SOME IDEMPOTENT MATRICES OF LARGE RANK

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Abstract We investigate the entries of idempotent matrices of large rank. In particular, we construct an example of a sequence of such matrices so that all the entries in most of the rows tend to zero. That answers a question previously raised by the author.

1. Motivation

Let \( \tilde{P} \) be a projection from \( C(-\pi, \pi) \) into itself such that the range of \( \tilde{P} \) is an \( n \)-dimensional subspace of \( \text{span}[e^{i\lambda_1 \theta}, \ldots, e^{i\lambda_m \theta}] \), where \( \lambda_1 < \lambda_2 < \ldots < \lambda_m \) are arbitrary integers. Throughout this note we will assume that

\[
m := m(n) = n + q(n),
\]

where \( q(n)/n \to 0 \) as \( n \to \infty \).

We wish to estimate the norm of \( \tilde{P} \) from below. The desired conclusion (cf. [2,5,4])

\[
\|\tilde{P}\| \geq c \log \frac{n}{q(n)}.
\]

Each \( \tilde{P} \) can be written as

\[
\tilde{P}f = \sum_{j=1}^{m} \left( \int f d\mu_j \right) e^{i\lambda_j \theta}
\]

where \( \mu_j \) are regular Borel measures on \( (-\pi, \pi) \).

Using the Littlewood inequality (cf. [3]) we obtain

\[
\|\tilde{P}\| \geq c \sum_{j=1}^{m} \|\mu_j\|/j.
\]

Hence further estimates depend on the bounds for the norms \( \|\mu_j\| \).
It is easy to observe that the $m \times m$ matrix $P = (p_{k,j})$; $k, j = 1, \ldots, m$, with $p_{k,j} = \mu_k(s^j\theta)$ is an idempotent matrix of rank $n$.

The estimate (1.1) implies

$$\|\tilde{P}\| \geq c \sum_{k=1}^{m} \frac{1}{k} \max_{j} |p_{k,j}|.$$  

Some time ago we conjectured (based on a very closely related Proposition 2 of [4]) that many rows of $P$ contain an element of large size.

**CONJECTURE 1.** Let $m(n) = n + q(n)$; $q(n)/n \rightarrow 0$. Then there exists a constant $c_0 > 0$ such that for any $m \times m$ idempotent matrix $P$ of rank $n$ there exist $n - q(n)$ rows $k_1, \ldots, k_{n-q(n)}$ such that

$$\max \{|p_{k_j,j}| : j = 1, \ldots, m\} \geq c_0.$$

In the next section we give a counterexample to this conjecture and its consequences. In the third section, we provide some positive results in this direction. First we need some notation.

Let $A = (a_{ij})$ be an $m \times n$ matrix. We use $\|A\|_{p,q}$ to denote the norm of an operator $A : \ell_p^n \rightarrow \ell_q^m$ induced by the matrix $A$. In particular

$$\|A\|_{1,\infty} = \max\{|a_{ij}| : i = 1, \ldots, m, j = 1, \ldots, n\}.$$  

If $p = q$ we use $\|A\|_p := \|A\|_{p,p}$.

We use $\nu_{p,q}(A)$ to denote the nuclear norm (cf. [1], section 6.3.1) of an operator $A : \ell_p^n \rightarrow \ell_q^m$. Also $\nu_p(A) := \nu_{p,p}(A)$. In particular,

$$\nu_1(A) = \sum_{i=1}^{m} \max_{j} |a_{ij}|; \quad \nu_\infty(A) = \sum_{j=1}^{n} \max_{i} |a_{ij}|.$$  

Further properties of the nuclear norm can be found in [1].

2. The Main Example

We will need the following:

**PROPOSITION 1** (cf. [1], section 11.11.11). There exists an $n \times n$ matrix $A$ such that rank $A = q(n)$ and

$$\|I - A\|_{1,\infty} \leq 3 \left[ \frac{\log(n + 1)}{q(n)} \right]^{\frac{1}{2}}.$$
With the aid of this proposition we can prove:

**PROPOSITION 2.** There exists an idempotent \( m \times m \) matrix \( P \) with rank \( P = n \) such that the first \( n \) rows in \( P \) have entries

\[
|p_{ij}| \leq 3 \left[ \frac{\log(n+1)}{q(n)} \right]^\frac{1}{3}.
\]

**PROOF.** Let \( A \) be the matrix of the Proposition 1. Then there exist a \( q(n) \times n \) matrix \( B \) and an \( n \times q(n) \) matrix \( C \) such that \( A = CB \). For arbitrary \( \epsilon > 0 \) consider a block-matrix

\[
P = \begin{bmatrix}
I - CB & \epsilon C \\
\frac{1}{\epsilon} B(I - CB) & BC
\end{bmatrix}.
\]

It is easy to see that \( P \) is an \( m \times m \) idempotent matrix with rank \( P = n \). The first \( n \) rows of this matrix consist of the entries of the matrices \( I - CB \) and \( \epsilon C \). By Proposition 1, every entry of \( I - CB \) satisfies the conclusion of the Proposition 2. Choosing \( \epsilon \) sufficiently small we obtain the desired result.

**COROLLARY 3.** The Conjecture 1 is false.

**PROOF.** It suffices to choose \( P \) as in Proposition 2 with \( q(n) = \log^2(n+1) \).

**COROLLARY 4.** There exists a subspace \( E \subset L_{\infty}^m \) such that \( \text{codim} \ E \leq 2q(n) \) and for every projection \( Q \) on \( L_{\infty}^m \) with \( \text{Range} \ Q = E \) we have

\[
\|Q\| \geq \frac{\sqrt{q(n)}}{\log^\frac{1}{2}(m+1)}.
\]

**PROOF.** Let \( G \subset L_{\infty}^m \) be a subspace of all vectors in \( L_{\infty}^m \) such that the last \( q(n) \) coordinates of these vectors are zero. Then \( \text{codim} \ G = q(n) \). Let \( P \) be as in Proposition 2. Then

\[
\text{codim} \ \text{Range} \ P^T < q(n).
\]

Hence,

\[
\text{codim}(\text{Range} \ P^T \cap G) \leq 2q(n).
\]

Define \( E := G \cap \text{Range} \ P^T \) and let \( Q \) be an arbitrary projection from \( L_{\infty}^m \) onto \( E \).

Consider a matrix \( P^1 \) with the first \( n \) columns being the same as in \( P^T \) and the last \( q(n) \) columns being zero. Then for every \( \epsilon \in E \)

\[
P^T \epsilon = P^1 \epsilon = \epsilon.
\]
Hence for every \( a \in \ell_\infty^{(m)} \)

\[ P^T Q a = P^1 Q a = a. \]

By trace-duality (cf. [1])

\[ \nu_\infty(P^1) \| Q \|_\infty \geq \text{tr}(P^1 Q) = \text{tr} Q \geq m - 2q(n). \]

Since the modulus of every element in \( P^1 \) is less than \( 4\sqrt{\log(n+1)/\sqrt{q(n)}} \) we obtain

\[ \nu_\infty(P^1) \leq 4m \sqrt{\log(n+1)/\sqrt{q(n)}}, \]

and from \( m/n \to 1 \)

\[ \| Q \|_\infty \geq C \sqrt{2q(m)/\log(m+1)}, \]

for some absolute constant \( C \).

Remark. It is known that for every \( E \subset \ell_\infty^{(m)} \) with \( \text{codim} E \leq 2q(n) \) there exists a projection \( Q \) onto \( E \) such that \( \| Q \|_\infty \leq \sqrt{2q(n) + 1} \). It would be interesting to know if the factor \( 1/\sqrt{\log(m+1)} \) can be removed in Corollary 2.

3. Positive Results

While the Conjecture 1 does not hold, in general the situation changes if we restrict ourself to the case of symmetric matrices.

**Proposition 5.** Let \( P \) be a symmetric \( m \times m \) idempotent matrix with rank \( P = n \). Then there exists \( n - q(n) \) rows of \( P : k_1, k_2, \ldots, k_{n-q(n)} \) such that

\[ \max \{ p_{k,i,j} : j = 1, \ldots, m \} \geq \frac{1}{2}, \]

for \( \ell = 1, \ldots, n - q(n) \).

**Proof.** Let \( K = \{ k : 1 \leq k \leq m \text{ and } \max \{ p_{k,j} : j = 1, \ldots, m \} \geq \frac{1}{2} \} \). Assume that

\[ s := \# K < n - q(n), \]

and consider a new matrix \( P^1 = (p^1_{k,j}) \) where

\[ p^1_{k,m} = \begin{cases} p_{k,i} & \text{if } k \not\in K \text{ and } j \not\in K, \\ 0 & \text{otherwise.} \end{cases} \]
Using (3.1) we obtain \(|p^1_{k,j}| < \frac{1}{2}\) for all \(k, j\). In order to prove the proposition we will contradict this inequality.

Observe that \(P\) is symmetric and idempotent, hence positive. Therefore, \(P^1\) is also positive. Let \(G = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_k = 0\) for all \(k \not\in K\}\). By assumption (3.2)
\[
\text{codim} G = s < n - q(n),
\]
while \(\dim \text{Range } P = n\). Hence for \(E := \text{Range } P \cap G\)
\[
\dim E > n - s.
\]
Since \(P\) is an identity on \(E\) so is \(P^1\). Hence \(P^1\) has at least \(n - s\) eigenvalues equal to 1.

Since \(P^1\) is positive, the rest of its eigenvalues are positive and hence
\[
n - s \leq \text{tr} P^1 = \sum_{j=0}^{n} p^1_{jj} \leq (n + q(n) - s) \max_{j \in K} p^1_{jj}.
\]
Thus by (3.2)
\[
\max_{j \in K} p^1_{jj} \geq \frac{n - s}{n + q(n) - s} \geq \frac{1}{2}.
\]

Let \(D = [d_{11}, \ldots, d_{mm}]\) be a diagonal \(m \times m\) matrix. It is clear that the expression (1.2) can be rewritten as
\[
\|\hat{P}\| \geq c \cdot \nu_1(DP).
\]
While we cannot evaluate this last expression from below, we finish this note with the following formally weaker inequality.

**PROPOSITION 6.** Let \(P\) be an \(m \times m\) idempotent matrix with rank \(P = n\). Then
\[
\nu_2(DP) \geq c \cdot \log \frac{n}{q(n)}
\]
for some absolute constant \(c\).

**PROOF.** Let \(Q\) be an orthogonal projection from \(\mathbb{R}^m\) onto the Range \(P\). Then \(PQ = Q\) and \(\|Q\|_2 = 1\). Using the ideal property of the nuclear norm we have
\[
\nu_2(DQ) = \nu_2(DPQ) \leq \nu_2(DP)\|Q\|_2.
\]
Thus by Proposition 5
\[
\nu_2(DP) \geq \frac{1}{2} \sum_{j=1}^{n} \frac{1}{k_j}.
\]
Since $1 < k_j < n + q(n)$ we obtain the result.

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Conjecture 1.

References

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