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Journal of Approximation Theory 139 (2006) 2–7
Journal of Approximation Theory
www.elsevier.com/locate/jat

On discrete norms of polynomials

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Received 26 September 2003; received in revised form 22 March 2004; accepted 14 June 2004

Communicated by Doron S Lubinsky
Available online 29 April 2005

Abstract

For a polynomial p of degree $n < N$ we compare two norms:

$$\|p\| := \sup\{|p(z)| : z \in C; |z| = 1\}$$

and

$$\|p\|_N := \sup\left\{\left|p\left(z_j\right)\right| : j = 0, \ldots, N-1\right\};$$

$z_j = e^{2\pi i \frac{j}{N}}$. We show that there exist universal constants C_1 and C_2 such that

$$1 + C_1 \log\left(\frac{N}{N-n}\right) \leqslant \sup\left\{\frac{\|p\|}{\|p\|_N} : p \in \mathbb{P}_n\right\} \leqslant C_2 \log\left(\frac{N}{N-n}\right) + 1.$$

MSC: 41A20; 42A05

Keywords: Discrete norm; Uniform norm; Roots of unity; Polynomials

1. Introduction

How well can one estimate the uniform norm of a polynomial by its values at a large number of points? In this article we answer the question in the case when the points are

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doi:10.1016/j.jat.2004.06.006

uniformly distributed on the unit circle. In other words, we compare the uniform norm of polynomials on the unit circle to its discrete analogue: maximum on the Nth roots of unity for N larger than the degree of polynomials. More precisely let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and $\mathbb{T}_N := \{z_j\}_{j=0,\dots,N-1} \subset \mathbb{T}$ where $z_j = e^{2\pi i \frac{j}{N}}$.

Let $\mathbb{P}_n$ be the set of polynomials of degree $n-1$, i.e.,

$$\mathbb{P}_n = \left\{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1};\ a_0, \dots, a_{n-1} \in \mathbb{C}\right\}.$$

For $p \in \mathbb{P}_n$ we define $\|p\| := \sup\{|p(z)| : z \in \mathbb{T}\}$ and $\|p\|_N := \sup\{|p(z)| : z \in \mathbb{T}_N\}$. Further define the quantity

$$K(N,n) := \sup\left\{\frac{\|p\|}{\|p\|_N} : p \in \mathbb{P}_n\right\}.$$

Clearly if $N \leqslant n$ then $K(N,n) = \infty$. For $N = n+1$, a well known theorem of Marcinkiewicz (cf. [3]) asserts

$$C_1 \log n \leqslant K(n+1,n) \leqslant C_2 \log n.$$

The purpose of this paper is to extend the last result for arbitrary $N > n$. Namely we prove the following

Theorem 1. *There exist positive constants C_1 and C_2 such that*

$$1 + C_1 \log\left(\frac{N}{N-n}\right) \leqslant K(N,n) \leqslant C_2 \log\left(\frac{N}{N-n}\right) + 1,$$

for all $N > n$.

Theorem 1 solves a special case of a conjecture of Erdős [2], mentioned in Section 4.

Before turning to the proof of this theorem we introduce a couple of integers that depend on N and n:

$$q = q(N,n) := N - n$$

and

$$m = m(N,n) := \left\lfloor \frac{N}{N-n} \right\rfloor = \left\lfloor \frac{N}{q} \right\rfloor.$$

We will prove the upper and lower bounds separately in the next two sections. The last section of the paper contains various conjectures related to the above theorem.

2. Upper bound

Let $p \in \mathbb{P}_n$ be a fixed polynomial such that

$$\left|p\left(z_j\right)\right| \leqslant 1 \text{ for } j = 0, \dots, N-1. \tag{1}$$

We want to show the existence of a constant C_2 such that $|p(z)| \leqslant 1 + C_2 \log\left(\frac{N}{N-n}\right)$ for all $z \in \mathbb{T}$. We fix a point $t \in \mathbb{T} \setminus \mathbb{T}_N$ and introduce polynomials

$$\begin{aligned} &T \in \mathbb{P} : T(z) := z^N - 1 \text{ and} \\ &Q = Q_t \in \mathbb{P}_{N-n} \text{ defined by } Q(z) = Q_t(z) := \frac{z^q - t^q}{z - t}. \end{aligned} \tag{2}$$

Then the polynomial $p(z)Q(z) \in \mathbb{P}_N$ and we can consider a rational function $R(z) = R_t(z)$ defined as

$$\begin{aligned} R(z) = R_t(z) &:= \frac{p(z)Q(z)}{T(z)} \\ &= \sum_{j=0}^{N-1} \frac{c_j}{z - z_j} \text{ where } c_j = \operatorname{res}\left(R, z_j\right) = \frac{p\left(z_j\right) Q\left(z_j\right)}{T'\left(z_j\right)}. \end{aligned} \tag{3}$$

From (1) and the obvious fact that $\left|T'\left(z_j\right)\right| = N$ we have $\left|c_j\right| \leqslant \frac{Q\left(z_j\right)}{N}$ and hence

$$|p(t)| = \left|\frac{R_t(t)T(t)}{Q_t(t)}\right| \leqslant \left|\frac{T(t)}{Q_t(t)}\right| \frac{1}{N} \sum_{j=0}^{N-1} \frac{\left|Q_t\left(z_j\right)\right|}{\left|t - z_j\right|}. \tag{4}$$

Since $|Q_t(t)| = q$ and from (2) we have

$$|p(t)| \leqslant \frac{1}{qN} \sum_{j=0}^{N-1} \left|\frac{t^N - 1}{t - z_j}\right| \left|\frac{t^q - z_j^q}{t - z_j}\right|. \tag{5}$$

Observe that $\left|\frac{t^N - 1}{t - z_j}\right| \leqslant \frac{2}{\left|t - z_j\right|}$ and $\left|\frac{t^N - 1}{t - z_j}\right| = \left|\frac{t^N - z_j^N}{t - z_j}\right| \leqslant N$.
Similarly $\left|\frac{t^q - z_j^q}{t - z_j}\right| \leqslant \min\left\{q, \frac{2}{\left|t - z_j\right|}\right\}$. Hence

$$\begin{aligned} |p(t)| &\leqslant \frac{1}{qN} \sum_{j=0}^{N-1} \min\left\{N, \frac{2}{\left|t - z_j\right|}\right\} \min\left\{q, \frac{2}{\left|t - z_j\right|}\right\} \\ &= \sum_{j=0}^{N-1} \min\left\{1, \frac{2}{N\left|t - z_j\right|}\right\} \min\left\{1, \frac{2}{q\left|t - z_j\right|}\right\}. \end{aligned} \tag{6}$$

Since the points z_j are uniformly distributed on the unit circle, there is no loss of generality in assuming that $t = e^{2\pi i\theta}$ lies between $z_0 = 1$ and z_{N-1}, i.e., $0 > \theta > -\frac{2\pi}{N}$. Therefore for $j = 0$, $\min\left\{1, \frac{2}{N\left|t - z_j\right|}\right\} \min\left\{1, \frac{2}{q\left|t - z_j\right|}\right\} \leqslant 1$ and for $j = 1, \ldots, \left\lceil\frac{N}{2}\right\rceil$;

$\left|t - z_j\right| \geqslant \left|1 - z_j\right| \geqslant \frac{2}{\pi}\frac{2\pi j}{N} = \frac{4j}{N}$. In conjunction with (6) we conclude that

$$\sum_{j=0}^{\lceil \frac{N}{2} \rceil} \min\left\{1, \frac{2}{N\left|t - z_j\right|}\right\} \min\left\{1, \frac{2}{q\left|t - z_j\right|}\right\}$$

$$\leqslant 1 + 8\sum_{j=1}^{\lceil \frac{N}{2} \rceil}\left(\frac{1}{j}\right)\min\left\{1, \frac{N}{qj}\right\}. \tag{7}$$

Once again by symmetry we have the same estimate for the $\sum_{\lceil \frac{N}{2} \rceil}^{N-1}$. A combination of (6) and (7) gives

$$|p(t)| \leqslant 2C\sum_{j=1}^{\lceil \frac{N}{2} \rceil}\left(\frac{1}{j}\right)\min\left\{1, \frac{N}{qj}\right\}. \tag{8}$$

For $j \leqslant m = \left\lfloor \frac{N}{q} \right\rfloor$ we have $\left(\frac{1}{j}\right)\min\left\{1, \frac{N}{qj}\right\} \leqslant \frac{1}{j}$. For $j > m$, we use $\left(\frac{1}{j}\right)\min\left\{1, \frac{N}{qj}\right\} \leqslant \frac{m}{j^2}$. From obvious inequality $\sum_{j=m+1}^{\infty}\frac{1}{j^2} \leqslant \frac{1}{m}$ and from (8) we conclude

$$|p(t)| \leqslant 2C\left(\sum_{j=1}^{m}\frac{1}{j} + m\sum_{j=m+1}^{\lceil \frac{N}{2} \rceil}\frac{1}{j^2}\right) \leqslant 2C\log m + \frac{m}{m+1} \leqslant C_2\log m + 1. \tag{9}$$

3. Lower bound

In this section, we exhibit a polynomial $p(z) \in \mathbb{P}_n$ such that

$$\left|p\left(z_j\right)\right| \leqslant 1 \text{ and } p(t) \geqslant C_1\log m \text{ for } t = e^{\frac{\pi i}{N}}. \tag{10}$$

To this end we will start with the polynomial

$$P(z) := \frac{z^N - 1}{N}\sum_{j=0}^{m}\frac{1}{z - z_j} = \frac{1}{N}\sum_{j=0}^{m}\frac{z^N - z_j^N}{z - z_j} \in \mathbb{P}_N. \tag{11}$$

It is easy to see that

$$P\left(z_j\right) = 0 \text{ if } j > m \text{ and } \left|P\left(z_j\right)\right| = \frac{1}{N}\left|\sum_{k=0}^{N-1} z^k z_j^{N-k-1}\right| \leqslant 1 \text{ for } j \leqslant m. \tag{12}$$

Furthermore, since $t = e^{\frac{\pi i}{N}}$, an easy computation shows that $\frac{1}{t - z_j} = -2ie^{\pi i \frac{2j+1}{2N}} \sin\left(\frac{2j-1}{2N}\right)$ and from (11)

$$\|P\| \geqslant |P(t)| \geqslant \frac{\left|t^N - 1\right|}{N} \left|-2ie^{\pi i \frac{2j+1}{2N}}\right| \left(\sum_{j=1}^{m} \frac{1}{\sin\left(\frac{\pi(2j-1)}{2N}\right)}\right)$$

$$\geqslant \frac{4}{N} \sum_{j=1}^{m} \frac{1}{\sin\left(\frac{\pi(2j-1)}{2N}\right)} \geqslant \frac{4}{\pi N} \sum_{j=1}^{m} \left(\frac{2N}{2j-1}\right) \geqslant C \log m. \tag{13}$$

Thus the polynomial $P(z)$ satisfies all the desired properties in (10) except that it is the polynomial of degree $N-1$ and not $n-1$, as promised.

So let $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$ and $r(z) = a_n z^n + a_{n+1} z^{n+1} + \cdots + a_{N-1} z^{N-1}$ be such that $P(z) = p(z) + r(z)$.

From (11) it is easy to see that

$$|a_k| = \frac{1}{N} \left| \sum_{j=0}^{m} z_j^{N-k-1} \right| \leqslant \frac{m}{N}. \tag{14}$$

Hence $|r(z)| \leqslant (N-n)\frac{m}{N} = \frac{qm}{N} \leqslant 2$. Therefore

$$\begin{aligned} & |p(z)| = |P(z) - r(z)| \geqslant C \log m - 2 \\ \text{and} \quad & \left|p\left(z_j\right)\right| = \left|P\left(z_j\right) - r\left(z_j\right)\right| \leqslant 1 + 1 = 2. \end{aligned} \tag{15}$$

4. Conjectures

We wish to conclude this note with various conjectures related to the estimates presented earlier. We introduce some additional notations: For $\mathbf{F} \subset \mathbb{T}$ define

$$\|P\|_{\mathbf{F}} := \sup\{|P(z)| : z \in \mathbf{F}\} \text{ and } K(n, \mathbf{F}) := \sup\left\{\frac{\|P\|}{\|P\|_{\mathbf{F}}} : P \in \mathbb{P}_n\right\}.$$

Conjecture 1 (*Erdős [2]*). *Let $N = \#F > n$ then $K(n, F) \geqslant C \log\left(\frac{N}{N-n}\right)$.*

This conjecture would follow from the following intuitively obvious

Conjecture 2. *Let $N = \#F > n$ then $K(n, F) \geqslant K(n, T_n)$, i.e., among all N-point sets, the roots of unity are optimal.*

The results of Section 3 can be easily extended to the following

Proposition 1. *Let* $0 \leqslant k_1, k_2, \ldots, k_n \leqslant N-1$ *be arbitrary* n *integers. Let* $X_n :=$ span $\left\{z^{k_s} : s = 1, \ldots, n\right\} \subset P_N$. *Then there exist a polynomial* $p \in X_n$ *such that* $\left|p\left(z_j\right)\right| \leqslant 1$ *and* $\|p\| \geqslant C \log\left(\frac{N}{N-n}\right)$.

Conjecture 3. *The above proposition remains valid if we replace the subspace* X_n *by an arbitrary* n*-dimensional subspace of* P_N.

Remark. After this paper was accepted for publication, the referee pointed out that similar results for trigonometric polynomials were obtained by Bernstein under the additional assumption that $\frac{N}{N-n}$ is an integer (cf. [1]).

Acknowledgements

We would like to take this opportunity to thank all the participants of the Tampa Analysis Seminar for their interest and participation in this research. We also thank Peter Vertesi for pointing out reference [2] and the referees for correcting errors in the original manuscript.

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