Discrete Approximating Operators on Function Algebras

Boris Shekhtman

Abstract. We give a new presentation and various extensions of a theorem of Somorjai. For any sequence of operators $L_n$, given by $L_n f = \sum_{k=1}^{n} f(z, k)l_{z, k}$ with $z \in \mathbb{T}$ and $l_{z, k} \in A(\mathbb{T})$, there exists a function $f \in A(\mathbb{T})$ such that $L_n f$ does not converge to $f$.

1. Introduction

The main purpose of this paper is to give a new presentation, as well as some extensions, of the result obtained in Somorjai's paper [So].

Let $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ be the unit circle and let $A(\mathbb{T})$ be the disk algebra. Linear operators $L_n : A(\mathbb{T}) \to A(\mathbb{T})$ are called discrete if they are of the form

(1.1) $L_n f = \sum_{k=1}^{n} f(z, k)l_{z, k},$

where $z \in \mathbb{T}$; $l_{z, k} \in A(\mathbb{T})$. Somorjai [So] gave an elegant proof that for any sequence $\{L_n\}$ of discrete operators, there exists a function $f \in A(\mathbb{T})$ such that $L_n f$ does not converge to $f$ in the topology of $A(\mathbb{T})$. The proof uses the translation-invariant property of $A(\mathbb{T})$.

Our analysis traces this result to some Banach space properties of $A(\mathbb{T})$, hence lends itself to extensions of the theorem to more general domains and more general function algebras. We consider (using the Rudin–Carleson theorem) $L_n$ as a composition of two maps

\[
A(\mathbb{T}) \longrightarrow A(\mathbb{T}) \quad \xrightarrow{U_n} \quad \ell_\infty,
\]

where

\[
A_n f = \{ f(z, k) \} \in \ell_\infty; \quad U_n \{ \zeta_k \} = \sum_{k=1}^{n} \zeta_k l_{z, k} \in A(\mathbb{T}).
\]

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Thus $L_n$ has a natural factorization through $l_\infty$ space. An easy proposition shows that if a sequence of operators $L_n$ on a Banach space $X$ factors through $l_\infty$ and serves as a nice approximation on $X$ (i.e., $L_n x \rightarrow x$ for all $x \in X$), then $X$ must inherit certain properties of $l_\infty$, namely $X$ must be an $L_\infty$ space. The Somorjai result follows from the fact that $A(T)$ is not an $L_\infty$ space.

In the next section we observe some simple facts related to approximation by operators that factor through $L_\infty$ spaces. In Section 3 we consider approximation on various subspaces $X \subset C(K)$ for which some analogs of the F. and M. Riesz theorem holds. On the one hand, this theorem implies the Rudin–Carleson theorem (see [B2]) and hence gives us the factorization of operators. On the other hand, it implies (see [P2], Corollary 5.1) that the space $X$ is not an $L_\infty$ space.

We use the rest of this section to state the theorem of Somorjai in full strength.

**Definition 1.** Let $X$ be a subspace of $C(K)$. Let $L$ be a linear operator on $X$ and let $H \subset K$ be a subset of $K$. We say that $L$ is determined on $H$ if $L f = L g$ for all $f, g \in X$, such that $f(k) = g(k)$ for all $k \in H$ (i.e., $f | H = g | H$).

**Theorem 1** (see [So]). Let $H_x$ be closed subsets of $T$ of Lebesgue measure zero. Let $L_n: A(T) \rightarrow A(T)$ be linear operators that are determined on $H_x$. Then there exists a function $f \in A(T)$ such that $L_n f$ does not converge to $f$.

2. $L_\infty$ Spaces

We use $l_\infty^n$ to denote $C^n$ equipped with the norm $\| (x_j) \|_\infty = \max |x_j|$. We define the Banach–Mazur distance from an arbitrary $n$-dimensional Banach space $E$ to $l_\infty^n$ as

$$d(E, l_\infty^n) = \inf \{ \| T \| \| T^{-1} \| : T \text{ is an isomorphism from } E \text{ onto } l_\infty^n \}.$$  

It is well known (see [LT]) that

$$d(E, l_\infty^n) \leq n$$  

for all $E$.

The rest of this section is also well known (see [LT]).

**Proposition 1.** Let $E$ be an $n$-dimensional subspace of a Banach space $X$. Then there exists a projection $P$ from $X$ onto $E$ such that

$$\| P \| \leq d(E, l_\infty^n).$$  

**Definition 2.** Let $1 \leq \lambda < \infty$. A Banach space $X$ is said to be an $L_{\infty, \lambda}$ space if for every finite dimensional subspace $E \subset X$ there exists a finite dimensional subspace $F \subset X$ such that $E \subset F$ and

$$d(F, l_\infty^n) \leq \lambda \quad \text{where} \quad m = \dim F.$$

A Banach space $X$ is an $L_\infty$ space if $X$ is an $L_{\infty, \lambda}$ space for some $\lambda < \infty$.  

Remark 1 (see [LR]). For every $\varepsilon > 0$ the spaces $l_\infty$, $C(K)$, $L_\infty(\mu)$ are $\mathcal{L}_{n,1+\varepsilon}$ spaces.

Remark 2 (see [LT], II.3.1). Let $X$ be a separable Banach space. Then $X$ is an $\mathcal{L}_{n,\lambda}$ space if and only if $X = U E_n$ where $\dim E_n = n$, $E_n \subset E_{n+1} \subset X$, and $d(E_n, E_{n+1}) < \lambda$.

Theorem 2 (see [LR], Theorem 4.3). A Banach space $X$ is an $\mathcal{L}_\infty$ space if and only if there exist constants $\lambda, K \geq 1$, such that for every finite dimensional subspace $E \subset X$ there exists an $\mathcal{L}_{n,\lambda}$ space $Y$ and operators $A : E \to Y$, $B : Y \to X$ such that $\|A\|B\| \leq K$ and $BAe = e$ for all $e \in E$.

The main tool in our investigation is the following:

Theorem 3. A Banach space $X$ is an $\mathcal{L}_\infty$ space if and only if there exists $\lambda \geq 1$, $K \geq 1$, a sequence of $\mathcal{L}_{n,\lambda}$ spaces $Y_n$, and a sequence of linear operators $A_n : X \to Y_n$ and $U_n : Y_n \to X$ such that

$$U_n A_n x \to x \quad \text{for all } x \in X$$

and $\|U_n\|\|A_n\| \leq K$.

Proof. If $X$ is an $\mathcal{L}_\infty$ space we choose $Y_n := X$; $A_n = U_n = I$. Conversely, let $E$ be a finite dimensional subspace of $X$ with $\dim E = N$. We use a standard perturbation argument (see [LT], p. 198). By (2.2) there exists a basis $e_1, \ldots, e_N \in E$ so that

$$\frac{1}{\sqrt{N}} \max\{\lambda_j\} \leq \|\sum \lambda_j e_j\| \leq \sqrt{N} \max\{\lambda_j\}$$

for all choices of $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$. Let $1 > \delta > 0$. Pick $\varepsilon = \delta/2N^{3/2}$ and choose $n$ so large that for $f_j := U_n A_n e_j$ we have $\|f_j - e_j\| < \varepsilon$. From

$$\|\sum \lambda_j f_j - e_j\| \leq \frac{1}{2\sqrt{N}} \max\{\lambda_j\} \leq \|\sum \lambda_j e_j\|$$

it now follows that

$$\frac{1}{2\sqrt{N}} \max\{\lambda_j\} \leq \|\sum \lambda_j f_j\| \leq 2\sqrt{N} \max\{\lambda_j\}.$$  

Let $F = \text{span}\{f_j\} \subset X$. Define functionals $\tilde{\mu}_k$ on $F$ by $\tilde{\mu}_k(f_j) = \delta_{kj}$, $k = 1, \ldots, N$, $j = 1, \ldots, N$. By (2.4)

$$\|\tilde{\mu}_k\| \leq 2\sqrt{N}.$$  

Let $\mu_k$ be Hahn–Banach extensions of $\tilde{\mu}_k$ onto $X$. Define $T : X \to X$ by

$$Tx = x + \sum \mu_k(x)(e_k - f_k).$$
Observe that $T_k = e_k$ for all $k = 1, \ldots, N$. It also follows from (2.6) that
\[ \| Tx \| \leq (1 + \delta) \| x \|. \]

Hence the operators $A := A_n$ and $B := TU_n$ satisfy the condition of Theorem 2 and $X$ is an $\mathcal{L}_\infty$ space.

For convenience we introduce:

**Definition 3.** Let $X$ be a subspace of a Banach space $Y$. We say that $X$ is near-complemented in $Y$ if there exists a sequence of operators $L_n: Y \rightarrow X$ such that $\| L_n \|$ are uniformly bounded and $L_n x \rightarrow x$ for all $x \in X$. We say that $X$ is locally complemented in $Y$ if there exists a sequence of finite dimensional operators $L_n: Y \rightarrow X$ such that $\| L_n \|$ are uniformly bounded and $L_n x \rightarrow x$ for every $x \in X$.

**Theorem 4.** Let $K$ be a compact metric space and let $X$ be a subspace of $C(K)$. The following are equivalent:

(a) $X$ is an $\mathcal{L}_\infty$ space;
(b) $X$ is locally complemented in $C(K)$;
(c) $X$ is near-complemented in $C(K)$.

**Proof.** If $X$ is an $\mathcal{L}_\infty$ space, then since $X$ is separable there exists a sequence of spaces $E_n \subset E_{n+1} \subset X \subset C(K)$ such that $\overline{UE_n} = X$, $d(E_n, E_{n+1}) \leq \lambda$. By Proposition 1 we can find a sequence of projections $P_n$ from $C(K)$ onto $E_n$ such that $\| P_n \| \leq \lambda$. Clearly, $P_n x \rightarrow x$ for all $x \in X$. The implication (b) $\Rightarrow$ (c) is trivial. To prove (c) $\Rightarrow$ (a) let $J: X \rightarrow C(K)$ be a natural embedding. We now use Theorem 3 with $A_n = J$, $U_n = L_n$, and $Y_n = C(K)$.

**Remark 3.** If $X$ is a complemented subspace of $C(K)$ then (see [LR], Theorem 3.2) it is an $\mathcal{L}_\infty$ space. The converse to that statement does not hold. Indeed we can find (see [LT], Proposition II.4.40) a subspace $X \subset C([-1, 1])$ which is an $\mathcal{L}_\infty$ space yet has no complement in $C([-1, 1])$. Hence the near-complemented subspaces form a larger class of subspaces than the complemented subspaces.

The argument in [So] and the remarks after the proof seem to indicate that all that was needed is the fact that $A(T)$ is not complemented in $C(T)$. This inconsistency with Theorem 4 can be explained by translation-invariant properties of $A(T)$. Indeed, for any compact abelian group $G$ a translation invariant subspace $X \subset C(G)$ is complemented if and only if $X$ is near-complemented if and only if $X$ is an $\mathcal{L}_\infty$ space if and only if $X$ is spanned by the characters in the dual group $\hat{G}$ from a coset ring in $\hat{G}$ (see [KP], pp. 311–312).

3. Extensions of Somorjai's Theorem

In this section we will extend Theorem 1 in several directions. The idea is to check that a given subspace $X \subset C(K)$ is not an $\mathcal{L}_\infty$ space on the one hand, and $X$ verifies
some analog of the Rudin–Carleson theorem on the other. Fortunately, there are conditions that imply both statements. One such condition is an F. and M. Riesz theorem. Here is a direct generalization of Theorem 1.

**Theorem 5.** Let $K$ be the closure of a domain $D \subset \mathbb{C}$ whose boundary $\Gamma$ consists of a finite number of nonintersecting analytic closed curves. Let $X$ be a subspace of $C(\Gamma)$ that consists of all functions in $C(\Gamma)$ that have analytic continuation in $D$. Let $H_n \subset \Gamma$ be closed sets of Lebesgue measure zero. Finally, let $L_n^*= X \rightarrow X$ be determined on $H_n$. Then there exists a function $f \in X$ such that $L_n f$ does not converge to $f$.

**Proof.** Let $\mu$ be a regular Borel measure on $X$ such that $\int f \, d\mu = 0$ for all $f \in X$. Then (see [R1], Theorem 3) the measure $\mu$ is absolutely continuous with respect to Lebesgue measure. Now that implies (see [P2], Corollary 5.1) that $X$ is not an $L_{\infty}$ space. On the other hand, the absolute continuity of $\mu$ also implies (see [B2]) that for any function $g \in C(H_n)$ there exists a function $f \in X$ such that $f(t) = g(t)$ for all $t \in H_n$ (i.e., $f|_{H_n} = g|_{H_n}$) and $\|f\| \leq \|g\|$. (This is the Rudin–Carleson theorem.)

Let $L_n^* X \rightarrow X$ be determined on $H_n$. Then for each $g \in C(\Gamma)$ we can define $\hat{L}_n g$ to be $L_n f$ where $f \in X$ is such that $g|_{H_n} = f|_{H_n}$. (Since $L_n$ are determined on $H_n$ the value $L_n f$ does not depend on the choice of $f$.) Hence $\|\hat{L}_n\| = \|L_n\|$. Suppose that $L_n f \rightarrow f$ for all $f \in X$. Then $\|L_n\|$ are uniformly bounded. Then the norms of $\hat{L}_n^*: C(\Gamma) \rightarrow X$ are also uniformly bounded. If $\hat{L}_n f \rightarrow f$ for all $f \in X$ then $X$ is near-complemented and is hence an $L_{\infty}$ space. We have the desired contradiction.

**Remark 4.** The cited result of Pełczyński actually states that $X$ does not have local unconditional structure. That clearly implies that $X$ is not an $L_{\infty}$ space.

We now prove another generalization of Theorem 1 where the analyticity of the boundary is not required. Let $\mathcal{C}$ denote the extended complex plane.

**Theorem 6.** Let $K$ be a compact set in $\mathbb{C}$ with nonempty connected interior and connected complement such that the boundary $\Gamma$ of $K$ is accessible from the complement $G := \mathbb{C} \setminus K$ through Jordan curves, i.e., every point $z \in \Gamma$ is the endpoint of the Jordan curves contained in $G \cup \{z\}$. Let $A(\Gamma)$ be the subalgebra of $C(\Gamma)$ of functions analytic in the interior of $K$. Then for every sequence of operators $L_n$ defined by

\[
L_n f = \sum_{k=1}^{n} f(z_{n,k}) l_{n,k}, \quad z_{n,k} \in \Gamma, \quad l_{n,k} \in A(\Gamma),
\]

there exists a function $f \in A(\Gamma)$ such that $L_n f$ does not converge to $f$.

**Proof.** It follows from the Rudin–Carleson theorem and the Carathéodory extension method (see [S–To], Proof of Lemma 1) that for any finite sequence of points $z_1, \ldots, z_n \in \Gamma$ and for any set of complex numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with $|\alpha_j| \leq 1$, there exists a function $f \in A(\Gamma)$ such that $f(z_j) = \alpha_j$ for $j = 1, \ldots, n$ and $\|f\| \leq 1$. 
Since the operators \( L_n \) are determined on the sets \( \{z_{n,1}, \ldots, z_{n,n}\} \) they can be extended to operators \( L_n \) on \( C(\Gamma) \) such that \( \|L_n\| \leq \|L_n\| \). Hence, if \( L_nf \to f \) for all \( f \in A(\Gamma) \), then \( A(\Gamma) \) is near-complemented in \( C(\Gamma) \) and thus is an \( L_\infty \) space. On the other hand, Bishop (see [B1], Theorem 3) proved an analog of the F. and M. Riesz theorem for \( A(\Gamma) \), and using the same result of Pelczynski (see [P2], Corollary 5.1) we learn that \( A(\Gamma) \) is not an \( L_\infty \) space.

Our final result extends Theorem 1 to several variables.

**Theorem 7.** Let \( U^N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N : |z_j| < 1\} \). Let \( A(U^N) \) be the algebra of all functions which are holomorphic in the polydisk \( U^N \) and continuous on its closure \( \overline{U}^N \).

Let \( H_n = H^2 \circ H^2 \times \cdots \times H^2_N \) where \( H^2_j \) are closed subsets of \( T \) with Lebesgue measure zero. Let operators \( L_n : A(U^N) \to A(U^N) \) be determined on \( H_n \). Then there exists a function \( f \in A(U^N) \) such that \( L_nf \) do not converge to \( f \).

**Proof.** An appropriate analog of the Rudin–Carleson theorem can be found in [R2], Example 6.3(8). The fact that \( A(U^N) \) is not an \( L_\infty \) space is proved in [P2], Theorem 11.2.

**Remark 5.** More exotic extensions of Theorem 1 can be obtained by combining the “Main Theorem” and its corollaries in [P1] with the results of Sections 5, 10, and 11 of [P2].

The extensions of Theorem 1 to certain translation-invariant subspaces on general compact Abelian groups can be obtained using the results in [KP], Section 2, in combination with the extensions of the F. and M. Riesz theorem (see [DLG]) as well as with Corollaries 1 and 2 of [P1].

**References**


B. Shekhtman
Department of Mathematics
University of South Florida
Tampa
Florida 33620
U.S.A.