Ideal Projections onto Planes

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Abstract. In this paper we classify all ideal projections from \( \mathbb{C}[x, y] \) onto the linear span of \( 1, x, y \). In particular we show that every such projection is a limit of interpolating projections. That verifies one particular case of a conjecture of C. deBoor.

§1. Introduction

This work was motivated by a question asked by Carl deBoor (see Conjecture below) at this conference. Thus we study finite-dimensional projections whose kernels are polynomial ideals of finite codimension. The general theory of polynomial ideals is well investigated in Algebraic Geometry (cf.[5]). The relationship between polynomial ideals and approximation theory had been emphasized, among others, in papers[2],[3] and [4]. In this note, we specify not only the codimension of an ideal, but also its complement i.e. the range of a projection. That restricts the number of ideals in question, and in some exceptional cases allows for a complete classification of all such ideals. In this paper we give a complete description of ideals that are complemented to the 3-dimensional space of polynomials of degree one.

Let \( \mathbb{C}[x, y] \) stand for the ring of polynomials of two variables with complex coefficients and let \( M[x, y] \subset \mathbb{C}[x, y] \) be the set of monomials. The elements of \( \mathbb{C}[x, y] \) will be written as a finite sum \( \sum a_{j,k}x^jy^k \) or \( \sum a_Ju^J \) with an understanding that the vector \( u = (x, y) \in \mathbb{C}_n \) and the multi-index \( J = (j, k) \in \mathbb{Z}_+^2 \), where \( \mathbb{Z}_+ \) stands for non-negative integers. Similarly \( M[x, y] = \{ x^jy^k = u^J : J = (j, k) \in \mathbb{Z}_+^2 \} \)

The symbol \( \mathbb{C}_n[x, y] \) denotes the polynomials of degree at most \( n \). Hence \( f \in \mathbb{C}_n[x, y] \) implies \( f = \sum_{j+k \leq n} a_{j,k}x^jy^k = \sum_{|J| \leq n} a_Ju^J \). Finally we let \( H_n[x, y] \) stand for homogeneous polynomials of degree \( n \) and
$M_n[x,y] \subset H_n[x,y]$ is the set of all monomials of degree $n$. Therefore $f \in H_n[x,y]$ if and only if $f = \sum_{j+k=n} a_{j,k} x^j y^k = \sum_{|J|=n} a_J u^J$ and $f \in M_n[x,y]$ if and only if $f = x^j y^{n-j} = u^J$ with $|J| = n$.

**Definition 1.** Let $P$ be a projection from $\mathbb{C}[x,y]$ onto a finite-dimensional subspace.

1) $P$ is called an ideal projection if

\[ P(f \cdot g) = P(f) \cdot P(g) \quad \forall f, g \in \mathbb{C}[x,y] \quad (1.1) \]

2) $P$ is called an interpolating projection if there exists a set $\Delta \subset \mathbb{C}$ such that

\[ Pf = 0 \iff f(u) = 0 \text{ for all } u \in \Delta. \]

In this case we denote the projection by $P_\Delta$.

3) $P$ is called a "lim-interpolating projection" (or LIP) if there exists a sequence $P_k$ of interpolating projections such that $P_k f \to Pf$ for every $f \in \mathbb{C}[x,y]$.

Clearly if $P$ is an interpolating projection then $P$ is an ideal projection. Hence every lim-interpolating projection is ideal. It is well-known and easy to see that, for polynomials of one complex variable, the converse is also true.

**Theorem 1.** Every ideal projection on $\mathbb{C}[x]$ is a Hermite interpolating projection, and therefore LIP.

**Proof.** Let $P$ be an ideal projection on $\mathbb{C}[x]$. Then $\ker P$ is an ideal in $\mathbb{C}[x]$, which is a principal ideal domain. Hence $\ker P$ is generated by one polynomial $f(x) = \prod_{j=0}^{n(j)} (x - x_j)^{n(j)}$. Hence $P$ interpolates the values at the points $x_j$ together with the values of the derivatives at these points up to the order $n(j)$. 

At the conference to which these proceedings are dedicated, Carl de-Boor proposed the following:

**Conjecture 1.** ([1]) Every ideal projection is LIP.

In this paper we will give a complete description of ideal projections $P : \mathbb{C}[x,y] \to \mathbb{C}_1[x,y]$ and use it to verify the above-mentioned conjecture for this special case as well as for all three dimensional projections in $\mathbb{C}[x,y]$.
§2. Ideal Projections.

The following proposition describes some obvious properties of the ideal projections:

**Proposition 2.** Let $P$ be a projection on $\mathbb{C}[x,y]$. Then

1. $P$ is ideal iff $\ker P$ is an ideal in $\mathbb{C}[x,y]$.
2. $P$ is ideal projection iff

$$ f_1, g_1, f_2, g_2 \in \mathbb{C}[x,y], \quad f_1 g_1 = f_2 g_2 \implies f_1 P g_1 - f_2 P g_2 \in \ker P. \quad (2.1) $$

3. If $P$ is an ideal projection and $f_1 P g_1 - f_2 P g_2 \in \ker P$ then $f_1 g_1 - f_2 g_2 \in \ker P$.
4. $P$ is ideal projection iff

$$ f_1, g_1, f_2, g_2 \in M[x,y], \quad f_1 g_1 = f_2 g_2 \implies f_1 P g_1 - f_2 P g_2 \in \ker P. \quad (2.2) $$

**Proof.** The statements 1), 2), 3) are trivial. To verify 4) assume $f = \sum_j a_j u^j$ and $g = \sum_k b_k u^K$. Then

$$ P(f g) = P(\sum_j \sum_k a_j b_k u^j u^K) = \sum_j \sum_k a_j b_k P(u^j u^K) $$

$$ = \sum_j \sum_k a_j b_k P(u^j) P(u^K) = P(\sum_j \sum_k a_j b_k (u^j P u^K)) = P((\sum_j a_j u^j) \cdot P(\sum_k b_k u^K)) = P(f P g). $$

Motivated by (2.2) we will localize the definition of an ideal projection.

**Definition 2.** Let $P$ be a projection on $\mathbb{C}_m[x,y]$. We say that $P$ is $m$-ideal if

$$ f_1, g_1, f_2, g_2 \in \mathbb{C}_m[x,y], \quad f_1 g_1 = f_2 g_2 \implies f_1 P g_1 - f_2 P g_2 \in \ker P. \quad (2.3) $$

Let $P$ be an $(m+k)$-ideal projection on $\mathbb{C}_{m+k}[x,y]$ with $\text{Im} P \subset \mathbb{C}_m[x,y]$. Then its restriction to $\mathbb{C}_m[x,y]$ is $m$-ideal. The converse is not necessarily true. It is possible that an $m$-ideal projection on $\mathbb{C}_m[x,y]$ does not have an $(m+k)$-ideal extension to $\mathbb{C}_{m+k}[x,y]$ for some $k$. However if such an extension exists, then the ideal property clearly guaranties its uniqueness.

The next Theorem shows that if a projection described above has an $(m+2)$-ideal extension, then it has an $(m+k)$-ideal extension for all $k$ and hence has an ideal extension to $\mathbb{C}[x,y]$ with the same range.

**Theorem 3.** Let $P$ be an $(m+2)$-ideal projection with $\text{Im} P \subset \mathbb{C}_m[x,y]$. Then $P$ has an ideal extension to $\mathbb{C}_n[x,y]$ for all $n \geq m+2$.

**Proof.** The proof proceeds by induction. Let $P$ be an ideal projection from $\mathbb{C}_n[x,y]$ onto $\text{Im} P \subset \mathbb{C}_m[x,y]$ and $n \geq m+2$. Define

$$ Q(x^j y^k) = \begin{cases} 
   P(x^j y^k) & \text{if } j + k \leq n \\
   P(y \cdot P(x^j y^{k-1})) & \text{if } j + k = n + 1; \ k \geq 1 \\
   P(x \cdot P(x^n)) & \text{if } j = n + 1, k = 0 \end{cases} \quad (2.4) $$
We need to show that for all \( f_1, g_1, f_2, g_2 \in M_{n+1}[x, y], \)
\[ f_1 g_1 = f_2 g_2 \implies f_1 Q g_1 - f_2 Q g_2 \in \ker Q. \] (2.5)

First assume that \( g_1 \) and \( g_2 \) \( / \in \text{Im} P \) and consider two cases:

Case 1. Suppose that \( f_1 = q h_1 \) and \( f_2 = q h_2 \) with \( \deg q \geq 1 \). Then
\[ Q(f_1 Q g_1 - f_2 Q g_2) = Q(q(h_1 P g_1 - h_2 P g_2)) \]
Since \( \deg(q(h_1 P g_1 - h_2 P g_2)) \leq m, h_1 g_1 = h_2 g_2 \in C_{m+1}[x, y] \)
using the inductive assumption twice we have
\[ Q(f_1 P g_1 - f_2 P g_2) = P(q(h_1 P g_1 - h_2 P g_2)) = 0, \]

since \( h_1 P g_1 - h_2 P g_2 \in \ker P \)

Case 2: Suppose that \( g_1 \) and \( g_2 \) have no common divisors. Then \( g_1 = x^s \) and \( g_2 = y^t \). Therefore
\[ f_1 = x^{n+1-j-s} y^j \quad \text{and} \quad f_2 = x^{n+1-j} y^{j-t}. \]

From the previous case we have
\[ (x y P(x^{m+1-j-1} y^j) - x^s P(x^{m+1-j-s} y^j)) \in \ker Q \]
and
\[ (y^t P(x^{m+1-j} y^{j-t}) - x y P(x^{m+1-j-1} y^{j-1})) \in \ker Q. \]

Adding these two equations together we get the desired conclusion.

Now suppose that \( g_1 \) and/or \( g_2 \) \( \in \text{Im} P \). Then choosing \( f_0 = x \) or \( y \) we find \( g_0 \) \( / \in \text{Im} P \) such that
\[ f_1 g_1 = f_0 g_0 = f_2 g_2. \]

We have
\[ Q(f_1 P(g_1)) = Q(f_1 g_1) = Q(f_0 P(g_0)). \] (2.6)

Similarly if \( g_2 \in \text{Im} P \), then
\[ Q(f_2 P(g_2)) = Q(f_2 g_2) = Q(f_0 P(g_0)). \] (2.7)

Otherwise (2.7) follows from the previous steps. Combining (2.6) and (2.7) we have the desired conclusion. ■

The importance of this Theorem can be illustrated by the following example.
Example 1. To obtain all ideal projections from $\mathbb{C}[x, y]$ onto $\mathbb{C}_m[x, y]$ we start with arbitrary $(m+1)$ polynomials $h_j \in \mathbb{C}_m[x, y]$, $j = 0, \ldots, m$. Define $P : \mathbb{C}_{m+1}[x, y] \to \mathbb{C}_m[x, y]$ by
\[ P(x^1y^{m+1-j}) = h_j \] (2.8)

Obviously the projection $P$ is $(m+1)$-ideal. To extend this projection to an ideal projection on $\mathbb{C}_{m+1}[x, y]$, we have to make sure that the polynomials $h_j$ satisfy the $m$ "consistency equations":
\[ P(xh_j-1) = P(yh_j); j = 1, \ldots, m. \] (2.9)

That guarantees that $P$ has an ideal extension to $\mathbb{C}_{m+2}[x, y]$ and hence has an ideal extension to $\mathbb{C}[x, y]$. In other words every ideal projection onto $\mathbb{C}[x, y]$ is completely determined by $(m+1)$ polynomials satisfying the consistency equations.

§3. Ideal varieties:

In this section we will switch our attention to the varieties determined by the kernels of ideal projections.

Definition 3. Let $P$ be an ideal projection. Define
\[ Z(P) := \{ u \in \mathbb{C}_2 : f(u) = 0 \text{ for all } f \in \ker P \}. \] (3.1)

To parallel the previous section we will start with a few observations:

Proposition 4. Let $P$ be an ideal projection with a finite-dimensional range. Then
1. $Z(P)$ is a finite algebraic variety. Moreover the cardinality of $Z(P)$
\[ 0 < \# Z(P) \leq \text{codim} (\ker P) = \dim (\text{Im} P). \] (3.2)
2. The projection $P$ is an interpolating projection if and only if
\[ \# Z(P) = \dim (\text{Im} P). \]
3. The projection $P$ is an interpolating projection if and only if the ideal $\ker P$ is radical.

Proof. Let $m = \text{codim} (\ker P) = \dim (\text{Im} P)$. Since $\ker P$ is an ideal, the set $Z(P) = V(\ker P)$ is a variety generated by the ideal. The left-hand side of (3.2) is the "The Weak Nullstellensatz" (cf.[5]). For the right-hand side, observe that point-evaluation functionals $\delta_u$ with $u \in Z(P)$ are
linearly independent and annihilate the ideal \( \ker P \). Hence the number of such point-evaluations can not be greater then \( \text{codim}(\ker P) \).

The second assertion is equally trivial. Indeed \((f - Pf)\) vanishes on \( Z(P) \), hence \( Pf \) interpolates \( f \) at \( m \) points and thus \( P \) is interpolating. Conversely, if \( P \) is an interpolating projection, then its \( m \) point-evaluation functionals annihilate \( \ker P \) and hence \( m \geq \#Z(P) \). Combined with (3.2) this proves 2.

Finally if \( P \) is interpolating and \( f \) vanishes on \( Z(P) \), hence \( Pf \) interpolates \( f \) at \( m \) points and thus \( P \) is interpolating.

Conversely, if \( P \) is an interpolating projection, then its \( m \) point-evaluation functionals annihilate \( \ker P \) and hence \( m \geq \#Z(P) \). Combined with (3.2) this proves 2.

We will now show that the variety \( Z(P) \) is completely determined by the zeroes of polynomials of lower order in \( \ker P \).

**Theorem 5.** Let \( P \) be an ideal projection from \( \mathbb{C}[x, y] \) onto \( \text{Im} P \subset \mathbb{C}_m[x, y] \). Then

\[
Z(\ker P) = \{(x, y) \in \mathbb{C}^2 : x^gy^k - P(x^gy^k) = 0 \text{ for } j + k \leq m + 1\}.
\]

**Proof.** Let \( W = \{(x, y) \in \mathbb{C}^2 : x^gy^k - P(x^gy^k) = 0 \text{ for } j + k \leq m + 1\} \). To show that \( W = Z(\ker P) \) it is clearly sufficient to prove that \((f - Pf)(x, y) = 0\) for every \((x, y) \in W\) and for every monomial \( f \). We proceed once again by induction on the degree of the monomial \( f \). Suppose the claim is proven for the monomials of degree \( n \geq m + 1 \), and let \( f \) be a monomial of degree \( n + 1 \). Then, without loss of generality, we assume that \( f = xg \). Let \( Pg = \sum_{j+k \leq m} a_{j,k}x^jy^k \). We have

\[
f - Pf = xg - xPg + xPg - P(xPg) = x(g - Pg) + x\left( \sum_{j+k \leq m} a_{j,k}x^jy^k \right) - P\left( x\left( \sum_{j+k \leq m} a_{j,k}x^jy^k \right) \right) = x(g - Pg) + \sum_{j+k \leq m} a_{j,k}(x^{j+1}y^k - P(x^{j+1}y^k)).
\]

Now if \((x, y) \in W\) then the first term vanishes by the inductive assumption and the rest vanish by definition of the set \( W \).

§ 4. Description of Ideal Projections

We will now specify the results of the previous sections to the ideal projections \( P \) onto \( \mathbb{C}_1[x, y] \). By Theorem 3 (see also Example 1) we conclude that all such projections can be described by three first degree polynomials: \( h_{2,0}, h_{1,1} \) and \( h_{0,2} \) (i.e. nine coefficients) provided they satisfy the
consistency equations (2.9). We use Theorem 5 to conclude that the variety $Z(P)$ is the set of zeroes of equations

$$x^j y^k - h_{j,k}(x, y) = 0 \text{ for } j + k = 2. \quad (4.1)$$

Assume that the "determining polynomials" $h_{2,0}, h_{1,1}$ and $h_{0,2}$ are given by

$$h_{2,0} = b_{2,0} + c_{2,0} x + d_{2,0} y$$
$$h_{1,1} = b_{1,1} + c_{1,1} x + d_{1,1} y$$
$$h_{0,2} = b_{0,2} + c_{0,2} x + d_{0,2} y. \quad (4.2)$$

We now rewrite the consistency equations (2.8) in terms of the coefficients of polynomials $h_{j,k}$ and we have

$$P(xP(y^2)) = P(xh_{0,2}) = P(b_{0,2} x + c_{0,2} x^2 + d_{0,2} x y) =$$
$$= b_{0,2} x + c_{0,2} h_{2,0} + d_{0,2} h_{1,1} =$$
$$= (b_{0,2} + c_{0,2} c_{2,0} + d_{0,2} c_{1,1}) x + (c_{0,2} d_{2,0} + d_{0,2} d_{1,1}) y + (c_{0,2} b_{2,0} + d_{0,2} b_{1,1}).$$

Similarly resolving the equation $P(yP(x^2)) = P(xP(yx))$ we obtain

$$P(yP(x^2)) =$$
$$= (c_{2,0} c_{1,1} + c_{0,2} d_{2,0}) x + (b_{2,0} + c_{2,0} d_{1,1} + d_{2,0} d_{0,2}) y + (c_{2,0} b_{1,1} + d_{2,0} b_{0,2}) =$$
$$= P(xP(yx)) =$$

$$(b_{1,1} + c_{2,0} c_{1,1} + c_{1,1} d_{1,1}) x + (c_{1,1} d_{2,0} + d_{1,1}^2) y + (c_{1,1} b_{2,0} + d_{1,1} b_{1,1})$$

Equating the coefficients leads to the following system of equations:

$$\begin{cases}
(b_{0,2} + c_{0,2} c_{2,0} + d_{0,2} c_{1,1}) = (c_{2,1}^2 + d_{1,1} c_{0,2}) \\
(c_{0,2} d_{2,0} + d_{0,2} d_{1,1}) = (b_{1,1} + c_{1,1} d_{1,1} + d_{0,2} d_{1,1}) \\
(c_{0,2} b_{2,0} + d_{0,2} b_{1,1}) = (c_{1,1} b_{1,1} + d_{1,1} b_{0,2}) \\
(c_{2,0} c_{1,1} + c_{0,2} d_{2,0}) = (b_{1,1} + c_{2,0} c_{1,1} + c_{1,1} d_{1,1}) \\
(b_{0,2} + c_{2,0} d_{1,1} + d_{2,0} d_{0,2}) = (c_{1,1} d_{2,0} + d_{1,1}^2) \\
(c_{2,0} b_{1,1} + d_{2,0} b_{0,2}) = (c_{1,1} b_{2,0} + d_{1,1} b_{1,1}).
\end{cases} \quad (4.3)$$

Solving for $b$s we obtain

$$\begin{cases}
b_{0,2} = (c_{2,1}^2 + d_{1,1} c_{0,2}) - (c_{0,2} c_{2,0} + d_{0,2} c_{1,1}) \\
b_{1,1} = (c_{0,2} d_{2,0} + d_{0,2} d_{1,1}) - (c_{1,1} d_{1,1} + d_{0,2} d_{1,1}) \\
b_{2,0} = (c_{1,1} d_{2,0} + d_{1,1}^2) - (c_{2,0} d_{1,1} + d_{2,0} d_{0,2}).
\end{cases} \quad (4.4)$$

It is easy to check that the rest of the equations (4.3) hold with the values assigned by (4.2). This leads to a complete description of ideal projections onto $\mathbb{C}_1[x, y]$:

**Theorem 6.** Every ideal projection $P$ from $\mathbb{C}[x, y] \to \mathbb{C}_1[x, y]$ is determined by polynomials $h_{2,0}, h_{1,1}$ and $h_{0,2}$, with coefficients satisfying (4.4).
More over if \( c_{0,2} \neq 0 \), the corresponding variety is \( Z(P) = \{(x_j, y_j); j = 1, 2, 3\} \), where

\[
x_j = -\frac{y_j^2 + c_{1,1}^2 + d_{1,1}c_{0,2} - c_{0,2}c_{2,0} - d_{0,2}c_{1,1} + d_{0,2}y_j}{c_{0,2}}
\]

and \( y_j \) are the solutions of the cubic equation

\[
0 = y^3 - (d_{0,2} + c_{1,1}) y^2 + \left(2d_{0,2}c_{1,1} - c_{1,1}^2 - 2d_{1,1}c_{0,2} + c_{0,2}c_{2,0}\right) y + \left(2c_{1,1}d_{1,1}c_{0,2} - c_{1,1}c_{0,2}c_{2,0} - d_{0,2}c_{1,1}^2 + c_{1,1}^3 - c_{0,2}c_{2,0}\right).
\]

If \( c_{0,2} = 0 \) then \( Z(P) = \{(d_{1,1}, d_{0,2} - c_{1,1}), (x_1, c_{1,1}), (x_2, c_{1,1})\} \) where \( x_1 \) and \( x_2 \) are the roots of quadratic equation

\[
x^2 - c_{2,0}x - 2c_{1,1}d_{2,0} - d_{1,1}^2 + c_{2,0}d_{1,1} + d_{2,0}d_{0,2} = 0.
\]

**Proof.** The description of \( Z(P) \) follows from solving equations (4.1) directly. \hfill \blacksquare

§5. Application to deBoor’s Conjecture.

In this section we will use description of ideal projections, to verify Conjecture 2 for ideal projections with three dimensional range.

**Theorem 7.** Let \( P \) be an ideal projection in \( \mathbb{C}[x, y] \) with \( \dim(\text{Im} P) = 3 \). Then \( P \) is a lim-interpolating projection.

**Proof.** An ideal projection \( P \) is an interpolating projection if (and only if) the corresponding ideal variety \( Z(P) \) consists of precisely three points. Next observe that for ideal projections described in the previous section, the variety \( Z(P) \) consists of three distinct points for a dense set of “free parameters” \( e-s \) and \( d-s \). Hence if \( P \) is an ideal projection onto \( \mathbb{C}_1[x, y] \), there exists a sequence of interpolating projections \( P_n \) such that \( P_n f \to Pf \) for every \( f \in \mathbb{C}[x, y] \). That means that the ideal \( J := \ker P \) can be approximated by radical ideals \( J_n = \ker P_n \). Thus if \( Q \) is a projection onto a different three dimensional subspace with \( \ker Q = \ker P \), then \( Q \) is LIP. In other words if \( Q \) is any three-dimensional ideal projection with

\[
\ker Q \cap \mathbb{C}_1[x, y] = \{0\}
\]  \hspace{1cm} (5.1)

then \( Q \) is a limit interpolating projection.

Assume now that \( 0 \in Z(\ker Q) \) and hence no polynomial with a constant coefficient belongs to \( \ker Q \). If (5.1) does not hold, then \( \ker Q \) contains one (and up to a constant multiple, only one) linear function. Indeed, if it contains two linearly independent linear functions, then \( \text{codim}(\ker Q) = 1 \).
So without loss of generality, let \( f(x, y) = y \in \ker Q \). Then \( \ker Q \) contains all polynomials that depend on \( y \), and \( x \notin \ker Q \). Since \( \text{codim}(\ker Q) = 3 \), it must not contain at least one other polynomial that does not depend on the indeterminate \( y \). Let \( E = \text{span}\{1, x, q(x)\} \) be a three dimensional space with \( E \oplus \ker Q = \mathbb{C}[x, y] \) and let \( R \) be a projection from \( \mathbb{C}[x, y] \) onto \( E \) with \( \ker R = \ker Q \). Then \( R \) is an ideal projection and it suffices to prove that \( R \) is a lim-interpolating projection.

Let \( \tilde{R} \) be the restriction of \( R \) onto \( \mathbb{C}[x] \). Then \( \tilde{R} \) is an ideal projection and by Theorem 1, there exist distinct points \( \Delta_n := \{x^{(n)}_1, x^{(n)}_2, x^{(n)}_3\} \subset \mathbb{C} \) such that the interpolating projections \( \tilde{R}_n := \tilde{R}_{\Delta_n} \rightarrow \tilde{R} \). That is

\[
\tilde{R}_n f \rightarrow \tilde{R} f \quad \text{for all } f \in \mathbb{C}[x]. \tag{5.2}
\]

For \( j = 1, 2, 3 \) define

\[
y_j^{(n)} = R(y)(x_j^{(n)}); u_j^{(n)} = (x_j^{(n)}, y_j^{(n)}) \in \mathbb{C}^2 \text{ and } \tau_n := \{u_j^{(n)}\}. \tag{5.3}
\]

Let \( R_n \) be projections onto \( E \) that interpolate at \( \tau_n \). We want to prove that

\[
R_n(x^k y^m) \rightarrow R(x^k y^m) \tag{5.4}
\]

for all \( k \) and \( m \).

Observe that \( (R(y) - y)(u_j^x) = R(y)(x_j^{(n)}) - y_j^{(n)} = 0 \), by (5.3). Hence

\[
R(y) = R_n(y). \tag{5.5}
\]

The rest of the proof proceeds by induction on \( m \). If \( m = 0 \) then (5.4) is the same as (5.2). Assume that (5.4) holds for \( m \). Then

\[
R_n(x^k y^{m+1}) = R_n(x^k y^m R_n(y)) = R_n(x^k y^m R(y)).
\]

Since \( R(y) \) is a polynomial in \( x \) only, we use the inductive assumption to conclude that \( R_n(x^k y^{m+1}) \rightarrow R(x^k y^{m+1}) \). \( \blacksquare \)

**Remark 8.** Using ”Maple” we can demonstrate that every 6-dimensional ideal projection \( P \) is LIP. We use the same procedure as the one described in the last two sections. First we describe all ideal projections onto \( \mathbb{C}_2[x, y] \). Depending on the coefficients of polynomials \( h_{j,k} \) there are 19 different descriptions of the varieties \( Z(P) \) (in Theorem 8, there are only two). However in each case the variety \( Z(P) \) depends on the roots of an equation of degree equal to \( \dim(\text{Im} P) \), which ”generically” has all distinct solutions. Next we use a procedure similar to the one involved in the proof of the last theorem to verify the claim for all 6-dimensional projections.

I am grateful to Professor deBoor for useful correspondence.
§6. References

1. deBoor, C. Presentation at the Eleventh International Conference in Approximation theory, Gatlinburg, 2004


