

We will identifying the dual space \( (C(X))^* \) with the space of regular Borel measures on \( X \) : \( \mathcal{B}(X) \), and the same letter may mean a measure or a functional.

Finally if \( A \) is a subset of \( C(X) \), and \( f \in C(X) \)

\[
d(f, A) := \inf \| f - a \| : a \in A \}.
\]

**Theorem 2.1.** Let \( f \in C(X) \) and \( G \subseteq C(X) \) be a subspace such that there exists \( g \in G \) with \( \| f - g^* \| = d(f, G) \). Then for every subspace \( H \subseteq C(X) \)

\[
d(f - g^*, R(G, H)) = \| f - g^* \|.
\]

Hence zero is a best approximation to \( f - g^* \) from \( R(G, H) \). Moreover if \( X = [a, b] \) and \( G \) is Chebyshev then zero is the unique best approximation from \( R(G, H) \) to \( f - g^* \).

**Proof.** Since \( g^* \) is the best approximation from \( G \) to \( f \), hence there exists a functional \( \mu \in \mathcal{B}(X) \) such that

\[
\mu \perp G \text{ i.e. } \mu(g) = 0 \text{ for all } g \in G. \tag{1}
\]

\[
\| \mu \| = 1; \quad \mu(f - g^*) = \| f - g^* \| \| \mu \|. \tag{2}
\]

We adopt the logic of [2] for this particular case.
Let $a > 0$ be such that $d(f - g^*, R(G, H)) < a$. Then there exists $g \in G$, $h \in H$, $h > 0$ such that for $f := g/h$ we have $\| (f - p^*) - \bar{f} \| < a$.

On the other hand,

$$0 \neq \| f - p^* \| |(\mu)(h)) = \int (f + p^*) h \, d\mu$$

$$= \int (f - p^*) h \, d\mu + \int \bar{f} h \, d\mu = \int (f - p^*) h \, d\mu + \int \bar{f} g \, d\mu$$

(since $\bar{f} = g$)

$$= \int ((f - p^*) - \bar{f}) h \, d\mu$$

(since $\mu \perp G$) $\leq \| (f - p^*) - \bar{f} \| \int h |\mu| (\text{since } h \text{ is positive}) < a |\mu|(h)$.

Thus $a > \| f - p^* \| = d(f, G)$. The “moreover” part of the Theorem was already proved in the Introduction.

**Remark.** Theorem 2.1 is a very simple observation. Yet even in the simple case of $C(X) = C([-1,1])$, $R(G, H) = R_n$, it is somewhat surprising. First of all it provides a large class of functions for which the best rational approximation is easily computed.

Second, it shows how easy it is to spoil a function for rational approximation.

For instance $\| f - p^* \| = e^{|x|}$, $\| f - p^* \| = \int \bar{f} g \, d\mu$.

Theorem 3.1.

Let $X$ be an infinite compact Hausdorff space. Let $G_n \subset C(X)$ be a sequence of finite-dimensional subspaces such that

$$d(f, G_n) \to 0 \quad \text{for all } f \in C(X).$$

Then for all finite-dimensional subspaces $H_n \subset C(X)$, the set

$$A := \{ f \in C(X): \text{lim sup}[d(f, R(G_n, H_n))/d(f, G_n)] \geq 1 \}$$

is the set of second category in $C(X)$.

3. Rates of Approximation

We now use the Theorem 2.1 to extend a result of P. Borwein and S. Zhou (cf. [1], Theorem 1) from $R_n$ to $R(G_n, H_n)$ for arbitrary $G_n$, $H_n \subset C(X)$.

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Proof. We proceed as in \cite{1}. Let $\tilde{G}_n := C(X) \setminus G_n$. Then $\tilde{G}_n$ is open and dense in $C(X)$. We consider sets

$$A_n = \{ f \in C(X) : \text{there exists } m(n) > n \text{ with } d(f, R(G_m, H_m)) / d(f, G_m) > 1 - \frac{1}{n}; d(f, G_m) \neq 0 \}.$$

Then $A = (\bigcap_{n=1}^\infty A_n) \cap (\bigcap_{n=1}^\infty \tilde{G}_n)$. It remains to prove that each set $A_n$ is open and dense in $C(X)$. The proof that $A_n$ is open is exactly the same as in \cite[Theorem 1]{1} and we refer to it for technical details. The idea, however, is very simple. For a fixed $f \in A_n$ choose $\varepsilon$ and $\delta$ so small that for all $\tilde{f}$ with $\| f - \tilde{f} \| < \delta$ we have

$$\tilde{f} \not\in G_n ; \quad |d(\tilde{f}, R(G_n, H_n)) - d(f, R(G_n, H_n))| < \varepsilon;$$

$$|d(f, G_n) - d(\tilde{f}, G_n)| < \varepsilon.$$

Since $\varepsilon$ is “very small” the ratio $d(\tilde{f}, R(G_n, H_n)) / d(f, G_n)$ is still greater than $1 - 1/n$.

We now turn to the density of $A_n$. Let $f \in C(X)$. For arbitrary $\varepsilon > 0$ pick $\eta = \varepsilon/2$ and let $g_m \in G_m$ be such that $\| f - g_m \| < \eta; m > n$. Let

$$E_m := \mathrm{span}\{ g_m \cdot h_m + g'_m \cdot h_m : g_m, g'_m \in G_m, h_m \in H_m \}.$$

Since $G_m$ and $H_m$ are of finite dimension, so is $E_m$. Let $F$ be an arbitrary function in $C(X) \setminus E_m$. Since $E_m$ is finite-dimensional, there exists $e_m^*$ which is a best approximation to $F$ from $E_m$. Denote

$$F := (F - e_m^*) / \| F - e_m^* \|.$$

We now consider the function

$$\varphi(x) = g_m(x) + \eta F^*(x).$$

Observe that $\| f - \varphi \| < \eta + \eta = \varepsilon$. It remains to show that $\varphi \in A_n$. Indeed, since $G_m \subset E_m$ we have

$$\eta \geq d(\varphi, G_m) = d(\eta F^*, G_m) \geq d(\eta F^*, E_m) = \eta.$$

Therefore

$$d(\varphi, G_m) = \eta. \quad (3)$$

Now let $e_m / h_m$ be an arbitrary element in $R(E_m, H_m)$. Then

$$\varphi - e_m / h_m = \eta F^* + g_m - e_m / h_m = \eta F^* + \frac{g_m h_m - e_m}{h_m} = \eta F - \frac{e_m}{h_m}$$

$$\varphi - e_m / h_m = \eta F^* + g_m - e_m / h_m = \eta F + \frac{g_m h_m - e_m}{h_m} = \eta F - \frac{e_m}{h_m}.$$
where $e_m' = e_m - g_m h_m$ is an arbitrary element in $E_m$. Thus
\[ d(\varphi, R(E_m, H_m)) = d(\varphi, R(E_m, H_m)) = \eta. \]
The last equality follows from the Theorem 2.1.
Since $G_m \subset E_m$ we have
\[ d(\varphi, R(G_m, H_m)) \geq d(\varphi, R(G_m, H_m)) \geq \eta; \]
which together with (3) implies
\[ d(\varphi, R(G_m, H_m))/d(\varphi, G_m) \geq \eta/\eta = 1 > 1 - \frac{1}{\eta} \]
and hence $\varphi \in A_n$. \[ \square \]

ACKNOWLEDGMENT

I am grateful to Professor Zhou for pointing out the reference [1] and [3] to me. A conversation with him sparked off my interest in this subject.

REFERENCES