BIVARIATE IDEAL PROJECTORS AND THEIR PERTURBATIONS

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Abstract. In this paper we present a complete description of ideal projectors from the space of bivariate polynomials $\mathbb{F}[x,y]$ onto its subspace $\mathbb{F}_{<n}[x,y]$ of polynomials of degree less than n. Several applications are given. In particular, we study small perturbations of ideal projectors as well as the limits of Lagrange projectors. The latter results verify one particular case of a conjecture of Carl de Boor.

1. Introduction

Ideal interpolation is an elegant form of multivariate approximation which encompasses classical tools of Numerical Analysis, such as Lagrange and Hermite interpolation as well as the Taylor polynomials. It provides a natural link between Multivariate Approximation Theory and Algebraic Geometry. The study of ideal projectors in approximation theory was initiated by G. Birkhoff [1] and continued by several authors. A comprehensive list of references can be found in the excellent surveys by C. de Boor [3] and by M. Gasca and T. Sauer [8].

In this paper we present a complete description of ideal projectors from the space of bivariate polynomials $\mathbb{F}[x,y]$ onto its subspace $\mathbb{F}_{<n}[x,y]$ of polynomials of degree less than n. (Here $\mathbb{F}$ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$). The reason for limiting ourself to the bivariate situation is rather simple: many of the results presented here are either false or not known for more than two variables. The reason for restricting our study to projectors onto $\mathbb{F}_{<n}[x,y]$, as opposed to the ideal projectors with arbitrary range, is three-fold. First, approximation by polynomials from $\mathbb{F}_{<n}[x,y]$ is the most common and classical form of polynomial approximation. Second, the class of ideal projectors onto $\mathbb{F}_{<n}[x,y]$ is sufficiently rich with patterns and anomalies to deserve an independent study. Third, the analogs of our results for ideal projectors with arbitrary range rely heavily on Grothendieck's description of Hilbert Schemes. I feel that the explicit "parameterization" of ideal projectors presented in this paper avoids the jungle of abstractions needed in the general situation and makes the meaning of the results easily accessible to an analyst (such as myself) not familiar with Algebraic Geometry.

As with any multivariate study, the first obstacle to overcome is the notations. Here is a small sample:

The symbol $\mathbb{F}$ stands for the real or complex field, $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, x_2, ..., x_d]$ stands for polynomials of d variables, the space of polynomials of one and two variables is denoted as $\mathbb{F}[x], \mathbb{F}[x,y]$ respectively. An element $f \in \mathbb{F}[\mathbf{x}]$ is written as a finite sum

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$\sum \hat{f}(k_1, ..., k_d) x_1^{k_1} x_2^{k_2} .... x_d^{k_d}$ or, in multiindex notation, $\sum \hat{f}(\mathbf{k})\mathbf{x}^{\mathbf{k}}$ with $\hat{f}(\mathbf{k}) \in \mathbb{F}$. By $\mathfrak{G}^N$ we denote the set (Grassmannian) of all N-dimensional (vector) subspaces of $\mathbb{F}[\mathbf{x}]$. The space of polynomials of degree less than n is denoted by $\mathbb{F}_{<n}[\mathbf{x}]$, the set of polynomials of degree n is $\mathbb{F}_{=n}[\mathbf{x}]$, while the space of homogeneous polynomials (forms) of degree n is denoted as $\mathbb{F}_{[n]}[\mathbf{x}]$. Finally the set of monomials of degree n is $M_n[\mathbf{x}]$, and $M[\mathbf{x}]$ is the set of all monomials in $\mathbb{F}[\mathbf{x}]$. For convenience we use $N(n) := \dim \mathbb{F}_{<n}[x, y]$, hence $\mathbb{F}_{<n}[x, y] \in \mathfrak{G}^{N(n)}$.

A polynomial $f \in \mathbb{F}[\mathbf{x}]$ can be written (uniquely) as a finite sum $f = \sum f^{[k]}$ with $f^{[k]} \in \mathbb{F}_{[k]}[\mathbf{x}]$, i.e., a homogeneous component of f. The non-zero homogeneous component that correspond to the largest k is called the leading form of the polynomial f and is denoted by $\mathrm{Lf}(f)$. Hence, the leading form of a nontrivial polynomial $f \in \mathbb{F}[\mathbf{x}]$ is the unique homogeneous polynomial $\mathrm{Lf}(f)$ such that $\deg(f - \mathrm{Lf}(f)) < \deg f$. Similarly, the non-zero homogeneous component that correspond to the least k, is the least form of f and is denoted by $\mathrm{lf}(f)$. For $f = 0$ we set $\mathrm{Lf}(f) = \mathrm{lf}(f) = 0$.

We will need some standard notions from Algebraic Geometry (cf. [5], [6]). Every subset $H \subset \mathbb{F}^d[\mathbf{x}]$ generates the corresponding ideal $\langle H \rangle$ consisting of all finite sums

$$\sum_{h \in H} q_h \cdot h, \ \ q_h \in \mathbb{F}[\mathbf{x}].$$

The set H is called a basis for the ideal $\langle H \rangle$. By the Hilbert Basis Theorem (cf. [5], p. 74), every ideal $J \subset \mathbb{F}[\mathbf{x}]$ is finitely generated, i.e., for every ideal $J \subset \mathbb{F}[\mathbf{x}]$, there exists a finite set $H \subset \mathbb{F}[\mathbf{x}]$ such that $J = \langle H \rangle$.

There are two distinct notions of minimality of a basis. Following [6], a basis H is called *unshortenable* if, for any proper subset $H_0 \subset H$, we have $\langle H_0 \rangle \neq \langle H \rangle$. A basis H is called *minimal* if $H_0 \subset \mathbb{F}[\mathbf{x}]$, $\#H_0 < \#H$ implies $\langle H_0 \rangle \neq \langle H \rangle$. In other words, a minimal basis is a basis with minimal number of terms.

A basis H for an ideal $J \in \mathbb{F}[\mathbf{x}]$ is called an H-basis if for every $f \in J$ there exists a (generalized) sequence ($g_{h,f} \in \mathbb{F}[\mathbf{x}]$, $h \in H$) such that

$$f = \sum_{b \in B} g_{h,f} h \text{ and } \deg g_{h,f} + \deg h \leq \deg f \text{ for all } h \in H.$$

For every ideal $J \subset \mathbb{F}[\mathbf{x}]$ we use $Z(J)$ to denote the associated variety

$$Z(J) = \{\mathbf{z} \in \mathbb{F}^d : f(\mathbf{z}) = 0, \forall f \in J\}.$$

The ideal J is called zero-dimensional (cf. [6]) if

$$\dim(\mathbb{F}[\mathbf{x}]/J) < \infty.$$

In algebraic terms, the quantity $\dim(\mathbb{F}[\mathbf{x}]/J)$ is called *colength* of J and measures the number of points in $Z(J)$ (with properly counted multiplicity). In particular zero-dimensionality of an ideal J implies (and for $\mathbb{F} = \mathbb{C}$ is equivalent to) the condition that the set $Z(J)$ is finite. On the other hand, $\dim(\mathbb{F}[\mathbf{x}]/J)$ measures the dimension of subspaces $G \subset \mathbb{F}[\mathbf{x}]$ that complement J, i.e.

$$\mathbb{F}[\mathbf{x}] = G \oplus J.$$

This decomposition defines a projector P onto G with $\ker P = J$. Since the paper focuses on the projectors, we will use the term *codimension* of J when referring to $\dim(\mathbb{F}[\mathbf{x}]/J) =: \mathrm{codim} J$.

Likewise, with every set $Z \subset \mathbb{F}^d$ we associate the ideal

$$J(Z) := \{f \in \mathbb{F}[\mathbf{x}] : f(\mathbf{z}) = 0, \forall \mathbf{z} \in Z\}.$$

It is easy to see (cf. [6]) that $J \subset J(Z(J))$. An ideal J is called a radical ideal if $f^m \in J$ for some integer m implies $f \in J$. For the complex field, the strong Nullstellensatz (cf. [5], Theorem 6, p. 174) states that the ideal $J \subset \mathbb{C}[\mathbf{x}]$ is radical if an only if $J(Z(J)) = J$.

It is well-known and easy to see (cf., e.g., [6], p.143) that, for every zero-dimensional $J \subset \mathbb{F}[\mathbf{x}]$, the cardinality $\#Z(J)$ of its associated variety is finite and bounded by the codimension of J, i.e.,

$$\#Z(J) \leq \operatorname{codim} J,$$

with equality holding if and only if J is a radical ideal. We use $\mathfrak{J}_N$ to denote the family (Hilbert scheme) of ideals $J \subset \mathbb{F}[\mathbf{x}]$ such that $\operatorname{codim} J = N$.

Finally, let G be a vector subspace of $\mathbb{F}[\mathbf{x}]$. We use $\mathfrak{J}_G$ to denote the family of all ideals, complementary to G. Clearly $\mathfrak{J}_G \neq \varnothing$ (cf. Proposition 4.4 below). Thus every ideal $J \in \mathfrak{J}_G$ uniquely determines a projector P_J from $\mathbb{F}[\mathbf{x}]$ onto G with $\ker P_J = J$.

Definition 1.1. *(Birkhoff, [1]). A linear idempotent operator P on $\mathbb{F}[\mathbf{x}]$ is called an ideal projector if $\ker P$ is an ideal in $\mathbb{F}[\mathbf{x}]$.*

This establishes a one-to-one correspondence between the family $\mathfrak{J}_G$ and the family $\mathfrak{P}_G$ of all ideal projectors onto G.

The standard example of an ideal projector is a *Lagrange projector*, i.e., a projector P for which Pf is the unique element in its range that agrees with f at a certain finite set Z in $\mathbb{F}^d$. For, its kernel consists of exactly those polynomials that vanish on Z, i.e., it is the zero-dimensional radical ideal $J(Z)$. An ideal $J \in \mathfrak{J}_N$ is called a *Lagrange ideal* if $J = J(Z)$ for some set Z of N distinct points in $\mathbb{F}^d$. As was noted earlier, if $\mathbb{F} = \mathbb{C}$ then Lagrange and radical ideals are one and the same.

Another important example of ideal projectors is the *Taylor projector* onto $\mathbb{F}_{<n}[\mathbf{x}]$ at a point $\mathbf{a} \in \mathbb{F}^d$ since its kernel is given by the (maximal) ideal

$$\left\langle (\mathbf{x} - \mathbf{a})^{\mathbf{k}}, |\mathbf{k}| = n \right\rangle.$$

In the complex case every maximal ideal is of this form, hence there is a one-to-one correspondence between the points $\mathbf{a} \in \mathbb{C}^d$ and maximal ideals

$$\mathfrak{m}_{\mathbf{a}} := \{f \in \mathbb{F}[\mathbf{x}] : f(\mathbf{a}) = 0\}$$

in $\mathbb{C}[\mathbf{x}]$. Thus we have an algebraic description of the Taylor projector as a projector P onto $\mathbb{C}_{<n}[\mathbf{x}]$ with $\ker P = \mathfrak{m}^n$ for some maximal ideal $\mathfrak{m} \in \mathbb{F}[\mathbf{x}]$.

A full description of "interpolation" properties of finite-dimensional ideal projectors is provided in [11] and [12]. Even if one is only interested in Lagrange interpolation, we feel that the "ideal" approach illuminates the underlying algebraic nature of a problem and provides a fruitful and convenient understanding of it. It emphasizes the process of division of a polynomial by an ideal (cf. [13]), the remainder of which is the desired interpolation.

We will use the rest of this extended introduction to explain the main result of this paper. As noted previously, we are only concerned with ideal projectors that map $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<n}[x, y]$. From the "de Boor's formula" (cf. [3])

$$P(fg) = P(fPg) \tag{1.1}$$

that characterizes ideal projectors P, it follows that, to describe the set of all ideal projectors onto $\mathbb{F}_{<n}[x,y]$, it is sufficient to describe the action of these projectors on monomials $x^{n-j}y^j$. In other words, a sequence

$$(P(x^{n-j}y^j), j=0,...,n)$$

of $n+1$ polynomials in $\mathbb{F}_{<n}[x,y]$ completely determines the ideal projector P, i.e., the set of polynomials $\{x^{n-j}y^j - P(x^{n-j}y^j), j=0,...,n\}$ is a basis for the ideal $\ker P$. Each of these polynomials depends on $\frac{n\times(n+1)}{2}$ parameters (coefficients of $P(x^{n-j}y^j)$). Hence every ideal projector onto $\mathbb{F}_{<n}[x,y]$ depends on $\frac{n\times(n+1)^2}{2}$ parameters. However, not every sequence of polynomials

$$(p_j \in \mathbb{F}_{<n}[x,y], j=0,...,n)$$

determines an ideal projector by $P(x^{n-j}y^j) = p_j$. The main theorem of this article tells exactly which sequences of polynomials do and which do not. In turns out that those that those that do, allow for an explicit description in terms of $n\times(n+1)$ parameters (coefficients of the leading form of $P(x^{n-j}y^j)$). That is, given $\mathrm{Lf}(P(x^{n-j}y^j))$ (homogeneous polynomial of degree $n-1$ or 0), we can compute the rest of the polynomials $P(x^{n-j}y^j)$. The converse also holds: Given an arbitrary sequence of $(n+1)$ homogeneous polynomials ($q_j \in \mathbb{F}_{[n-1]}[x,y]$, $j=0,...,n$), there exists a unique ideal projector P onto $\mathbb{F}_{<n}[x,y]$ such that $\mathrm{Lf}(\ P(x^{n-j}y^j)) = q_j$.

For instance, let P be any projector from the quadratic polynomials $\mathbb{F}_{\leq 2}[x,y]$ onto $\mathbb{F}_{<2}[x,y]$ =span$\{1,x,y\}$. Let

$$Px^{2-j}y^j = a_j + b_j x + c_j y,\ \ j=0,1,2.$$

Then P can be extended to an ideal projector from $\mathbb{F}[x,y]$ onto $\mathbb{F}_{<2}[x,y]$ if and only if

$$\begin{aligned} a_0 &= -c_0c_2 + c_0b_1 - c_1b_0 + c_1^2,\\ a_1 &= -b_1c_1 + c_0b_2,\\ a_2 &= -b_2b_0 + b_2c_1 - b_1c_2 + b_1^2. \end{aligned}$$

In the next section we will state and prove the main theorem. The rest of the sections explore various applications of the theorem. Section 3 establishes an "ideals-ideal projectors" dictionary. That is, we describe a correspondence between approximation properties of ideal projectors and algebraic properties of the related ideals. In section 4 we study the perturbation of ideal projectors. Section 5 uses the result of section 4. We prove that in bivariate setting a given $N(n)$-dimensional subspace $G \subset \mathbb{F}[x,y]$ complements the generic ideal in $\mathfrak{J}_{\mathbb{F}_{<n}[x,y]}$ and that the generic projector onto $\mathbb{F}_{<n}[x,y]$ is a Lagrange projector. Both properties fail for three or more variables.

2. The Main Theorem

We will start with a proposition that summarizes some of the simple properties of ideal projectors.

Proposition 2.1. *Let P be a linear mapping on $\mathbb{F}[\mathbf{x}]$. Then*

1) P is an ideal projector if and only if

$$P(fg) = P(f\cdot Pg) \tag{2.1}$$

for all $f,g \in \mathbb{F}[\mathbf{x}]$.

2) P is an ideal projector if and only if the mapping $P' := I - P$ satisfies

$$P'(fg) = f \cdot P'(g) + P'(f \cdot Pg) \tag{2.2}$$

for all $f, g \in \mathbb{F}[\mathbf{x}]$.

The first part of this proposition was observed by de Boor (cf. [3]). The second is easily deduced from the first (cf. [15]).

Let G be an N-dimensional subspace in $\mathbb{F}[\mathbf{x}]$ with basis $\{g_1, ..., g_N\}$ and let P be an ideal projector onto G. The proposition implies that the sequence

$$(P(1) \text{ and } p_{j,k} := P(x_j \cdot g_k), k = 1, ..., N; j = 1, ..., d) \tag{2.3}$$

of $Nd + 1$ polynomials in G determine the projector P. Equivalently, (2.2) implies that the set of polynomials

$$\{h_{j,k} := (x_j \cdot g_k - p_{j,k}) \in \ker P, k = 1, ..., N; j = 1, ..., d\} \cup \{1 - P(1)\}$$

is a basis for the ideal $\ker P$.

In particular, if $G = \mathbb{F}_{<n}[x, y] \subset \mathbb{F}[x, y]$, hence $1 \in G$, then the sequence

$$(p_j := P(x^{n-j} y^j) \in \mathbb{F}_{<n}[x, y], j = 0, ..., n) \tag{2.4}$$

of $n + 1$ polynomials determines P on the whole space $\mathbb{F}[x, y]$ and the set $\{h_j := (x^{n-j} y^j - p_j) \in \ker P,\ j = 0, ..., n\}$ is a basis for the ideal $\ker P$. Formula (2.1) also shows that the polynomials p_j must satisfy additional relations. Indeed, we have

$$P(x^{n-j} y^{j+1}) = P(xP(x^{n-j-1} y^{j+1})) = P(yP(x^{n-j} y^j)).$$

Hence

$$P(xp_{j+1}) = P(yp_j) \text{ for all } j = 1, ..., n$$

or equivalently

$$yp_j - xp_{j+1} \in \ker P \text{ for all } j = 0, ..., n - 1. \tag{2.5}$$

Consider now the reverse situation. Suppose we start with an arbitrary sequence of polynomials $p_0, ..., p_n \in \mathbb{F}_{<n}[x, y]$ and define a projector

$$P : \mathbb{F}_{\leq n}[x, y] \to \mathbb{F}_{<n}[x, y]$$

by letting $P(x^j y^{n-j}) = p_j$. We can attempt to use the ideal property (2.1) to extend this projector to the whole of $\mathbb{F}[x, y]$. However, such an extension may not be well defined. First of all, the polynomials p_j must satisfy (2.4). That would guarantee the well-defined extension of P to $\mathbb{F}_{\leq n+1}[x, y]$. To have the well-defined extension to arbitrary $\mathbb{F}_{\leq m}[x, y]$ we have to check that

$$f_1 g_1 = f_2 g_2 \in M_m \Longrightarrow f_1 P g_1 - f_2 P g_2 \in \ker P. \tag{2.6}$$

The next proposition shows that (2.4) implies (2.5) for all m. This result is not new. It follows from the fact that the space $\mathbb{F}_{<n}[x, y]$ satisfies the “Mourrain condition” (cf. [3]). However, in our case the proof is straightforward:

Proposition 2.2. *Let P be a projector from $\mathbb{F}_{\leq n}[x, y]$ onto $\mathbb{F}_{<n}[x, y]$. If the corresponding p_j as defined in (2.4) satisfy (2.6) then P has an ideal extension to $\mathbb{F}_{\leq m}[x, y]$ for all m and hence P has an ideal extension to $\mathbb{F}[x, y]$.*

Proof. The proof proceeds by induction on m. Let $m \geq n+1$ and let P be a projector from $\mathbb{F}_{\leq m}[x,y]$ onto $\mathbb{F}_{<n}[x,y]$ satisfying

$$f_1 g_1 = f_2 g_2 \in \mathbb{F}_{\leq m}[x,y] \Longrightarrow f_1 P g_1 - f_2 P g_2 \in \ker P. \tag{2.7}$$

Define

$$Q(x^j y^k) = \begin{cases} P(x^j y^k) & \text{if } j+k \leq m; \\ P(yP(x^j y^{k-1})) & \text{if } j+k = m+1;\ k \geq 1; \\ P(xP(x^m)) & \text{if } j = m+1, k = 0. \end{cases} \tag{2.8}$$

We need to show that

$$f_1 g_1 = f_2 g_2 = x^j y^{m+1-j} \Longrightarrow f_1 Q g_1 - f_2 Q g_2 \in \ker Q. \tag{2.9}$$

First assume that $g_1$and g_2 $\notin$ran$P$ and consider two cases:

Case 1. Suppose that $f_1 = qh_1$ and $f_2 = qh_2$ with $\deg q \geq 1$. Then

$$Q(f_1 Q g_1 - f_2 Q g_2) = Q(q(h_1 P g_1 - h_2 P g_2)).$$

Since

$$\deg(q(h_1 P g_1 - h_2 P g_2)) \leq m \text{ and } h_1 g_1 = h_2 g_2 \in \mathbb{F}_{\leq m}[x,y]$$

and using the inductive assumption twice we have

$$Q(f_1 P g_1 - f_2 P g_2) = P(q(h_1 P g_1 - h_2 P g_2)) = P(qP(h_1 P g_1 - h_2 P g_2)) = 0$$

since $h_1 P g_1 - h_2 P g_2 \in \ker P$.

Case 2: Suppose that f_1 and f_2 have no common divisors. Then $f_1 = x^s$ and $f_2 = y^t$, say. Therefore

$$g_1 = x^{j-s} y^{m-1-j} \text{ and } g_2 = x^j y^{m-1-j-t}.$$

It follows from the previous case that

$$(xyP(x^{j-s-1} y^{m-2-j}) - x^s P(x^{j-s} y^{m+1-j})) \in \ker Q$$

and

$$(y^t P(x^j y^{m+1-j-t}) - xyP(x^{j-s-1} y^{m-2-j})) \in \ker Q.$$

Adding these two equations we get the desired conclusion.

Now suppose that $g_1$and/or g_2 $\in$ran$P$. Then choosing $f_0 = x$ or y we find g_0 $\notin$ran$P$ such that

$$f_1 g_1 = f_0 g_0 = f_2 g_2.$$

We have

$$Q(f_1 P(g_1)) = Q(f_1 g_1) := Q(f_0 P(g_0)). \tag{2.10}$$

Similarly if g_2 $\in$ran$P$, then

$$Q(f_2 P(g_2)) = Q(f_2 g_2) := Q(f_0 P(g_0)). \tag{2.11}$$

Otherwise (if g_2 $\notin$ran$P$), (2.11) follows from the previous steps. Combining (2.10) and (2.11) we have the desired conclusion. □

This proposition shows that every ideal projector P onto $\mathbb{F}_{<n}[x,y]$ is uniquely determined by the sequence (2.4) of polynomials (p_j, $j = 0, ..., n$) satisfying (2.6) and conversely every sequence of polynomials ($p_j \in \mathbb{F}_{<n}[x,y]$, $j = 0, ..., n$) that satisfies (2.6) uniquely defines an ideal projector by (2.4). Hence, our goal of describing all ideal projectors onto $\mathbb{F}_{<n}[x,y]$ is reduced to describing all polynomial sequences (p_j, $j = 0, ..., n$) in $\mathbb{F}_{<n}[x,y]$ that satisfy (2.6).

Proposition 2.3. *Let P be an ideal projector onto $\mathbb{F}_{<n}[x,y]$; set $P(x^{n-j}y^j) =: p_j$. Then there exists a sequence of $n(n+1)$ "determining coefficients"*

$$(a_{j,k}, j = 1, ..., n, k = 0, ..., n)$$

such that

$$yp_j - xp_{j+1} = \sum_{k=0}^{n} a_{j,k}(x^{n-k}y^k - p_k),\ \ j = 0, ..., n-1. \tag{2.12}$$

Conversely, if the p_j satisfy (2.12) for some constants $a_{j,k}$ then the $p_j = P(x^{n-j}y^j)$ determine an ideal projector.

Proof. Consider P as a projector from $\mathbb{F}_{\le n}[x,y]$ onto $\mathbb{F}_{<n}[x,y]$. Since the functions $(x^{n-k}y^k - p_k)$ are in $\ker P$ and are linearly independent, we have

$$\begin{aligned} n+1 &= \dim\operatorname{span}\{x^{n-k}y^k - p_k,\ k = 0, ..., n\} \\ &= \dim \mathbb{F}_{\le n}[x,y] - \dim \mathbb{F}_{<n}[x,y] \end{aligned}$$

and hence

$$\operatorname{span}\{x^k y^{n-k} - p_k,\ k = 0, ..., n\} = \ker P.$$

This together with (2.12) implies (2.4). The converse is obvious. □

In view of the importance of the equations (2.12) we call them "the consistency equations". We think of (2.12) as equations for p_j or as a system of *linear equations* for the coefficients of the polynomials p_j. Observe that there are $(n+1)$ polynomials p_j each having $\frac{n(n+1)}{2}$ coefficients. Hence the number of unknowns in the consistency equations is $\frac{n(n+1)^2}{2}$. There are n equations in (2.12) for polynomials of degree n. Equating the terms in front of each monomial gives us $\frac{n(n+1)(n+2)}{2}$ equations. Thus there are more equations than unknowns, $\frac{n(n+1)}{2}$ more, to be precise. The next, main theorem of this section shows that there is enough redundancy built into the equations to overcome this discrepancy. Moreover, it gives an explicit description of all sequences of polynomials p_j that determine an ideal projector onto $\mathbb{F}_{<n}[x,y]$.

For a given n, we will need a fixed $n \times (n+1)$ matrix:

$$\Xi = \begin{bmatrix} y & -x & 0 & 0 & ... & 0 & 0 \\ 0 & y & -x & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & ... & y & -x \end{bmatrix}$$

and an arbitrary $n \times (n+1)$ matrix A, with the following enumeration of the entries:

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & ... & a_{0,n-1} & a_{0,n} \\ a_{1,0} & a_{1,1} & ... & a_{1,n-1} & a_{1,n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{n-1,0} & a_{n-1,1} & ... & a_{n-1,n-1} & a_{n-1,n} \end{bmatrix}$$

For an arbitrary $n \times (n+1)$ matrix B we use B_k to denote its square submatrix obtained from B by deleting the k-th column.

Theorem 2.4. *Every $n \times (n+1)$ matrix A defines a unique sequence*

$$p_k := (x^{n-k}y^k - (-1)^{n+k}\det(\Xi + A)_k),\ k = 0, ..., n$$

of polynomials satisfying (2.12). Hence every $n \times (n+1)$ matrix A defines a unique ideal projector P_A from $\mathbb{F}[x, y]$ onto $\mathbb{F}_{<n}[x, y]$ by letting $P_A(x^{n-k}y^k) = p_k$.

The coefficients of the resulting polynomials p_j are (homogeneous) polynomials in the $a_{j,k}$.

Proof. To write (2.12) in matrix form, let $\mathbf{p} = (p_0, p_1, ..., p_n) \in (\mathbb{F}_{<n}[x, y])^{n+1}$ be the vector of polynomials to be found. Let $\mathbf{q} = (x^n, x^{n-1}y, ..., y^n)$. Then the consistency equations are written as $\Xi\mathbf{p} = A(\mathbf{q} - \mathbf{p})$ or equivalently $(\Xi+A)\mathbf{p} = A\mathbf{q}$. Observe that $\Xi\mathbf{q}$ =0, which implies that the equations (2.8) are equivalent to

$$(\Xi + A)(\mathbf{q} - \mathbf{p}) = 0 \tag{2.13}$$

i.e. the $(n+1)$-dimensional vector $\mathbf{q} - \mathbf{p}$ is orthogonal to the n rows of the matrix $(\Xi + A)$. By Linear Algebra, the vector

$$((-1)^k \det((\Xi + A)_k, k = 0, ..., n)$$

is one such vector. From the form of the matrix Ξ, $\det(\Xi_k) = (-1)^{n+k}x^{n-k}y^k$, thus $\det((\Xi+A)_k) \in \mathbb{F}_{\le n}[x, y]$ and $(-1)^{n+k}x^{n-k}y^k$ is the only monomial in $\det(\Xi+A)_k$ of degree n, hence the vector of polynomials

$$p_k := x^{n-k}y^k - (-1)^{n+k}\det(\Xi + A)_k) \in \mathbb{F}_{<n}[x, y] \tag{2.14}$$

is a solution to equations (2.13).

If $\mathbf{p}'$ is another solution to (2.12) then $(\Xi + A)(\mathbf{p} - \mathbf{p}') = 0$ which easily leads to $\mathbf{p} = \mathbf{p}'$. Indeed, since $\det(\Xi + A)_k$ is a non-zero polynomial, the matrix $\Xi + A$ has rank n for almost all x and y, thus the rows of the matrix $(\Xi + A)$ are linearly independent for almost all values of x and y, hence $p_k - p'_k = c(-1)^k \det((\Xi + A)_k)$ where the constant c is chosen independently of k. But $p_k - p'_k \in \mathbb{F}_{<n}[x, y]$ while $(-1)^k \det((\Xi + A)_k)$ is of degree n. That implies that $c = 0$ and $p_k = p'_k$.

Formula (2.14) shows that the coefficients of p_j are polynomials in the $a_{j,k}$. □

3. Corollaries

The equation (2.14) provides a simple way of computing (using a Computer Algebra System) the family of all ideal projectors onto $\mathbb{F}_{<n}[x, y]$. For instance:

Example 3.1. *To compute all ideal projectors onto $\mathbb{F}_{<2}[x, y]$ =span$\{1, x, y\}$ all we need to do, is to write down the matrix*

$$\begin{aligned} \Xi + A &= \begin{bmatrix} y & -x & 0 \\ 0 & y & -x \end{bmatrix} + \begin{bmatrix} a_{0,0} & a_{0,1} & a_{0,2} \\ a_{1,0} & a_{1,1} & a_{1,2} \end{bmatrix} \\ &= \begin{bmatrix} y + a_{0,0} & -x + a_{0,1} & a_{0,2} \\ a_{1,0} & y + a_{1,1} & -x + a_{1,2} \end{bmatrix} \end{aligned}$$

and compute all three of its subdeterminants. For every matrix A, formula (2.14) gives ideal projectors P_A:

$$\begin{aligned} &P_Ax^2 = a_{0,2}y + (a_{1,2} + a_{0,1})x + (a_{0,2}a_{1,1} - a_{0,1}a_{1,2}), \\ &P_Axy = a_{1,2}y - a_{0,0}x + (a_{0,0}a_{1,2} - a_{0,2}a_{1,0}), \\ &P_Ay^2 = -(a_{1,1} + a_{0,0})y - a_{1,0}x + (a_{1,0}a_{0,1} - a_{0,0}a_{1,1}). \end{aligned}$$

Comparing the leading forms in the equations (2.12) we obtain an easy relationship between the coefficients of the leading forms of $p_j = P_A(x^{n-j}y^j)$ and the determining coefficients $a_{j,k}$. We have

Corollary 3.2. *Let P_A be the ideal projector onto $\mathbb{F}_{<n}[x,y]$ determined by the matrix A. Let*

$$\mathrm{Lf}(P_A(x^{n-j}y^j)) = \mathrm{Lf}(p_j) = \sum_{m=0}^{n-1} \hat{p}_j(m, n-m)x^m y^{n-m}.$$

Then

$$a_{i,j} = \hat{p}_i(n-j, j-1) - \hat{p}_{i+1}(n-j-1, j), i = 0, ..., n-1, \ j = 0, ..., n, \tag{3.1}$$

where $\hat{p}_i(m,l) = 0$ if $m < 0$ or $l < 0$.

In particular, the leading forms of the polynomials $P_A(x^{n-j}y^j)$ uniquely determine these polynomials and hence the ideal projector P_A. This phenomenon is unique to two variables.

The formulas (2.14) and (2.12) provide a correspondence between the family of ideal projectors onto $\mathbb{F}_{<n}[x,y]$ and $n \times (n+1)$ matrices.

Corollary 3.3. *Every ideal projector P onto $\mathbb{F}_{<n}[x,y]$ generates an $n \times (n+1)$ matrix A_P by (2.12) and every $n \times (n+1)$ matrix A generates an ideal projector P_A onto $\mathbb{F}_{<n}[x,y]$. Clearly*

$$A_{P_A} = A \text{ and } P_{A_P} = P.$$

In particular:

Corollary 3.4. *Let P_A be the ideal projector onto $\mathbb{F}_{<n}[x,y]$ specified by A and let $f \in \mathbb{F}[x,y]$ be a fixed polynomial. Then the coefficients $(\widehat{P_A f}(j,k),\ j+k < n)$ of the polynomial $P_A f$ are themselves polynomials of the entries of the matrix A, i.e.,*

$$P_A f = \sum_{j+k<n}^{n-1} q_{j,k,f}(A)x^j y^k$$

where $q_{j,k,f} \in \mathbb{F}[A]$ are polynomials in $n(n+1)$ variables (entries of A).

Proof. By Theorem 2.4, for every $f \in M_n$, the coefficients of $P_A f$, obtained from the determinants (2.14), are (homogeneous) polynomials in the entries of the matrix A. Inductively, (2.1) implies that for every m and every $f \in M_m$, the coefficients of $P_A f$ are polynomials in the entries of the matrix A. Thus the same is true for all $f \in \mathbb{F}[x,y]$. □

In particular, we wish to associate each A with the $(\frac{n(n+1)^2}{2})$-sequence

$$\mathbf{w}_A := (\widehat{P_A(x^{n-j}}y^j)(m,k),\ m+k < n, j = 0, ..., n)$$

of polynomial coefficients of the p_j.

The set

$$W_n := \{\mathbf{w}_A,\ A \in \mathbb{F}^{n(n+1)}\}$$

of all such sequences "parameterizes" the set $\mathfrak{P}_{\mathbb{F}_{<n}[x,y]}$ of all ideal projectors onto $\mathbb{F}_{<n}[x,y]$. In fact, a lot more can be said about the set W_n :

Corollary 3.5. *The set W_n is a polynomial image of $\mathbb{F}^{n(n+1)})$ and thus an irreducible $n(n+1) = 2 \times \dim \mathbb{F}_{<n}[x,y]$-dimensional affine algebraic variety in $\mathbb{F}^{\frac{n(n+1)^2}{2}}$.*

Proof. The equations (2.12) gives W_n the structure of an affine algebraic variety in $\mathbb{F}^{\frac{n(n+1)^2}{2}}$. Since, by Theorem 2.4, the coefficients of the polynomials $P_A(x^j y^{n-j})$ are polynomials in the entries of the matrix $A \in \mathbb{F}^{n(n+1)}$, hence W_n is a polynomial image of $\mathbb{F}^{n(n+1)}$ in $\mathbb{F}^{\frac{n(n+1)^2}{2}}$. Thus (cf. [5], Proposition 5, p. 196), this image is an irreducible affine algebraic variety of the same dimension as $\dim \mathbb{F}^{n(n+1)}$. □

Proposition 3.6. *Let P be an ideal projector onto $\mathbb{F}_{<n}[x,y]$. The set*

$$H_P = \{h_j := P'(x^{n-j}y^j) = (I-P)(x^{n-j}y^j), j = 0, ..., n\} \subset \mathbb{F}[x,y]$$

is an H-basis for the ideal $\ker P$ *as well as a Groebner basis for* $\ker P$ *with respect to any graded monomial order (cf. [11]).*

Proof. It follows from (2.2) that H_P is a basis for the ideal $\ker P$. It is known (cf. [13]) that a basis B is an H-basis if and only if $\langle \mathrm{Lt}(b), b \in B\rangle = \langle \mathrm{Lt}(f), f \in \langle B\rangle\rangle$, where $\mathrm{Lt}(f)$ stands for the leading term of f with respect to a graded monomial order (cf. [13]). Since $\mathrm{Lt}(P'(x^{n-j}y^j)) = x^{n-j}y^j$ for $j = 0, ..., n$, it follows that

$$\langle \mathrm{Lt}(h), h \in H_P\rangle = \langle M_n[x,y]\rangle\,.$$

Suppose that for some non-zero $f \in \langle H_P\rangle$, we have $\mathrm{Lt}(f) \notin \langle M_n[x,y]\rangle$. Then $\deg f < n$ and hence $f \in \langle H_P\rangle \cap \mathbb{F}[x,y] = \ker P \cap \mathrm{ran} P$ which is a contradiction.

The proof that H_P is a Groebner basis with respect to graded monomial order is similar. □

Since every ideal $J \in \mathfrak{J}_{\mathbb{F}_{<n}[x,y]}$ generates an ideal projector $P_J \in \mathfrak{P}_{\mathbb{F}_{<n}[x,y]}$, we have

Corollary 3.7. *Every ideal $J \in \mathfrak{J}_{\mathbb{F}_{<n}[x,y]}$ is generated by polynomials*

$$(h_k := \det(\Xi + A_J)_k, k = 0, ..., n),$$

for some $n \times (n+1)$ matrix A_J. Conversely, for every $n \times (n+1)$ matrix A the polynomials $h_k := \det((\Xi + A_J)_k$ generate an ideal

$$J_A := \langle h_k, k = 0, ..., n\rangle \in \mathfrak{J}_{\mathbb{F}_{<n}[x,y]}.$$

When is the basis $\{h_j = P_A'(x^{n-j}y^j), j = 0, ..., n\}$ an unshortenable basis for the ideal P_A? The next corollary answers this question.

Corollary 3.8. *The basis $\{P_A'(x^j y^{n-j}), j = 0, ..., n\}$ is an unshortenable basis for the ideal J_A if and only if A is bidiagonal, i.e.,*

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & 0 & ... & 0 & 0 \\ 0 & a_{1,1} & a_{1,2} & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a_{n-1,n-1} & a_{n-1,n} \end{bmatrix}. \tag{3.2}$$

If an ideal projector P_A is a Lagrange projectors then the family of Lagrange projectors P_A with bidiagonal A coincides with the family of Lagrange projectors that interpolate on a triangular subgrid of a tensor-product grid, studied in [14].

Proof. By (2.12) we have

$$yh_j - xh_{j+1} = \sum_{k=0}^{n} a_{j,k} h_k,\ \ j = 0, ..., n-1$$

or equivalently

$$(y - a_{j,j})h_j - (x - a_{j,j+1})h_{j+1} = \sum_{k\neq j.j+1}^{n} a_{j,k}h_k,\ j = 1, ..., n.$$

If one of the coefficients $a_{j,m},\ m \neq j, j+1$ is non-zero then the corresponding polynomial

$$h_m = \frac{1}{a_{j,m}}((y - a_{j,j})h_j - (x - a_{j,j+1})h_{j+1} - \sum_{k\neq j,j-1,m}^{n} a_{j,k}h_k) \in \langle h_s, s \neq m\rangle$$

and the basis $\{P'_A(x^jy^{n-j}), j = 0, ..., n\}$ is not unshortenable.

Conversely, suppose A is bidiagonal. Then every submatrix of $\Xi + A$ is a triangular matrix and by the theorem 2.4 we have

$$\pm h_m = \prod_{j=0}^{m-1}(y + a_{j,j}) \cdot \prod_{j=m}^{n-1}(-x + a_{j,j+1}), m = 0, ..., n, \tag{3.3}$$

where the empty products are set to be 1.

First assume that the scalars $a_{j,j}, j = 0, ..., n-1$ are distinct, and let $a_{n,n}$ be yet another scalar not in this set. Similarly, assume that the scalars $a_{j,j+1}, j = 0, ..., n-1$ are distinct and $a_{-1,0}$ is not in this set. Then $h_m(a_{m-1,m}, -a_{m,m}) \neq 0$ while $h_s(a_{m-1,m}, -a_{m,m}) = 0$ for all $s \neq m$. In particular $h_m \notin \langle h_s, s \neq m\rangle$ and hence the basis is unshortenable. In the general case the same argument works, provided we replace the point evaluation by the evaluation of the appropriate derivative, i.e.,

$$D^{(\alpha(m),\beta(m))}h_m(a_{m-1,m}, -a_{m,m}) \neq 0 \text{ while } D^{(\alpha(m),\beta(m))}h_s(a_{m-1,m}, -a_{m,m}) = 0$$

for all $s \neq m$, where $\alpha(m)$ is the number of scalars $a_{j-1,j}$ such that $j > m$ and $a_{j-1,j} = a_{m-1,m}$, and $\beta(m)$ is the number of scalars $a_{j,j}$ such that $j < m$ and $a_{j,j} = a_{m,m}$. □

Corollary 3.9. *The basis $\{P'(x^jy^{n-j}), j = 0, ..., n\}$ is a minimal basis for the ideal* $\ker P$ *if and only if P is a Taylor projector at some point $(a, b) \in \mathbb{F}^2$, i.e., if and only if A is a bidiagonal matrix and the coefficients in (3.2): $a_{k,k} = b$ and $a_{k,k+1} = a$ for all $k = 0, ..., n-1$.*

Proof. If $\{P'(x^jy^{n-j}), j = 0, ..., n\}$ is a minimal basis then it is unshortenable and hence $P = P_A$ is generated by a bidiagonal matrix A of the form (3.2). Assume, by way of contradiction, that $a_{j,j+1} \neq a_{k,k+1}$ for some $j \neq k$, $j, k = 0, ..., n-1$. Then it follows from (3.3) that

$$(-a_{j,j}, a_{j,j+1}), (-a_{j,j}, a_{k,k+1}) \in Z(\ker P).$$

Similarly, assuming that $a_{j,j} \neq a_{k,k}$ we conclude that the variety $Z(\ker P)$ contains two points with either the same x-coordinates or with the same y-coordinates. Now, we can always make the change of variables

$$X = c_1x + c_2y, Y = c_3x + c_4y$$

so that, in the new variables X and Y, the variety $Z(\ker P)$ does not contain points with identical coordinates. Repeating the argument we arrive at a contradiction. □

The next corollary is just a reformulation of the Corollary 3.4:

Corollary 3.10. *The set* $\mathfrak{J}_{\mathbb{F}_{<n}[x,y]}$ *is parameterized by an irreducible affine algebraic variety in* $\mathbb{F}^{\frac{n(n+1)^2}{2}}$ *of dimension* $2 \times \dim \mathbb{F}_{<n}[x,y]$.

4. Perturbations of Ideal Projectors

Given an ideal projector P, we want to know whether there exists a small perturbation of this projector with some additional properties. Specifically, we are interested in two types of properties:

First, given an ideal projector P onto an N-dimensional subspace $E \in \mathfrak{G}^N$ and given another subspace $G \in \mathfrak{G}^N$, we investigate the existence of ideal projectors P_m onto E such that

$$P_m f \to Pf \text{ for all } f \in \mathbb{F}[\mathbf{x}] \tag{4.1}$$

and $\ker P_m \in \mathfrak{J}_G$. Equivalently we want to know if an ideal $J \in \mathfrak{J}_G$ can be approximated by ideals $J_m \in \mathfrak{J}_G \cap \mathfrak{J}_E$.

Second, given an ideal projector P onto an N-dimensional subspace $E \subset \mathbb{F}[\mathbf{x}]$, we investigate the existence of Lagrange projectors P_m onto E satisfying (4.1). Equivalently, we want to know if an ideal $J \in \mathfrak{J}_E$ can be approximated by Lagrange ideals $J_m \in \mathfrak{J}_G$.

The main result of this section (Theorem 4.8) shows that in the complex case, the two properties are equivalent.

We start with a discussion of the notion of convergence in $\mathfrak{J}_N$ and its relationship to convergence of ideal projectors. For the sake of specificity, for every $f = \sum_{\mathbf{k}} \hat{f}(\mathbf{k})\mathbf{x}^{\mathbf{k}} \in \mathbb{F}[\mathbf{x}]$ we define the norm

$$\|f\| := \sum_{\mathbf{k}} \left|\hat{f}(\mathbf{k})\right|,$$

which turns $\mathbb{F}[\mathbf{x}]$ into a normed linear space with continuous multiplication. Let $(\mathbb{F}[\mathbf{x}])'$ be the algebraic dual of $\mathbb{F}[\mathbf{x}]$. Every $J \in \mathfrak{J}_N$ induces an N-dimensional subspace $J^{\perp} \subset (\mathbb{F}[\mathbf{x}])'$defined as

$$J^{\perp} := \{\lambda \in (\mathbb{F}[\mathbf{x}])' : \lambda(f) = 0, \forall f \in J\}$$

that uniquely identifies the ideal J via $(J^{\perp})^{\top} = J$. We will adopt the following definition of convergence:

Definition 4.1. *Let* $(J_m,\, m \in \mathbb{N})$ *be a sequence of ideals in* $\mathfrak{J}_N$ *and let* $J \in \mathfrak{J}_N$. *We say that* $J_m \to J$ *if for every* $\lambda \in J^{\perp}$ *there exists* $\lambda_m \in J_m^{\perp}$ *such that*

$$\lambda_m(f) \longrightarrow \lambda(f) \tag{4.2}$$

for every $f \in \mathbb{F}[\mathbf{x}]$.

Next, we wish to relate convergence of ideals to the convergence of ideal projectors. A linear projector is entirely determined by the interplay between its range and the range of its dual. One would similarly expect the distance between two linear projectors with the same range to be entirely determined by the distance between the ranges of their duals. This is indeed the case for our particular definition of the "distance".

Theorem 4.2. *Let* $(J_m,\, m \in \mathbb{N})$ *be a sequence of ideals in* $\mathfrak{J}_N$ *and let* $J \in \mathfrak{J}_N$ *be such that* $J_m \to J$. *If* $E \in \mathfrak{G}^N$ *complements* J *then the space* E *complements* J_m *for sufficiently large* m *and*

$$P_m f \to Pf \text{ for all } f \in \mathbb{F}[\mathbf{x}], \tag{4.3}$$

where $P_m, P \in \mathfrak{P}_E$ *are such that* $\ker P_m = J_m$, $\ker P = J$.

Conversely, let P_m *and* P *be ideal projectors onto a space* E *such that (4.3) holds. Then* $\ker P_m \to \ker P$.

Proof. Assume that $E \in \mathfrak{G}^N$ complements J. For a functional $\lambda \in (\mathbb{F}[\mathbf{x}])'$ we use $\tilde{\lambda}$ to denote restriction of λ onto E. Suppose that

$$e_m \in E \cap J_m \tag{4.4}$$

and $\|e_m\| = 1$. Since E is finite dimensional, hence, passing to a subsequence, we can assume that $e_m \to e \neq 0$, $e \in E$. Let $\lambda \in J^\perp$ and $\lambda_m \in J_m^\perp$ satisfy (4.2). Then $\lambda_m \to \lambda$ uniformly on the bounded subsets of E and in particular $\lambda_m(e_m) \to \lambda(e)$. From (4.4) it follows that $\lambda_m(e_m) = 0$ hence $\lambda(e) = 0$ for all $\lambda \in J^\perp$ and $0 \neq e \in E \cap J$ which contradicts the assumption that E complements J.

Next, define ideal projectors $P_m, P \in \mathfrak{P}_E$ with $\ker P_m = J_m$, $\ker P = J$. Let $\{e_1, ..., e_N\}$ be a fixed basis in E. Then P can be written in the form

$$P = \sum_{j=1}^{N} \lambda_j \otimes e_j$$

(or equivalently, $Pf = \sum_{j=1}^{N} \lambda_j(f) e_j$) for some basis $\{\lambda_1, ..., \lambda_N\}$ of $J^\perp$, dual to $\{e_1, ..., e_N\}$, i.e., satisfying $\lambda_j(e_k) = \delta_{j,k}$. Define a finite-dimensional subspace $F \subset \mathbb{F}[\mathbf{x}]$ that contains E by

$$F := \operatorname{span}\{1, e_k, x_j \cdot e_k, k = 1, ..., N; j = 1, ..., d\}.$$

Choose $\lambda_j^{(m)} \in J_m^\perp$ satisfying (4.2). Since the set $\{\lambda_1, ..., \lambda_N\}$ is linearly independent over F and $\lambda_j^{(m)} \to \lambda_j$ uniformly on the bounded subsets of F, hence, for sufficiently large m, the set $\{\lambda_j^{(m)}, j = 1, ..., N\}$ is linearly independent over F, thus over $\mathbb{F}[\mathbf{x}]$, and form a basis for $J_m^\perp$. Therefore

$$P_m = \sum_{j=1}^{N} \lambda_j^{(m)} \otimes e_j^{(m)}$$

for some basis $\{e_j^{(m)}, j = 1, ..., N\}$ for E. Consider the operators $T_m : F \longrightarrow F$ defined by

$$T_m f = f - \sum_{j=1}^{N} (\lambda_j - \lambda_j^{(m)})(f) e_j, \forall f \in F.$$

Clearly

$$\begin{aligned} T_m e_k^{(m)} &= e_k^{(m)} - \sum_{j=1}^{N} \lambda_j(e_k^{(m)}) e_j + \sum_{j=1}^{N} \lambda_j^{(m)}(e_k^{(m)}) e_j \\ &= e_k^{(m)} - P e_k^{(m)} + e_k = e_k \end{aligned}$$

and $\|I_F - T_m\| \to 0$. Hence, for sufficiently large m, the operators T_m are invertible and $T_m^{-1} \to I_F$. Thus $e_j^{(m)} = T_m^{-1} e_j \to e_j$ and together with $\lambda_j^{(m)} \to \lambda_j$ on F we have $\sum_{j=1}^N \lambda_j^{(m)} \otimes e_j^{(m)} \to \sum_{j=1}^N \lambda_j \otimes e_j$ as $m \to \infty$ or equivalently $P_m \to P$ uniformly on the bounded subsets of F.

To prove (4.3) we need to show that $P_m u \to Pu$ for any monomial u of degree k. Since $1 \in F$, $P_m 1 \longrightarrow P1$, and it is so for $k = 0$. Assuming that it is true for a given k, let v be a monomial of degree $k+1$. Then $v = x_j u$ for some monomial u of degree k and some $j = 1, ..., d$. We have $x_j P_m u \to x_j P u$ and since

$$x_j P_m u, x_j P u \in x_j E \subset F,$$

the uniform convergence of P_m on the unit ball of F implies

$$P_m(v) - P(v) = P_m(x_j P_m u) - P(x_j P u) \to 0.$$

To prove the converse, let $\lambda \in (\ker P)^\perp$ and P_m satisfy (4.3). Then $P^*\lambda = \lambda$ and, since the restriction of λ onto the finite-dimensional subspace E is a continuous functional, (4.3) implies $\lambda(P_m f) \to \lambda(Pf)$ and $(P_m^*\lambda)f \to (P^*\lambda)f = \lambda(f)$ for every $f \in \mathbb{F}[\mathbf{x}]$. Thus

$$\lambda_m := (P_m^*\lambda) \in (\ker P_m)^\perp$$

satisfies (4.2). □

Remark 4.3. *If $P_m f \to Pf$ for all $f \in \mathbb{F}[\mathbf{x}]$ then for $f \in \ker P$ we have*

$$f_m := f - P_m f \in \ker P_m$$

and $(f - P_m f) \to f$. In other words $J_m \to J$ implies that

(4.5) *for every $f \in J$ there exists $f_m \in J_m$ such that $f_m \to f$.*

Turning to investigation of the two properties mentioned in the introduction to this section, we start by showing that any two N-dimensional subspaces of $\mathbb{F}[\mathbf{x}]$ can be simultaneously complemented by an ideal $J \in \mathfrak{J}_N$. In fact we prove a bit more:

Proposition 4.4. *Given G,$E \in \mathfrak{G}^N$, there exists an open and dense set $\mathcal{Z} \subset (\mathbb{F}^d)^N$ such that for every $Z \in \mathcal{Z}$ the ideal $J(Z)$ is a Lagrange ideal in $\mathfrak{J}_N$ (Z consists of N distinct points) and $J(Z)$ complements G and E simultaneously.*

Proof. Let $\{g_1, ..., g_N\}$ be a basis for G. For an ordered set

$$Z = (\mathbf{z}_1, ..., \mathbf{z}_N) \in (\mathbb{F}^d)^N$$

consider the determinant function of Z:

$$g(Z) := \det(g_j(\mathbf{z}_k),\ j, k = 1, ..., N).$$

It is a standard result that a sequence of N scalar-valued function is linearly independent if and only if their restrictions to some set of N distinct points in their common domain is linearly independent. Hence $g(Z)$ is not identically zero and for some $Z \in (\mathbb{F}^d)^N$ the ideal $J(Z)$ complements G. Observe that $g \in \mathbb{F}[Z]$ is a polynomial in dN variables $Z = (\mathbf{z}_1, ..., \mathbf{z}_N)$ and thus is different from zero for "almost all" $Z \in (\mathbb{F}^d)^N$. That is the set

$$\mathcal{Z}_1 := \{Z \in (\mathbb{F}^d)^N : g(Z) \neq 0\} = \{Z \in (\mathbb{F}^d)^N : G \oplus J(Z) = \mathbb{F}[\mathbf{x}]\}$$

is an open and dense set in $(\mathbb{F}^d)^N$. Similarly the set

$$\mathcal{Z}_2 := \{Z \in (\mathbb{F}^d)^N : G \oplus J(Z) = \mathbb{F}[\mathbf{x}]\}$$

is an open and dense. It remains to set $\mathcal{Z} := \mathcal{Z}_1 \cap \mathcal{Z}_2$. □

Next we will describe ideal projectors onto one very specific space

$$E := \text{span}\{1, x_1, x_1^2, ..., x_1^{N-1}\}$$

and show that ideal projectors onto this space in d variables bear similarity to the ideal projectors onto $\mathbb{F}_{<n}[x, y]$ in the bivariate setting.

We view E as an N-dimensional subspace of $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, x_2, ..., x_d]$. From the de Boor's formula, every ideal projector P onto E is determined by d polynomials $p_1, ..., p_d$ in E:

$$\begin{aligned} P(x_1^N) = p_1 = \textstyle\sum_{j=0}^{N-1} b_{1,j} x_1^j, \\ P(x_2) = p_2 = \textstyle\sum_{j=0}^{N-1} b_{2,j} x_1^j, \\ \vdots \\ P(x_d) = p_d = \textstyle\sum_{j=0}^{N-1} b_{d,j} x_1^j. \end{aligned} \tag{4.6}$$

Equivalently, every ideal $J \in \mathfrak{J}_E$ is generated by polynomials of the form

$$\{x_1^N - p_1(x_1), x_2 - p_2(x_1), ..., x_d - p_d(x_1)\}. \tag{4.7}$$

and thus $J \in \mathfrak{J}_E$ is completely determined by a (generalized) sequence of $d \times N$ scalars

$$B = (b_{k,j}, k = 1, ..., d; j = 0, ..., N - 1). \tag{4.8}$$

It is a unique feature of the space E that the converse is also true.

Proposition 4.5. *Every sequence $(p_1, ..., p_d)$ of polynomials in E defines an ideal $J = \langle x_1^N - p_1, x_2 - p_2, ..., x_d - p_d \rangle$ that complements E.*

Proof. It is clear from the construction of E that $E \cap J = \{0\}$. Let $f \in \mathbb{F}[x_1]$ be a polynomial in x_1 only. Using the division algorithm, we have $f = q(x_1^N - p_1) + r$ with $\deg r < N$. Thus the ideal $\langle x_1^N - p_1 \rangle$ complements $\mathbb{F}_{<N}[x_1]$ in $\mathbb{F}[x_1]$ and $E + J \supset \mathbb{F}[x_1]$. Inductively, we assume that $E + J \supset \mathbb{F}[x_1, ..., x_k]$, $k < d$ and prove that $E + J \supset \mathbb{F}[x_1, ..., x_k, x_{k+1}]$, i.e., we need to show that $x_{k+1}^n \mathbb{F}[x_1, ..., x_k] \subset E + J$ for all n. For $f \in \mathbb{F}[x_1, ..., x_k]$ we have

$$x_{k+1} f = (x_{k+1} - p_{k+1}) f + p_{k+1} f \in E + J \tag{4.9}$$

since the first term is in the ideal J and the second belongs to $E + J$ by the inductive assumption. Using induction on n, assume that $f \in x_{k+1}^n \mathbb{F}[x_1, ..., x_k]$ and conclude that $x_{k+1} f \in x_{k+1}^{n+1} \mathbb{F}[x_1, ..., x_k]$ has a representation (4.9). □

In view of this proposition we can identify projectors onto the space E with $\mathbb{F}^{dN}$. With every sequence of parameters $B \in \mathbb{F}^{dN}$ we associate a unique ideal projector Q_B onto E by (4.6). In accordance with (2.1):

$$Q_B f = \sum_{j=0}^{N-1} r_{j,f}(B) x_1^j, r_{j,f} \in \mathbb{F}^{dN}[B], \tag{4.10}$$

i.e., $r_{j,f}$ are polynomials in the dN variables B.

Theorem 4.6. *Let E =span$\{1, x_1, x_1^2, ..., x_1^{N-1}\} \subset \mathbb{F}[\mathbf{x}]$ and let $J \in \mathfrak{J}_E$. For every $G \in \mathfrak{G}^N$ there exists a sequence (J_m) of ideals in $\mathfrak{J}_E \cap \mathfrak{J}_G$ such that $J_m \to J$.*

Proof. By Proposition 4.6, there exists an open and dense family of sets

$$Z = \{\mathbf{z}_1, ..., \mathbf{z}_N\} \subset \mathbb{F}^d$$

of points such that the ideal

$$J(Z) = \{f \in \mathbb{F}^d : f(\mathbf{z}_1) = f(\mathbf{z}_2) = ... = f(\mathbf{z}_N) = 0\} \tag{4.11}$$

complements G and E at the same time. Thus there exists a $B \in \mathbb{F}^{dN}$ such that the kernel of the ideal projection Q_B onto E complements G. Let $g_1, ..., g_N$ be a basis for G. The fact that $\ker Q_B$ complements G is equivalent to $\ker Q_B \cap G = \{0\}$, which means that the assumption:

$$Q_B(\sum_{j=1}^{N} \alpha_j g_j) = \sum_{j=1}^{N} \alpha_j Q_B g_j = 0$$

implies that $\alpha_j = 0$ for all $j = 1, ..., N$. That is, the functions $Q_B g_1, ..., Q_B g_N$ are linearly independent. But

$$Q_B g_j = \sum_{n=0}^{N-1} r_{n,g_j}(B) x_1^n$$

where $r_{n,g_j}(B)$ are polynomials in $d \times N$ variables B. The linear independence is equivalent to

$$\det(r_{n,g_j}(B), j = 1, ...N; n = 0, ..., N-1) \neq 0.$$

This determinant is itself a polynomial in B and since it is non-vanishing for one particular choice of B, the set of all B such that this determinant is non-zero is open and dense. Hence starting with an arbitrary ideal

$$J_B = \left\langle x_1^N - \sum_{j=0}^{N-1} b_{1,j} x_1^j, x_k - \sum_{j=0}^{N-1} b_{k,j} x_1^j, k = 2, ..., d \right\rangle$$

complemented by E we can perturb scalars in B so that the resulting ideal $J_{\tilde{B}}$ simultaneously complements E and G. □

In complex case $\mathbb{F} = \mathbb{C}$ we have

Theorem 4.7. *Every ideal $J \subset \mathbb{C}[\mathbf{x}]$, complemented by the space E described above, is a limit of radical ideals.*

Proof. Let $J \subset \mathbb{C}[\mathbf{x}]$ be given by (4.6). Then

$$x_1^N - p_1(x_1) = \prod_{j=0}^{N-1} (x_1 - z_{1,j})$$

and once again we perturb p_1 to obtain polynomials $p_1^{(m)} \in E$ such that

$$x_1^N - p_1^{(m)}(x_1) = \prod_{j=0}^{N-1} (x_1 - z_{1,j}^{(m)})$$

with distinct zeroes $\{z_{1,0}^{(m)}, ..., z_{1,N-1}^{(m)}\}$ and

$$z_{1,j}^{(m)} \to z_{1,j} \tag{4.12}$$

for every $j = 0, ..., n-1$.

Let $z_{k,j}^{(m)} = p_k(z_{1,j}^{(m)})$ for $k = 2, ..., d$. Then the ideal

$$J_m := \left\langle x_1^N - p_1^{(m)}(x_1), x_2 - p_2(x_1), ..., x_d - p_d(x_1) \right\rangle$$

has the associated variety

$$Z(J_m) = \{\mathbf{z}_0 := (z_{1,0}^{(m)}, z_{2,0}^{(m)}, ..., z_{d,0}^{(m)}), ..., \mathbf{z}_{N-1} := (z_{1,N-1}^{(m)}, ..., z_{d,N-1}^{(m)})\}$$

that consists of precisely N distinct points. Thus J_m is a radical ideal and by (4.12) and Definition 4.1 we conclude that $J_m \to J$. □

Theorem 4.8. *Let $J \in \mathfrak{J}_N$ be an ideal in $\mathbb{C}[\mathbf{x}]$ and E be as above. The following are equivalent:*

(1) J is a limit of radical ideals in $\mathfrak{J}_N$.
(2) For any $G \in \mathfrak{G}^N$ there exists a sequence $(J_m \in \mathfrak{J}_G)$ such that $J_m \to J$.
(3) There exists a sequence $(J_m \in \mathfrak{J}_E)$ such that $J_m \to J$.

Proof. (1)⇒(2) follows from the Proposition 4.4. The implication (2)⇒(3) is obvious. To prove that (3)⇒(1) we only need to prove that J is a limit of radical ideals which is nothing but the Theorem 4.7. □

5. Perturbations of Bivariate Ideal Projectors

In one variable, every ideal $J \in \mathfrak{J}_N$ complements $\mathbb{F}_{<N}[x]$, i.e., $\mathfrak{J}_N = \mathfrak{J}_{\mathbb{F}_{<n}[x]}$. In two or more variables this is not the case. In fact (cf. [16]), no N-dimensional subspace $G \subset \mathbb{F}[\mathbf{x}]$ has the property that $\mathfrak{J}_N = \mathfrak{J}_G$. In this section we will use Theorem 2.4 to show that $\mathfrak{J}_{\mathbb{F}_{<n}[x,y]}$ is "dense" in $\mathfrak{J}_{N(n)}$. This property fails in three or more variables (cf. [19]). We will show that given any $N(n)$-dimensional space $G \subset \mathbb{F}[\mathbf{x}]$ and any ideal projector P onto $\mathbb{F}_{<n}[x,y]$ there exists a sequence of ideal projectors P_m onto $\mathbb{F}_{<n}[x,y]$ such that $P_m f \to Pf$ for all $f \in \mathbb{F}[x,y]$ and $\ker P_m \in \mathfrak{J}_G$.

Proposition 5.1. *Let P_A and P_{A_m} be projectors onto $\mathbb{F}_{<n}[x,y]$ as defined in Corollary 2.3. Then $A_m \to A$ if and only if $P_{A_m}(f) \to P_A(f)$ for all $f \in \mathbb{F}[x,y]$.*

Proof. By Corollary 3.4

$$P_A f = \sum_{j+k<n}^{n-1} q_{j,k,f}(A) x^j y^k$$

where $q_{j,k,f} \in \mathbb{F}[A]$ are polynomials (therefore continuous functions) in $n(n+1)$ variables (entries of the matrix A). This implies the sufficiency. Conversely, $P_{A_m}(f) \to P_A(f)$ implies $P_{A_m}(x^j y^{n-j}) \to P_A(x^j y^{n-j})$ for every $j = 0, ..., n$. Thus the coefficients of the leading form of $P_{A_m}(x^j y^{n-j})$ converge to the coefficients of the leading form of $P_A(x^j y^{n-j})$. By Corollary 3.2, this implies that $A_m \to A$. □

We are now ready for the main theorem of this section:

Theorem 5.2. *Let G be a subspace of $\mathbb{F}[x,y]$ with $\dim G = N(n) = \dim \mathbb{F}_{<n}[x,y]$ and let P be an ideal projector onto $\mathbb{F}_{<n}[x,y]$. Then there exists a sequence of ideal projectors (P_m) onto $\mathbb{F}_{<n}[x,y]$ such that $P_m(f) \to P(f)$ for all $f \in \mathbb{F}[x,y]$ and $\ker P_m$ complements G for all m.*

Proof. Let A be an $n \times (n+1)$ matrix and P_A be the corresponding projector onto $\mathbb{F}_{<n}[x,y]$. Let $\{g_1, ..., g_{N(n)}\}$ be a linear basis for G. To prove that $\ker P_A$ is complements G we only need to prove that $\ker P_A \cap G = \{0\}$. In other words we need to prove that for every $g \in G$, $P_A(g) = 0$ implies $g = 0$. Equivalently, we need to show that the polynomials

$$P_A g_j = \sum_{m+k<n} \widehat{P_A g_j}(m,k) x^m y^k$$

are linearly independent. This is equivalent to proving that the $N(n) \times N(n)$ matrix C_A whose j-th row consists of ordered coefficients $\widehat{P_A g_j}(m,k)$ has a non-zero determinant. Since, by Corollary 3.4, $\widehat{P_A g_j}(m,k)$ are fixed polynomial in the entries of the matrix A, hence $\det C_A$ is a polynomial in the entries of the matrix A. By Proposition 4.4, this polynomial is not identically zero, and thus $\det C_A \neq 0$ for all A in some open and dense set. In particular, it means that for a fixed matrix A and a fixed $m > 0$, there exists a matrix A_m such that $\|A - A_m\| < \frac{1}{m}$ and $\ker P_{A_m}$ is complemented by G. Thus, by proposition 5.1, we conclude that $P_{A_m}(f) \to P_A(f)$ for all $f \in \mathbb{F}[x,y]$. □

In one dimension every ideal projector P from $\mathbb{C}[x]$ onto $\mathbb{C}_{<N}[x]$ is a limit of Lagrange projectors (Theorem 4.7). Combination of Theorem 4.8 and Theorem 5.2 yield the same conclusion in two variables:

Corollary 5.3. *Every ideal projector P from $\mathbb{C}[x,y]$ onto $\mathbb{C}_{<n}[x,y]$ is a limit of Lagrange projectors.*

This provides a partial answer to a question, posed by de Boor [3]. Actually, it is known (cf. [15]) that the result is true for a general finite-dimensional ideal projectors in $\mathbb{C}[x,y]$ and is false for projectors in $\mathbb{C}[\mathbf{x}]$ for $d > 2$. Unlike the general case, the proof presented here does not rely on the notion of Hilbert schemes and algebraic geometry. For this reason, I feel it to be worthwhile.

We finish this short section with a few remarks and open problems.

1) The combination of the results of [15] and Theorem 4.8 provides an extension of Theorem 5.2 in the complex case:

Corollary 5.4. *Let $J \in \mathfrak{J}_N$ be an ideal in $\mathbb{C}[x,y]$ and let $G \in \mathfrak{G}^N$. Then there exists a sequence of (radical) ideals $J_m \in \mathfrak{J}_G$ such that $J_m \to J$.*

The combination of the same two theorems implies

Corollary 5.5. *For $d > 2$, there exists an ideal $J \in \mathfrak{J}_N$ such that J can not be approximated by ideals in $\mathfrak{J}_E$.*

The results of [19] allow to choose the ideal J in the previous corollary in $\mathfrak{J}_{\mathbb{F}_{<n}[\mathbf{x}]}$ for real and complex field $\mathbb{F}$. Related, is the question, posed by de Boor (cf. [2]): What properties of $(\ker P)^\perp$ imply that P is a limit of Lagrange projectors?

2) The Theorem 2.4 provides a parametrization of $\mathbb{F}_{<n}[x,y]$ by an irreducible affine variety. It is shown in [7] that, in the complex case, the family $\mathfrak{J}_N$ of ideals in $\mathbb{C}[x,y]$ of codimension N can be parameterized by an irreducible *algebraic* (as oppose to affine) variety (cf. [10], Definition 2, p.52) of dimension $2N$. As a consequence every finite-dimensional ideal projector in $\mathbb{C}[x,y]$ is a limit of Lagrange projectors (cf [15]). Equivalently, for every finite-dimensional space $G \subset \mathbb{C}[x,y]$ and every ideal J with $\operatorname{codim} J = \dim G$, there exists a small perturbation $\tilde{J}$ of J with

$\tilde{J} \in \mathfrak{P}_G$. It would be nice to find a direct proof of this fact without the use of [7]. In particular, it would be nice to know the spaces $G \subset \mathbb{F}[x, y]$ such that $\mathfrak{J}_G$ can be parameterized by an irreducible affine variety of dimension $2N$. As of now, I do not have a proof of the analogue of Corollary 5.5 (as opposed to Theorem 5.2) in the real case.

3) It would be interesting to determine the relationship between the properties of the matrix A and the properties of the ideal projector P_A. For instance: What properties of A imply that P_A is a Lagrange projector and how to determine the interpolation sites of P_A from the entrees of A? More generally, what is the relationship between the entries of A and the space $(\ker P_A)^{\perp}$? Corollary 3.8 is a small step in that direction.

4) Corollary 3.9 has an interesting application in the study of the "error formulas" for ideal projectors (cf. [18]). I don't know if the analogous result is true in an arbitrary number of variables.

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